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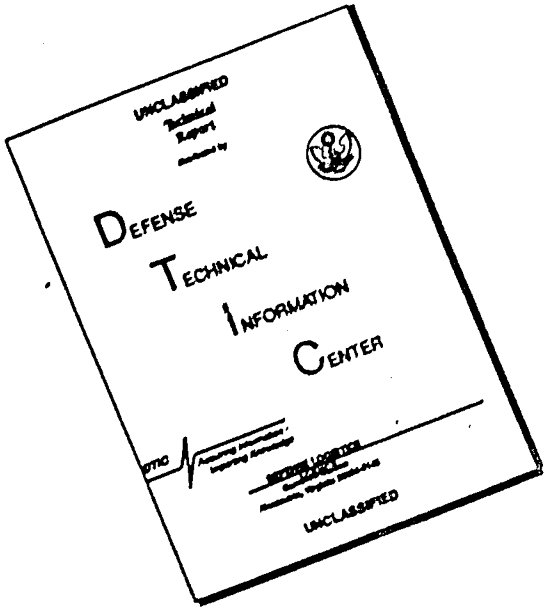

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THE HYDRODYNAMICS OF THE HYDROFOIL

[Panchenkov, A. N., Gidrodinamika podvodnogo kryla, Institute of Hydromechanics, Ukr.S.S.R. Academy of Sciences, Kiev, 1965, 539 pp.; Russian]

[ABSTRACT] [2*

The results of the investigation in the area of the hydrofoil theory are systematized in this monograph. A great deal of attention is given to the mechanical-mathematical aspects of the hydrofoil theory and to the development of the effective methods for solving the basic equations in the hydrofoil theory.

The book considers the theory of steady motion of a wing with an arbitrary profile in a plane-parallel flow; the linear theory of thin hydrofoils in the two-dimensional and three-dimensional flows; the theory of a hydrofoil in an unsteady flow; problems of the interaction of hydrofoils and motion of hydrofoils near the interface of fluids with different densities.

The monograph is intended for scientists and engineering personnel at scientific research and design organizations specializing in the area of hydrodynamics of the submerged hydrofoil, aerodynamics, and aerohydrodynamics.

Chief Editor: Corresponding Member of the Ukr.S.S.R. Academy of Sciences N. A. Kil'chevskiy.

PREFACE [3

One of the most important problems in the area of transportation, set by the Program of the Communist Party of the Soviet Union, is to increase considerably the speed of the railroad, sea, and river transports.

At the present time the problem of building high-speed ships is being successfully solved through the extensive

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*Numbers in the right margin indicate pagination in the original text.

use of hydrofoils. As a result, an intensive development of yet another area of hydrodynamics, the hydrodynamics of the hydrofoil, is being pursued. It uses widely the methods of the wave and aerodynamic theories.

The general state of studies in the area of hydrodynamics of the hydrofoil makes it possible, as early as now, to start using the analytical methods in designing lifting systems for high-speed ships. In this respect hydrofoil ships have an advantage over displacement ships.

In spite of the considerable progress made in the general wave and wave drag theories, the transition from the general theoretical solutions to the analytical methods of determining the wave drag and to the design of ship lines still involves considerable difficulties due to the complexity of ship forms, large relative thicknesses, maximum errors in the linear wave theory for the speed range used and the viscosity considerations. The hydrofoils are easier to analyze because they are submerged under a free surface and are of relatively small thickness. On the other hand, the analytical methods of the hydrodynamics of the submerged hydrofoil are, by far, more complicated than the aerodynamic methods and more time-consuming. For example, in order to obtain the information on the lifting force and induced drag on an airplane wing, it is sufficient to solve one integral-differential equation for one set of initial data. The corresponding information for the hydrofoil requires 20-30 sets of data describing the various types of motion. Naturally, these computations can be performed only with the extensive use of computers.

In the monograph, the questions which by now have re- [4
ceived considerable attention and which describe basically the subject of hydrodynamics of the hydrofoil are analyzed in sequence. Analysis of motion of submerged bodies with circulation under a free surface began almost simultaneously with the wave theory studies. The first major theoretical investigations of the hydrofoil submerged under the surface of a heavy fluid were presented by N. Ye. Kochin [56], M. V. Keldysh and M. A. Lavrent'yev [41].

In 1937, N. Ye. Kochin [56] offered a general solution of the two-dimensional problem of motion of a hydrofoil with an arbitrary profile submerged under a free surface of a heavy fluid. The method, developed by him, played an important part in further studies of the two-dimensional problems and is being widely used at present. In his study, N. Ye. Kochin, for the first time, introduces the function $H(\lambda)$, which subsequently is used by him and others

in studying the hydrodynamic grid theory, the motion of bodies near an interface, and the wing motion in shallow water.

M. V. Keldysh and M. A. Lavrent'yev [41] present a solution of the linear problem of motion for a thin hydrofoil. They obtained a general integral equation and gave an approximate solution for the case of the deeply submerged hydrofoil.

In 1958, the solution of the problem of the deeply submerged hydrofoil was offered by T. Nishiyama [209]. For the special case of thin hydrofoils, his results agreed with those obtained by M. V. Keldysh and M. A. Lavrent'yev.

The generalization of the N. Ye. Kochin solution for the case of fluids of finite depth was presented by M. D. Khaskind [156], while that of the M. V. Keldysh and M. A. Lavrent'yev solution was done by A. I. Tikhonov [147].

On the basis of the theoretical results obtained by M. V. Keldysh and M. A. Lavrent'yev, a practical design method for deeply submerged hydrofoils was offered by A. N. Vladimirov [14]. After World War II, along with the development of high-speed ships, an intensive theoretical and experimental research was conducted on shallowly submerged hydrofoils, which served as the basis for our country's wing configurations.

The experimental methods for determining the wing lifting force near a free surface were developed by S. D. Chudinov [167], M. G. Kulayev [11], and V. T. Sokolov.

The theoretical studies of the hydrodynamics of the wing of finite span were initiated in 1956 by M. D. Khaskind [155]. The problems of motion of the submerged hydrofoil in the three-dimensional fluid flow of infinite depth were also investigated by T. Nishiyama [213-216], Vu [196], G. A. Goshev [21, 22]. The study of this problem in the case of fluid of finite depth was carried out by Breslin [233].

The questions of the hydrodynamic interaction of thin [5
wings were analyzed by Isay [193, 194]. The problem of the unsteady motion of the hydrofoil in the two-dimensional flow was considered in a number of studies by I. T. Yegorov [29-30] and A. N. Shebalov [170-172].

A number of other problems of the hydrodynamics of the hydrofoil have been solved by T. Nishiyama, A. B.

Lukashevich, L. A. Epshteyn, M. B. Maseyev, G. V. Logvinovich, V. S. Voytsenya and others. The problem of motion of the vortex under a free fluid surface of finite depth with nonlinear boundary conditions on such surface was considered by N. N. Moiseyev, A. M. Ter-Krikorov, I. G. Filippov [88-90, 151, 152].

In these studies particular attention was paid to the existence of the bifurcation points when the Froude numbers are close to unity.

In spite of a large number of studies available on the hydrodynamics of the submerged hydrofoil, no study was ever published in which the basic questions are treated systematically and fully.

This monograph is the first attempt to fill this gap. The problems treated in the monograph are, in one way or another, related to those studies by the author. A number of topics such as the study of finite-span wings, of unsteady motion, and the theory of the wing near an interface, are being published for the first time.

This monograph also includes information which was obtained by the author in cooperation with A. I. Yukhimenko, S. V. Koval'chuk, A. V. Miodushevskaya, I. P. Tkachenko, P. I. Zinchuk and V. A. Stepanov (see Ch. VIII).

The author expresses deep gratitude to the Corresponding Member of the Ukr.S.S.R. Academy of Sciences N. A. Kil'chevskiy; Corresponding Member of the Ukr.S.S.R. Academy of Sciences P. F. Fil'chakov; Dr. of Tech. Sciences I. T. Yegorov; Dr. of Tech. Sciences M. Ya. Alfer'yev; Candidates of Phys. and Math. Sciences M. M. Sidlyar and V. I. Merkulov; Candidates of Tech. Sciences A. V. Vasil'yev, M. G. Starostin, V. I. Putyata; as well as to A. I. Yukhimenko, S. V. Koval'chuk, D. V. Rode, A. V. Miodushevskaya, I. P. Tkachenko, A. G. Gubish and V. M. Roman for the discussion of the results obtained.

INTRODUCTION [7

During the motion of a hydrofoil under a free surface of a heavy fluid, the surface of the fluid undergoes deformations, producing a system of waves which propagate behind the wing. The existence of the free fluid surface changes the nature of the flow around the hydrofoil, which results in the change of the lift, the induced drag and the generation of the wave drag.

Let us examine the potential flow of an ideal fluid caused by the motion of a submerged hydrofoil by introducing the following two systems of coordinates: a system, moving together with the wing (x, y, z) and a stationary system (x_1, y_1, z_1), which coincides with the axes of coordinates x, y, z at time t = 0. The free surface of the fluid at rest is perpendicular to the Oz axis and lies in the Oxy plane.

For the potential flow the velocity vector is equal to

$$\vec{v} = \nabla\varphi_1,$$

and for the incompressible fluid the continuity equation $\operatorname{div}\vec{v} = 0$ leads to the Laplace equation for the velocity potential

$$\Delta\varphi_1 = 0. \tag{1}$$

To determine the velocity potentials φ_1(x, y, z, t) let us write the boundary conditions.

On the surface of the wing s we have the following streamline condition

$$\varphi_{1n} = v_n(Q, t), \tag{2}$$

where Q - a point on the surface s;
v_n - the normal velocity component of point Q.

The normal component is directed into the fluid. The second boundary condition is obtained from the dynamic and kinematic conditions on the free surface of the fluid. The dynamic boundary condition is obtained by applying the Cauchy-Lagrange integral to the points on the free surface.

By assuming that the pressure on the free surface is [8
constant we obtain the following equation:

$$g\eta + \varphi_{1t} + \frac{1}{2}(\nabla\varphi_1)^2 = 0. \tag{3}$$

The kinematic relationship will be as follows:

$$\varphi_{1x}\eta_x + \varphi_{1y}\eta_{1y} + \eta_t = \varphi_{1z}. \tag{4}$$

Here η(x, y, z, t) denotes the elevation of the free surface.

The problem of determining the potential from relations (3) and (4) is nonlinear and these relations are satisfied at the boundary unknown beforehand. The linearization of the boundary conditions on the free surface results in the corresponding linear conditions being satisfied at the known boundary (at the unperturbed level). In this case the statement of the problem corresponds to the classical statement of the problem in the theory of potential flow. Only a few solutions to the nonlinear problems in the wave and the hydrofoil theories exist in literature. Most fully treated are the methods dealing with the theory of small waves. At the present time, they are practically the only methods which permit the study of the important practical problems in the submerged hydrofoil theory.

The ship wave theory shows, in a number of cases (large values of $\frac{B}{L}$ and Fr_b), the inadequacy of linear approximations within the boundary conditions on the free surface. The hydrodynamics of the hydrofoil may, however, give more accurate results in the linear consideration, because the profiles used in practice have small relative thicknesses (5-6%) and are submerged below the free surface.

The linear wave theory considers the infinitely small movements of the fluid with respect to the equilibrium position. In this case, the deviation of particles from the equilibrium position and the velocities produced will be small (of the first order of magnitude). Then, by disregarding the insignificantly small quantities of the second order, let us, in the first approximation, replace the total derivatives with respect to time by partial derivatives to obtain linear conditions on the free surface by satisfying the conditions at the undisturbed level.

The systematic derivation of linear boundary conditions on the free surface can be made by expanding, in terms of the small parameter, the solution of equation (1) into series and taking conditions (3) and (4) into account. For a submerged hydrofoil, this approach leads to the evaluation of the approximation resulting from the linear conditions obtained for various Froude numbers [113].

Let us consider the steady motion of a submerged hydrofoil with a velocity v_0. Equations (3) and (4), expressed in the dimensionless form, will be [9

$$\frac{1}{Fr_b^2}\bar{\eta} - \bar{\varphi}_x + \frac{1}{2}(\nabla\bar{\varphi})^2 = 0, \tag{5}$$

$$\bar{\varphi}_x\bar{\eta}_x + \bar{\varphi}_y\bar{\eta}_y + \bar{\eta}_t = \bar{\varphi}_z, \tag{6}$$

where $Fr_b = \frac{v_0}{\sqrt{gb}}$;

φ - the velocity potential in the movable coordinate system;

b - the chord of the wing.

By writing $\bar{\eta}$ and $\bar{\varphi}$ as the power series in terms of the small parameter ε we have the following:

$$\bar{\eta} = \varepsilon\bar{\eta}_1 + \varepsilon^2\bar{\eta}_2 + \varepsilon^3\bar{\eta}_3 + \ldots \tag{7}$$

$$\bar{\varphi} = \varepsilon\bar{\varphi}_1 + \varepsilon^2\bar{\varphi}_2 + \varepsilon^3\bar{\varphi}_3 + \ldots \tag{8}$$

Substitution of expressions (7) and (8) into equations (5) and (6) produces a series of boundary conditions for determining $\bar{\varphi}_j$; however, depending on the order of magnitude of $\frac{1}{Fr_b{}^2}$, these conditions will be different.

Depending on the order of magnitude of the quantity $\frac{1}{Fr_b{}^2}$, the following three modes of motion can be selected:

I. $\frac{1}{Fr_b^2} \sim \frac{1}{\varepsilon^n}$ - low velocities.

II. $\frac{1}{Fr_b^2} \sim 1$ - transitional mode.

III. $\frac{1}{Fr_b^2} \sim \varepsilon^n$ - high velocities.

Let us analyze all three cases.

I.

$$\frac{1}{Fr_b^2} \sim \frac{1}{\varepsilon^n}$$

1. $\bar{\eta}_1 = 0$

1. $\bar{\eta}_2 = 0$

. . . .

$$
\left.\begin{array}{ll}
n+1. & \dfrac{1}{\mathrm{Fr}_b^2}\varepsilon^n\bar{\eta}_{n+1}-\bar{\varphi}_{1x}=0 \\
n+2. & \dfrac{1}{\mathrm{Fr}_b^2}\varepsilon^n\bar{\eta}_{n+2}-\bar{\varphi}_{2x}+\dfrac{1}{2}(\nabla\bar{\varphi}_1)^2-\bar{\eta}_{n+1}\bar{\varphi}_{xy_1}=0 \\
 & \qquad\qquad (z=0)\ldots \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
n+k. & \dfrac{1}{\mathrm{Fr}_b^2}\varepsilon^n\bar{\eta}_{n+k}-\bar{\varphi}_{kx}+H_{(k-1)}=0 \\
1. & \bar{\eta}_{1t}=\bar{\varphi}_{1z} \\
2. & \bar{\eta}_{2t}=\bar{\varphi}_{2z} \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
n+1. & \bar{\eta}_{n+1,t}=\bar{\varphi}_{n+1,z} \\
n+2. & \bar{\eta}_{n+2,t}=\bar{\varphi}_{n+2,z}-\bar{\varphi}_{1,x}\bar{\eta}_{n+1,x}+ \\
 & \qquad +\bar{\varphi}_{1zz}\bar{\eta}_{n+1}-\bar{\varphi}_{1y}\bar{\eta}_{n+1,y} \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
n+k & \bar{\eta}_{n+k_t}=\bar{\varphi}_{n+k_z}+G_{n+k-1}
\end{array}\right| \qquad (9)
$$

II.

$$
\left.\begin{array}{ll}
 & \dfrac{1}{\mathrm{Fr}_b^2}\sim 1 \\
1. & \dfrac{1}{\mathrm{Fr}_b^2}\bar{\eta}_1-\bar{\varphi}_{1x}=0 \\
2. & \dfrac{1}{\mathrm{Fr}_b^2}\bar{\eta}_2-\bar{\varphi}_{2x}+\dfrac{1}{2}(\nabla\bar{\varphi}_1)^2=0 \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
n. & \dfrac{1}{\mathrm{Fr}_b^2}\bar{\eta}_n-\bar{\varphi}_{nx}+F_{(n-1)}=0 \\
 & \bar{\eta}_{1t}=\bar{\varphi}_{1z} \\
 & \bar{\eta}_{2t}=\bar{\varphi}_{2z}-\bar{\varphi}_{1x}\eta_{1x}+\bar{\varphi}_{1zz}\bar{\eta}_1-\bar{\varphi}_{1y}\bar{\eta}_{n+1,y} \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\
 & \bar{\eta}_{nt}=\bar{\varphi}_{nz}+G_{(n-1)}
\end{array}\right| \qquad (10)
$$

$$\frac{1}{\mathrm{Fr}_b^2} \sim \varepsilon^n$$

$$\left.\begin{array}{l}
1.\quad \bar{\varphi}_{1x} = 0 \\
2. - \bar{\varphi}_{2x} + \frac{1}{2}(\nabla\bar{\varphi}_1)^2 = 0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n+1.\quad \frac{1}{\mathrm{Fr}_b^2}\varepsilon^n\bar{\eta}_1 - \bar{\varphi}_{n+1_x} + N_{(n)} = 0 \\
1.\quad \bar{\eta}_{1t} = \bar{\varphi}_{1z} \\
2.\quad \bar{\eta}_{2t} = \bar{\varphi}_{2z} - \bar{\varphi}_{1x}\eta_{1x} + \bar{\varphi}_{1zz}\bar{\eta}_1 - \varphi_{1y}\eta_{1y} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n.\quad \bar{\eta}_{nt} = \bar{\varphi}_{nz} + G_{(n-1)}
\end{array}\right\}. \tag{11}$$

By determining $\bar{\varphi}_j$ in succession, using the linear boundary conditions (9-10), it is possible to obtain an asymptotic solution, in the form of the series (8), for the hydrofoil with nonlinear boundary conditions on the free surface. However, this method does not make it possible to obtain the general solution, for any Froude number and higher degrees of approximations, because even in the first approximation, the boundary conditions for determining potentials $\bar{\varphi}_j$ are different for different types of motion.

From relations (9) it follows that, for low velocities, the boundary conditions on the free surface are as follows:

$$\mathrm{Fr}_b < \sqrt{\varepsilon},\quad \bar{\eta} = 0,\quad \bar{\eta}_t = \bar{\varphi}_z. \tag{12}$$

with the dropped terms being of the order of ε^{n+1}.

In the transitional mode, the first approximation gives us the classical linear boundary condition

$$\frac{1}{\mathrm{Fr}_b^2}\bar{\eta} - \bar{\varphi}_x = 0,\quad \bar{\eta}_t = \bar{\varphi}_z \quad (\mathrm{Fr}_b \sim 1). \tag{13}$$

At high velocities the boundary conditions in the first approximation are:

$$\mathrm{Fr}_b > \frac{1}{\sqrt{\varepsilon}},\quad \bar{\varphi}_x = 0,\quad \bar{\eta}_t = \bar{\varphi}_z. \tag{14}$$

The dropped terms are always of the ε^2 order.

J. Stoker derived the boundary conditions in the dimensional form without evaluating the order of magnitude of terms in equations (13-14) [142]. Specifically, he assumed that the value φ_t in the expression (3) can be expanded into the following series: [12

$$\varphi_t = \varepsilon\varphi_{1t} + \varepsilon^2\varphi_{2t} + \varepsilon^3\varphi_{3t} + \dots$$

The boundary conditions of the type given in (9-11) can also be obtained, without any major complications, for the unsteady motion. Considering the unsteady motion of the hydrofoil with a finite constant velocity and infinitely small variations of this velocity, we obtain the same approximations which were derived for the steady motion.

The boundary conditions (12) and (14) and their coresponding solutions can be obtained in the form of limiting conditions from equation (13). The linear problem, therefore, will be formulated using the general terms of (13). For the arbitrary motion of the hydrofoil the linear conditions on the free surface can be combined into one expression and can be written as follows:

$$\varphi_{1tt} + g\varphi_{1z} = 0. \qquad (15)$$

Conditions (2) and (15) are not the only solutions of equation (1). The unique solution of the problem can be obtained by considering a series of additional conditions which describe the type of flow at infinity and near the edges of the wing. The condition for the absence of perturbations at infinity can be satisfied by considering the following:

$$\nabla\varphi \to 0 \text{ at } x \to \infty, \quad z \to \pm\infty. \qquad (16)$$

Condition (16) excludes from computations the appearance of the free waves that satisfy equation (1). This can also be accomplished by the introduction into the equation of small dissipating forces which are proportional to the velocity of fluid particles. Designating these forces by the formula

$$\vec{P} = -\mu\nabla\varphi,$$

where μ is the positive coefficient, which in the final solution should tend to zero, one can again assume the

existence of the velocity potential. In this case the boundary conditions (15) will take a more general form:

$$\varphi_{1tt} + \mu\varphi_{1t} + \varphi_{1z} = 0. \tag{17}$$

The problem of the hydrofoil motion can be formulated, similarly to the aerodynamic problem for the wing moving in the three-dimensional flow, as the problem of motion of lifting surfaces. During the motion of a lifting surface in a fluid, a velocity discontinuity surface is produced on which certain boundary conditions must also be satisfied. Let s be the given pressure and velocity discontinuity surface and let Σ be the semi-infinite velocity discontinuity surface which extends beyond the surface s and adjoins this surface along line L.

On surface s the impermeability condition is satisfied while on surface Σ the following two conditions have to be met [126]: [13

a) pressure continuity in the transitional zone of Σ;

b) coincidence of normal velocities of points on the surface with the corresponding velocities of the surrounding fluid.

During the motion of a lifting surface through a fluid at rest these conditions will be expressed in the following form:

$$s(x, y, z, t) = 0, \qquad s_t + \nabla\varphi_1\nabla s = 0; \tag{18}$$

$$\Sigma(x, y, z, t) = 0, \qquad \Sigma_t + \nabla\varphi_{1\pm}\nabla\Sigma = 0, \qquad p_+ = p_-. \tag{19}$$

Let us introduce a surface $s_p(x, y, z, t)$, which moves in a given manner, without perturbing the fluid. According to equation (18) on this surface

$$s_{pt} - \vec{v}\nabla s_p = 0, \tag{19'}$$

where $\overleftarrow{v}$ is the velocity of points on the surface in the fixed system of coordinates.

If the perturbations caused by the moving surface s are small, then surface s can be described as follows:

$$s = s_p - s_L = 0, \tag{20}$$

where s_L is infinitely small.

It follows that if the small values of the higher order are neglected, the expression (18) can be written as follows:

$$s_p(x, y, z, t) = 0, \qquad -s_{Lt} + \vec{v}\nabla s_L + \nabla\varphi\nabla s_p = 0. \tag{21}$$

By dropping in the expression (19) the terms that depend on the induced velocities, the condition on the surface Σ can be rewritten in the following form:

$$\Sigma_t - \vec{v}\nabla\Sigma = 0. \tag{22}$$

From relationship (22) it becomes evident that, in the linear theory, the shape of surface Σ is independent of the velocities induced.

Based on equation (22) the conditions on surface Σ can be written as follows:

$$\Sigma(x, y, z, t) = 0, \qquad (\nabla\varphi_+ - \nabla\varphi_-)\nabla\Sigma = 0,$$
$$p_+ - p_- = 0. \tag{23}$$

The flow pressure will be determined by the Lagrange-Cauchy integral

$$p = -\varrho\varphi_{1t} - \frac{\varrho}{2}|\nabla\varphi_1|^2 + F(t).$$

Neglecting the small values of the second order as well as value F(t), which is independent of coordinates, we get

$$p = -\varrho\varphi_{1t}. \tag{24}$$

The boundary conditions on surfaces s_p and Σ can then be set by the following relations: [14

$$s_p = 0, \qquad \varphi_n|\nabla s_p| = -\vec{v}\nabla s_L + s_{Lt}, \tag{25}$$
$$\Sigma = 0, \qquad \varphi_{n+} = \varphi_{n-}, \qquad \varphi_{1t+} = \varphi_{1t-}. \tag{26}$$

It is necessary to add to these conditions, the conditions which determine the nature of the flow around the edges of the surface s.

The flow conditions around edge L can be determined from the N. Ye. Zhukovskiy-S. A. Chaplygin postulate. The conditions of the postulate are satisfied if $\nabla\varphi$ is finite

at L.

For the leading edge of the wing let us assume the following condition: $\nabla\varphi \sim \frac{1}{\delta^{\alpha}}$, $\alpha < 1$, where δ is the distance to the leading edge. Let us examine a lifting surface and surface s_p, which is in the Oxy plane, both moving in the positive direction of the x axis at a velocity v_0. In this case the boundary conditions (25) and (26) will simplify and take the following form:

$$s_p = 0, \qquad \varphi_z = s^0_{Lt} - v_0 s^0_{Lx}, \tag{27}$$

$$\Sigma = 0, \qquad \varphi_{z+} = \varphi_{z-}, \qquad (\varphi_t - v_0\varphi_x)_+ = (\varphi_t - v_0\varphi_x)_-. \tag{28}$$

Now, a combination boundary problem for the Laplace equation can be formulated. To formulate this problem, which is the basic problem in the hydrodynamics of the hydrofoil, is to find a solution to the following equation

$$\Delta\varphi = 0 \tag{29}$$

for a region confined between the horizontal plane Oxy and surface $s + \Sigma$ with the following boundary conditions:

$$\varphi_{tt} - 2v_0\varphi_{tx} + v_0^2\varphi_{xx} + g\varphi_z = 0 \quad \text{at } z = 0, \tag{30}$$

$$\varphi_{z+} = s_{Lt} - v_0 s_{Lx} \text{ on } s_p, \tag{31}$$

$$\varphi_{z+} = \varphi_{z-},$$

$$(\varphi_t - v_0\varphi_x)_+ = (\varphi_t - v_0\varphi_x)_- \text{ on } \Sigma \tag{32}$$

and conditions

$$\nabla\varphi \to 0 \text{ at } x \to \infty, \qquad z \to \pm\infty, \tag{33}$$

$\nabla\varphi$ is finite at the trailing edge of surface s, (34)

$\nabla\varphi \sim \frac{1}{\delta^{\alpha}}$, $\alpha < 1$ at the leading edge of surface s. (35)

Here s_L is the initial distance from a point on the lifting surface to the x,y,0 plane.

If the coefficient of the dissipating forces is introduced, conditions (30) and (33) are substituted by a single expression

$$\varphi_{tt} - 2v_0\varphi_{tx} + v_0^2\varphi_{xx} + \mu\varphi_t - v_0^2\mu\varphi_x + g\varphi_z = 0. \qquad (36)$$

Other generalizations of the formulated problem will be given below.

TABLE OF CONTENTS

PART ONE [15

THE THEORY OF THE SUBMERGED HYDROFOIL IN A PLANE-PARALLEL FLOW

CHAPTER I. LINEAR THEORY OF A THIN HYDROFOIL IN A PLANE-PARALLEL FLOW [17

1.1. Integral Equations for a Thin Hydrofoil

For the steady wing motion in a plane-parallel flow, conditions (32) change into conditions which are satisfied at all points in the flow $(\varphi_y = 0;\ \varphi_{z+} = \varphi_{z-};\ \varphi_{x+} = \varphi_{x-})$, and the formulated problem is greatly simplified. The solution of this problem can be obtained by using the effective methods used in the theory of analytical functions [25, 91].

Let us examine the complex potential for the given flow:

$$W(z) = -v_0 z + W(z);$$
$$W(z) = \varphi(x, y) + i\psi(x, y).$$

Here W(z), $\varphi(x, y)$ and $\Psi(x, y)$ are the complex potential, the velocity potential, and the stream function of the induced velocities, respectively.

For the W(z) function conditions (30) and (33) can be written in the following form:

$$\operatorname{Im}[iW_{zz}(z) - \nu W_z(z)] = 0, \quad (y = 0), \tag{I.1}$$

where $\nu = \frac{g}{v_0^2}$;

$$\lim_{x\to\infty} W_z(z) = 0. \tag{I.2}$$

Expression (I.1) can also be written as follows:

$$\operatorname{Im}[iW_z(z) - \nu W(z)] = 0. \tag{I.3}$$

Let C be the given velocity discontinuity line. Then the boundary condition (31) on line C will be expressed in the following form:

$$\varphi_y = -v_0 f'(x) = F(x). \tag{I.4}$$

Let us introduce here function $\Phi(z)$: [18

$$\Phi(z) = iW_z(z).$$

Then expressions (I.4) will become the boundary conditions for the Dirichlet function $\Phi(z)$:

$$\operatorname{Re}\Phi_+(x) = \operatorname{Re}\Phi_-(x) = F(x), \tag{I.5}$$

where F(x) is the given function for C that satisfies the Gelder condition [15].

The solution of equation (I.5) will be sought in the following form:

$$\Phi(z) = \frac{1}{2\pi}\int_C \gamma(s)\left[\frac{1}{z-s} + k(s, z)\right]ds, \tag{I.6}$$

where k(s, z) is an analytical function in the lower half-plane.

From the Yu. Sokhotsky formula for the limiting values of integral (I.6) we have the following:

$$\begin{aligned}\Phi_+(x) &= \frac{i}{2}\gamma(x) + \frac{1}{2\pi}\int_L \gamma(s)\left[\frac{1}{x-s} + k(s, x)\right]ds,\\ \Phi_-(x) &= -\frac{i}{2}\gamma(x) + \frac{1}{2\pi}\int_L \gamma(s)\left[\frac{1}{x-s} + k(s, x)\right]ds,\end{aligned} \tag{I.7}$$

from which, using expression (I.5), the following integral equation is derived:

$$\begin{gathered}\frac{1}{2\pi}\int_C \gamma(s)\left[\frac{1}{x-s} + G(s, x)\right]ds = F(x);\\ G(s, x) = \operatorname{Re}k(s, x).\end{gathered} \tag{I.8}$$

The complex potential W(z) will be in the form

$$W(z) = \frac{1}{2\pi i}\int_C \gamma(s)\left[\ln(z-s) + k_1(s, z)\right]ds, \tag{I.9}$$

with $k_1(s, z) = k_z(s, z)$.

If the function $k_1(s, z)$ satisfies the conditions (I.1) and (I.2), then the expression (I.9) satisfies all the conditions of the problem and is the only expression which satisfies these conditions. The proof of this is given by M. V. Keldysh and M. A. Lavrent'yev [41].

The problem examined can be generalized for the case of a wing with the discontinuity of normal velocity. For this problem the complex potential can be expressed in the [19
following form:

$$W(z) = \frac{1}{2\pi i}\int_C \{\gamma(s)[\ln(z-s) + k_1(s, z)] + ig(s)[\ln(z-s) + k_2(s, z)]\}\, ds, \qquad (I.10)$$

where $g(s) = v_{n-}(s) - v_{n+}(s)$ is the discontinuity of the normal velocities.

1.2. Source and Vortex Flow

A convenient method for solving the W(z) function by using condition (I.3) belongs to M.V. Keldysh [40].

If the imaginary term of a certain function $\Omega(z)$ is equal to zero on the OX axis, then by using the Schwartz's symmetry principle, it can be determined in the following manner:

$$\Omega(z) = \Psi_-(z) + \bar{\psi}_-(z), \qquad (I.11)$$
$$\mathrm{Im}\,\Omega(z) = 0. \qquad (y = 0),$$

where $\Psi_-(z)$ is the given function of point z in the half-plane s.

M. V. Keldysh introduces a function $\Omega(z)$ which can be determined from the following combination:

$$\Omega(z) = iW_z(z) - \nu W(z). \qquad (I.12)$$

According to conditions (I.3), function $\Omega(z)$ satisfies the given conditions and consequently it can be constructed on the basis of formula (I.11).

For the moving vortex under a free surface, the function $\Psi_-(z)$ will have the following form:

$$\psi_{-}(z) = \frac{\Gamma}{2\pi i}\left[\frac{i}{z-\zeta} - \nu \ln(z-\zeta)\right] \tag{I.13}$$

and the relations (I.11) and (I.12) will result in the ordinary differential equation of the first order with respect to function W(z) as follows:

$$iW_z(z) - \nu W(z) = \frac{\Gamma}{2\pi i}\left[\left(\frac{i}{z-\zeta} + \frac{i}{z-\bar{\zeta}}\right) - \nu \ln\frac{z-\zeta}{z-\bar{\zeta}}\right]. \tag{I.14}$$

By integrating this equation and solving for the arbitrary constant in equation (I.2), we receive the following:

$$W(z) = \frac{\Gamma}{2\pi i}\left(\ln\frac{z-\zeta}{z-\bar{\zeta}} + 2e^{-\gamma iz}\int\limits_{+\infty}^{z}\frac{e^{\gamma it}}{t-\bar{\zeta}}\,dt\right). \tag{I.15}$$

For the source the differential equation will have the following form:

$$iW(z) - \nu W(z) = \frac{Q}{2\pi}\left\{\frac{i}{z-\zeta} - \frac{i}{z-\bar{\zeta}} - \nu \ln[(z-\zeta)(z-\bar{\zeta})]\right\}. \tag{I.16}$$

the solution of which is given by the following formula: [20

$$W(z) = \frac{Q}{2\pi}\left\{\ln[(z-\zeta)(z-\bar{\zeta})] - 2e^{-\gamma iz}\int\limits_{+\infty}^{z}\frac{e^{\nu it}\,dt}{t-\bar{\zeta}}\right\}. \tag{I.17}$$

The solution of the two-dimensional problems with the condition (I.3) taken into account can be also obtained by the Fourier method. The Fourier method will be applied to a number of problems below.

Let us introduce some additional formulas to describe forces due to the moving source and vortex under the free surface:

$$\left.\begin{aligned} P_{\text{в}} &= +\varrho\Gamma\left[V_0 - \frac{\Gamma}{4\pi h} + \nu\frac{\Gamma}{\pi}e^{-2\nu h}E_i(2\nu h)\right] \\ Q_{\text{в}} &= \varrho\nu\Gamma^2 e^{-2\nu h}\end{aligned}\right\}; \tag{I.18}$$

$$\left.\begin{aligned} P_{\text{и}} &= -\frac{Q^2}{2\pi}\left[\frac{1}{2h} - 2\nu e^{-2\nu h}E_i(2\nu h)\right] \\ Q_{\text{и}} &= \varrho Q[Q\nu e^{-2\nu h} + V_0]\end{aligned}\right\}. \tag{I.19}$$

Here $E_{i1}(x) = \mathrm{Re}E_i(x)$;

$E_i(x)$ - the integral exponential function [181];

h - the distance from the undisturbed free surface;

$P_{И}$, $Q_{И}$ - lift and drag of the disturbance.

From condition (13) follows the equation for the shape of the free surface:

$$\eta = +\frac{v_0}{g}\,\mathrm{Re}\,[W_z(z)] \qquad (y = 0). \tag{I.20}$$

The remainder computation gives the following:

$$\int_{-\infty}^{+\infty} \frac{e^{i\nu t}}{t-\xi}\,dt = 2\pi i e^{i\nu\xi}.$$

Using this expression we obtain, for large negative values of x, an asymptotic formula for the wave surface with the moving vortex as follows:

$$\eta = \frac{2\Gamma}{v_0} e^{-\nu h} \sin \nu (x - \zeta). \tag{I.21}$$

With the source in motion, the shape of the free surface at a large distance behind it will be determined by the following formula:

$$\eta = \frac{2Q}{v_0} e^{-\nu h} \cos \nu (x - \zeta). \tag{I.22}$$

Formulas (I.21) and (I.22) indicate that at large distances beyond the disturbance, the free surface has a sinusoidal shape. [21

Solutions (I.15) and (I.17) determine the following functions $k_1(s, z)$ and $k_2(s, z)$:

$$\left.\begin{aligned} k_1(s, z) &= -\ln(z - \bar{s} - 2ih) + 2e^{-\gamma i z}\int_{+\infty}^{z} \frac{e^{\gamma i t}}{t - \bar{s} - 2ih}\,dt \\ k_2(s, z) &= \ln(z - \bar{s} - 2ih) - 2e^{-\gamma i z}\int_{+\infty}^{z} \frac{e^{\gamma i t}}{t - s - 2ih}\,dt \end{aligned}\right\}, \tag{I.23}$$

from which it follows that $k_2(s, z) = -k_1(s, z)$.

1.3. Formulas for Forces Acting on a Thin Hydrofoil. Deeply Submerged Hydrofoils.

The lifting force of a thin wing is determined from the expression

$$P = \int_{-a}^{+a} (p_- - p_+)\, ds,$$

however, for a steady motion the flow pressure will be determined by the following formula:

$$p = \rho v_0 \varphi_x.$$

Then

$$P = \rho v_0 \int_{-a}^{+a} (\varphi_{s-} - \varphi_{s+})\, ds,$$

however

$$\varphi_{x-} - \varphi_{x+} = \gamma(x),$$

and the formula for the lifting force will acquire the following form:

$$= \rho v_0 \int_{-a}^{+a} \gamma(s)\, ds. \qquad (I.24)$$

The moment of the hydrodynamic forces relative to the center of a thin hydrofoil is determined in the same manner. It is considered to be positive if it is directed clockwise:

$$M = \rho v_0 \int_{-a}^{+a} s\gamma(s)\, ds, \qquad (I.25)$$

where a is the half-chord of the wing.

The asymptotic expression for the large negative values of x for the shape of the wave profile has the following form: [22

$$\eta = \frac{2}{v_0} \operatorname{Re} e^{-\nu i t} \int_{-a}^{+a} [g(s) + i\gamma(s)]\, e^{\nu i \zeta s}\, ds. \qquad (I.26)$$

To calculate the wave drag, one may utilize the well-known formula which describes it by means of the plane wave amplitude as follows [47, 139]:

$$Q = \frac{1}{4}\varrho g a^2. \tag{I.27}$$

In accordance with expressions (I.26) and (I.27)

$$Q = \frac{\varrho g}{V_0^2} e^{-2vh} \left| \int_{-a}^{+a} [g(s) + i\gamma(s)] e^{vis} ds \right|^2. \tag{I.28}$$

Expressing formulas (I.24), (I.25) and (I.28) in the dimensionless form, we have the following:

$$P = \varrho a v_0^2 \int_{-1}^{+1} \bar{\gamma}(\bar{s})\, d\bar{s}; \tag{I.29}$$

$$M = \varrho a^2 v_0^2 \int_{-1}^{+1} \bar{s}\bar{\gamma}(\bar{s})\, d\bar{s}; \tag{I.30}$$

$$Q = \varrho g^2 a^2 e^{-\lambda} \left| \int_{-1} [\bar{g}'(\bar{s}) + i\bar{\gamma}(\bar{s})] e^{i\frac{1}{4 \cdot Fr_b^2}\bar{s}} ds \right|^2. \tag{I.31}$$

M. V. Keldysh and M. A. Lavrent'yev have derived formulas which can determine the lift P and the moment M to the second degree of accuracy [41].

Equation (I.28) can be written as

$$\int_{-1}^{+1} \bar{\gamma}(s) \left[\frac{1}{\bar{x} - \bar{s}} + k(\bar{x} - \bar{s}) \right] d\bar{s} = -2\pi \bar{f}'(x), \tag{I.32}$$

where

$$\bar{k}(\bar{x}) = \frac{1}{\bar{x} - 4\bar{h}i} - \frac{\omega}{2} i e^{-\frac{\omega}{2}xi} \int_{\infty}^{x} \frac{e^{\frac{\omega}{2}i\zeta}}{\zeta - 4\bar{h}i} d\zeta, \tag{I.33}$$

and $h = \frac{h}{2a}$.

Equation (I.32) is singular with the main expression containing the regular part. [23

The theory of singular equations is very well presented in the monograph by N. I. Muskhelishvili [91].

The singular equation does not contain only one solution, and the singularity of the solution is determined by the additional conditions at both ends of the interval. In our problem, the singularity of the solution will be determined by the conditions postulated by N. Ye. Zhukovskiy and S. A. Chaplygin, which require that the solution be restricted to point x = -1. Consequently, we will seek a solution to equation (I.32) in the class of functions limited at point x = -1.

If the regular part of the main expression is absent, then the solution of equation (I.32) is a closed one and is given by the Cauchy's integral transformation formulas [15, 91]. The presence of the regular part in the main expression complicates the equation considerably. To solve the equation a number of various approximation methods are used [41, 112]. One approximate solution to equation (I.32) was offered by M. V. Keldysh and M. A. Lavrent'yev [42]. They seek the solution in terms of a power series $1/\bar{h}$, which converges for deeply submerged hydrofoils and, therefore, determines the characteristics of the latter. Let us cite certain points in this solution.

Assuming that the solution to equation (I.32) exists and represents the regular function of the submerged wing h, let us seek this solution in the following form:

$$\bar{\gamma}(\bar{s}) = \gamma_0(\bar{s}) + \frac{1}{\bar{h}}\bar{\gamma}_1(\bar{s}) + \frac{1}{\bar{h}^2}\bar{\gamma}_2(\bar{s}) + \ldots \qquad (I.34)$$

The function (I.33) can be expanded as follows:

$$\bar{k}(\bar{x}) = \frac{1}{\bar{x}} + \sum_{n=0}^{\infty} k_n \frac{\bar{x}^n}{\bar{h}^{n+1}}. \qquad (I.35)$$

Relations (I.34) and (I.35) produce a system of recurrent equations

$$\int_{-1}^{+1} \frac{\gamma_0(\bar{s})\,\partial\bar{s}}{\bar{x}-\bar{s}} = -2\pi\bar{f}'(\bar{x}),$$

$$\int_{-1}^{+1} \frac{\gamma_n(\bar{s})\,ds}{\bar{x}-\bar{s}} = -\int_{-1}^{+1} \sum_{m=0}^{m=n-1} k_m(\bar{x}-\bar{s})\,\gamma_{n-m-1}(\bar{s})\,ds, \qquad (m=1,2\ldots). \qquad (I.36)$$

The problem of evaluating each function $\gamma_n(x)$ reduces, therefore, to the solution of the singular equation

$$\int_{-1}^{+1} \frac{\varphi(\bar{s})\, d\bar{s}}{\bar{x} - \bar{s}} = \Psi(\bar{x}),$$ [24

the solution of which, limited at point $\bar{x} = -1$, is described by the Cauchy's integral transformation formulas

$$\varphi(\bar{x}) = -\frac{1}{\pi^2} \sqrt{\frac{1+\bar{x}}{1-\bar{x}}} \int_{-1}^{+1} \sqrt{\frac{1-\bar{s}}{1+\bar{s}}} \frac{\Psi(\bar{s})}{\bar{x}-\bar{s}} d\bar{s}. \qquad (I.37)$$

The formulas which describe forces acting on a thin foil and which were derived by this method, have the following form:

$$P = 2\pi\rho a v_0^2 \left[\alpha + p_{11}\frac{\alpha}{\bar{h}} + \left(p_{21}\alpha + p_{22}\frac{\alpha_0}{2}\right)\frac{1}{\bar{h}^2} + \left(p_{31}\alpha + p_{32}\frac{\alpha_0}{2}\right)\frac{1}{\bar{h}^3}\right], \qquad (I.38)$$

$$M_f = \pi\rho a^2 v_0^2 \left[\frac{1}{4}\alpha_0 + m_{21}\alpha\frac{1}{\bar{h}^2} + \left(m_{31}\alpha + m_{32}\frac{\alpha_0}{2}\right)\frac{1}{\bar{h}^3}\right], \qquad (I.39)$$

$$Q = 8\pi^2 a^2 \rho g e^{-\lambda}\left[\alpha^2 + g_1\frac{\alpha^2}{\bar{h}} + \left(g_{21}\alpha^2 + g_{22}\frac{\alpha\alpha_0}{2} + g_{23}\frac{\alpha_0^2}{4}\right)\frac{1}{\bar{h}^2} + \right.$$

$$\left. + \left(g_{31}\alpha^2 + g_{32}\frac{\alpha\alpha_0}{2}\right)\frac{1}{\bar{h}^3}\right], \qquad (I.40)$$

where M_f is the moment with respect to the point located at a distance $\frac{1}{2}a$ from the leading edge of the wing:

$$M_f = M + \frac{a}{2}P;$$

$$p_{11} = -\frac{\pi}{2}\lambda e^{-\lambda};$$

$$p_{21} = \frac{1}{8}\left[2\pi^2\lambda^2 e^{-2\lambda} - \frac{1}{2} - \lambda + \lambda^2 e^{-\lambda} E_{i1}(\lambda)\right];$$

$$p_{22} = \frac{1}{2^6}[1 + 2\lambda - 2\lambda^2 e^{-\lambda} E_{i1}(\lambda)];$$

$$p_{31} = \frac{\pi\lambda e^{-\lambda}}{16}\left[1 + 2\lambda + \frac{3}{8}\lambda^2 - 2\lambda^2 E_{i1}(\lambda) e^{-\lambda} - 2\pi^2\lambda^2 e^{-2\lambda}\right];$$

$$p_{32} = -\frac{\pi\lambda e^{-\lambda}}{2^7}\left[1 + 2\lambda + \frac{3}{4}\lambda^2 - 2\lambda^2 e^{-\lambda} E_{i1}(\lambda)\right];$$

$$m_{21} = -\frac{1}{2^5}[1 + 2\lambda - 2\lambda^2 e^{-\lambda} E_{i1}(\lambda)];$$ [25

$$m_{31} = \frac{\pi\lambda e^{-\lambda}}{2^6}\left[1 + 2\lambda + \frac{3}{4}\lambda^2 - 2\lambda^2 e^{-\lambda} E_{i1}(\lambda)\right];$$

$$m_{32} = -\frac{1}{2^8}\pi\lambda^3 e^{-\lambda};$$

$$g_1 = -\pi\lambda e^{-\lambda};$$

$$g_{21} = -\frac{1}{8}\left[1 + 2\lambda + \frac{\lambda^2}{8} - 2\lambda^2 e^{-\lambda} E_{i1}(\lambda) - 6\pi^2\lambda^2 e^{-2\lambda}\right];$$

$$g_{22} = \frac{2}{2^5}\left[1 + 2\lambda - \frac{\lambda^2}{4} - 2\lambda^2 e^{-\lambda} E_{i1}(\lambda)\right];$$

$$g_{23} = \frac{\lambda^2}{2^8};$$

$$g_{31} = \frac{\pi\lambda e^{-\lambda}}{2^4}\left[-8\pi^2\lambda^2 e^{-2\lambda} + 3 + 6\lambda + \lambda^2 - 6\lambda^2 e^{-\lambda} E_{i1}(\lambda)\right];$$

$$g_{32} = -\frac{\pi\lambda e^{-\lambda}}{2^5}\left[1 + 2\lambda + \frac{\lambda^2}{4} - 2\lambda^2 e^{-\lambda} E_{i1}(\lambda)\right];$$

$$\lambda = \frac{2gh}{v_0^2}; \qquad \alpha = \alpha_\kappa + \alpha_0;$$

where α_k - the edge angle of attack;

α_0 - the angle of zero lift.

The asymptotic equations for the hydrodynamic characteristics of the foil for large Froude numbers are obtained in the following form:

$$P = 2\pi\rho a v_0^2\alpha\left[1 - \frac{1}{16\bar{h}^2} - \frac{1}{2\bar{h}}\alpha\right]; \tag{I.41}$$

$$M_t = -\pi\rho a v_0^2\alpha\frac{1}{32\bar{h}^2}. \tag{I.42}$$

The problem of a moving, deeply submerged hydrofoil was analyzed by T. Nishiyama [210, 213]. In particular, for large Froude numbers accurate to the terms of the first order, the formula obtained by him agrees with that obtained by M. V. Keldysh and M. A. Lavrent'yev (I.41).

1.4. Hydrofoils Submerged to An Arbitrary Depth. Expansion in Terms of Parameter τ [26

If a certain parameter is introduced which satisfies the condition $\tau < 1$ throughout the entire lower side of the half-plane and which is related to the submergence of the hydrofoil, then one may attempt to seek a solution to equation (I.32) in the form of a series in terms of this parameter.

Let us analyze the function reflecting the shape of a cylinder of radius R onto the shape of a thin plate and establish the relationship among points located on the imaginary axis of the cylinder and plate planes. We have the following:

$$f(z) = z + \frac{R^2}{z}; \qquad z_0 = iH; \qquad ih = i\left(H - \frac{R^2}{H}\right).$$

Assuming

$$\bar{H} = \frac{H}{R}; \qquad \bar{h} = \frac{h}{4R},$$

then

$$4\bar{h} = \bar{H} - \frac{1}{\bar{H}}; \qquad \frac{1}{\bar{H}} = \sqrt{4\bar{h}^2 + 1} - 2\bar{h}. \tag{I.43}$$

It is clear that parameter $1/\bar{H}$ satisfies the prescribed conditions and, for large values of $\bar{h}$, is of the order of $\frac{1}{\bar{H}} \sim \frac{1}{4\bar{h}}$. Let us look for the solution of equation (I.32) in the form of a series using the even powers of the parameter $\tau = \sqrt{4\bar{h}^2 + 1} - 2\bar{h}$:

$$\gamma(s) = \bar{\gamma}_0(\bar{s}) + \tau^2 \bar{\gamma}_1(s) + \tau^4 \bar{\gamma}_2(s) + \ldots \tag{I.44}$$

The expansion of the regular part of the function nucleus will be written as follows:

$$k(x) = \sum_{n=1}^{\infty} k_n \tau^{2m}. \tag{I.45}$$

Then equation (I.32) breaks down into the following system of recurrence equations:

$$\int_{-1}^{+1} \frac{\bar{\gamma}_0(\bar{s})}{\bar{x} - \bar{s}} d\bar{s} = -2\pi \bar{f}'(x),$$

$$\int_{-1}^{+1} \frac{\bar{\gamma}_n(\bar{s})}{\bar{x} - \bar{s}} = -\int_{-1}^{+1} \sum_{m=1}^{n} k_m (\bar{x} - \bar{s})\, \bar{\gamma}_{n-1-m}(\bar{s})\, ds, \quad (m = 1, 2 \ldots), \tag{I.46}$$

the solution of which is also determined by formula (I.37).

The expansion of the regular nucleus k(x) in terms of powers of parameter τ is formed without any difficulties: [27

$$\bar{k}(\bar{x}) = \pi\omega e^{-2\bar{h}\omega}\cos\frac{\omega x}{2} + \bar{k}_1(\bar{x});$$

$$\bar{k}_1(\bar{x}) = \sum_{n=2,4}^{\infty}\tau^h \sum_{k=0}^{\frac{n}{2}-1}\frac{(n-1-k)\ldots(k+1)(-1)^{n/2-k+1}\bar{x}^{n-1-2k}}{(n-1-2k)!}\times$$
$$\times\left[1 + 2\,\mathrm{Re}\,F_{n-1-k}\left(\frac{\omega}{2\tau}\right)\right];$$
$$\omega = \frac{1}{\mathrm{Fr}_b^2};\quad \mathrm{Fr}_b = \frac{v_0}{\sqrt{2ga}}. \tag{I.47}$$

Here $F_n(\lambda)$ is a function which is determined by the following formula:

$$F_n(\lambda) = \lambda\left\{\frac{1}{n} + \frac{\lambda}{n(n-1)} + \frac{\lambda^2}{n(n-1)(n-2)} + \ldots\right.$$
$$\left.\ldots - \frac{\lambda e^{-\lambda}}{n!}[E_i(\lambda) - i\pi]\right\}. \tag{I.48}$$

The integral form of the function is as follows:

$$F_n(\lambda) = \lambda i^n \int_0^{-\infty}\frac{e^{-\lambda\alpha i}}{(\alpha + i)^{n+1}}\,d\alpha.$$

It is not difficult to establish the recurrence relationships for $\mathrm{Re}F_n(\lambda)$ and $\mathrm{Im}F_n(\lambda)$ as follows:

$$\mathrm{Re}\,F_n(\lambda) = \frac{\lambda}{\pi}[1 + \mathrm{Re}\,F_{n-1}(\lambda)];$$

$$\mathrm{Re}\,F_0 = -\lambda e^{-\lambda}E_{i1}(\lambda);$$

$$\mathrm{Im}\,F_n(\lambda) = \frac{\lambda}{n!}\mathrm{Im}\,F_{n-1}(\lambda);$$

$$\mathrm{Im}\,F_0(\lambda) = \lambda e^{-\lambda}\pi.$$

With $\lambda \to 0$, $F_n(\lambda) \to 0$, and with $\lambda \to \infty$, $F_n(\lambda) \to -1$. The values of the first six functions $F_n(\lambda)$ are listed in Table 1, while Figures 1 and 2 show curves plotted using these values.

Table 1 [28

λ	F_0		F_1		F_2		F_3
	Re	Im	Re	Im	Re	Im	Re
0,00	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
0,01	0,0399	0,0311	0,0104	0,0003	0,0051	0,0000	0,0034
0.02	0,0650	0,0616	0,0213	0,0012	0,0102	0,0000	0,0067
0,03	0,0844	0,0914	0,0325	0,0027	0,0155	0,0000	0,0102
0,04	0,1000	0,1207	0,0440	0,0048	0,0209	0,0001	0,0136
0,05	0,1126	0,1494	0,0556	0,0075	0,0264	0,0002	0,0171
0,06	0,1229	0,1775	0,0674	0,0107	0,0320	0,0003	0,0206
0,07	0,1312	0,2050	0,0792	0,0144	0,0378	0,0005	0,0242
0,08	0,1379	0,2320	0,0910	0,0186	0,0436	0,0007	0,0278
0,09	0,1430	0,2584	0,1029	0,0233	0,0496	0,0011	0,0314
0,1	0,1468	0,2843	0,1147	0,0284	0,0557	0,0014	0,0352
0,2	0,1346	0,5144	0,2691	0,1029	0,1269	0,0103	0,0748
0.3	0,0673	0,6982	0,3202	0,2095	0,1980	0,0314	0,1198
0,4	−0,0281	0,8425	0,3888	0,3369	0,2778	0,0674	0,1704
0,5	−0,1378	0,9527	0,4311	0,4764	0,3578	0,1191	0,2263
0,6	−0,2535	1,0345	0,4479	0,6207	0,4344	0,1862	0,2869
0,7	−0,3702	1,0921	0,4409	0,7644	0,5043	0,2676	0,3510
0,8	−0,4843	1,1293	0,4125	0,9034	0,5650	0,3614	0,4173
0,9	−0,5938	1,1496	0,3656	1,0346	0,6445	0,4656	0,4844
1,0	−0,6972	1,1557	0,3028	1,1557	0,6514	0,5779	0,5505
1,2	−0,8826	1,1355	0,1408	1,3625	0,6845	0,8175	0,6738
1,4	−1,0382	1,0846	−0,0535	1,5184	0,6626	1,0629	0,7759
1,6	−1,1647	1,0149	−0,2635	1,6238	0,5892	1,2990	0,8476
1,8	−1,2645	0,9347	−0,4761	1,6825	0,4715	1,5143	0,8829
2,0	−1,3410	0,8504	−0,6820	1,7007	0,3180	1,7007	0,8787
2,2	−1,3974	0,7658	−0,8742	1,6842	0,1384	1,8532	0,8348
2,4	−1,4371	0,6840	−1,0491	1,6416	−0,0589	1,9699	0,7529
3,6	−1,4632	0,6067	−1,2044	1,5774	−0,2657	2,0506	0,6364
2,8	−1,4778	0,5349	−1,3379	1,4977	−0,4730	2,0969	0,4919
3,0	−1,4837	0,4692	−1,4511	1,4077	−0,6767	2,1115	0,3233
3,2	−1,4828	0,4098	−1,5444	1,3113	−0,8710	2,0981	0,1376
3,4	−1,4765	0,3565	−1,6199	1,2120	−1,0539	2,0603	−0,0611
3,6	−1,4662	0,3090	−1,6785	1,1125	−1,2212	2,0025	−0,2655
3,8	−1,4532	0,2671	−1,7223	1,0148	−1,3723	1,9282	−0,4716
4.0	−1,4382	0,2302	−1,7526	0,9206	−1,5053	1,8413	−0,6737
4,2	−1,4220	0,1979	−1,7722	0,8310	−1,6216	1,7451	−0,8702
4,4	−1,4050	0,1697	−1,7818	0,7467	−1,7200	1,6428	−1,0560
4,6	−1,3878	0,1453	−1,7840	0,6682	−1,8031	1,5368	−1,2315
4,8	−1,3706	0,1241	−1,7791	0,5956	−1,8698	1,4294	−1,3916
5,0	−1,3538	0,1060	−1,7690	0,5300	−1,9225	1,3225	−1,5375
6	−1,2814	0,0467	−1,6885	0,2804	−2,0653	0,8410	−2,1307
7′	−1,2224	0,0201	−1,5569	0,1404	−1,9490	0,4913	−2,2144
8	−1,1818	0,0084	−1,4547	0,0675	−1,8189	0,3098	−2,1837
9	−1,1528	0,0035	−1,3753	0,0314	−1,6888	0,1413	−2,0665
10	−1,1315	0,0014	−1,3146	0,0143	−1,5729	0,0713	−1,9097
11	−1,1154	0,0006	−1,2698	0,0063	−1,4841	0,0349	−1,7752
12	−1,1030	0,0002	−1,2360	0,0028	−1,4160	0,0163	−1,6647
13	−1,0930	0,0001	−1,2092	0,0012	−1,3597	0,0078	−1,5586
14	−1,0849	0,0000	−1,1886	0,0005	−1,3199	0,0036	−1,4929
15	−1,0781	0,0000	−1,1717	0,0002	−1,2874	0,0017	−1,4360
∞	−1,0000	0,0000	−1,0000	0,0000	−1,0000	0,0000	−1,0000

Table 1 (cont.) [29

	F_4		F_5		F_6	
Im	Re	Im	Re	Im	Re	Im
0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
0,0000	0,0025	0,0000	0,0020	0,0000	0,0017	0,0000
0,0000	0,0050	0,0000	0,0040	0,0000	0,0033	0,0000
0,0000	0,0076	0,0000	0,0060	0,0000	0,0050	0,0000
0,0000	0,0101	0,0000	0,0081	0,0000	0,0067	0,0000
0,0000	0,0127	0,0000	0,0101	0,0000	0,0084	0,0000
0,0000	0,0153	0,0000	0,0122	0,0000	0,0102	0,0000
0,0000	0,0179	0,0000	0,0143	0,0000	0,0118	0,0000
0,0000	0,0206	0,0000	0,0163	0,0000	0,0136	0,0000
0,0000	0,0232	0,0000	0,0184	0,0000	0,0153	0,0000
0,0000	0,0259	0,0000	0,0205	0,0000	0,0170	0,0000
0,0007	0,0537	0,0000	0,0421	0,0000	0,0347	0,0000
0,0031	0,0840	0,0002	0,0650	0,0000	0,0533	0,0000
0,0090	0,1170	0,0009	0,0894	0,0001	0,0726	0,0000
0,0199	0,1533	0,0025	0,1133	0,0003	0,0929	0,0000
0,0372	0,1930	0,0056	0,1416	0,0007	0,1142	0,0000
0,0624	0,2364	0,0092	0,1731	0,0013	0,1369	0,0001
0,0964	0,2835	0,0193	0,2054	0,0031	0,1607	0,0004
0,1397	0,3340	0,0329	0,2401	0,0059	0,1802	0,0009
0,1926	0,3876	0,0482	0,2775	0,0096	0,2129	0,0016
0,3270	0,5021	0,0981	0,3605	0,0235	0,2721	0,0047
0,4960	0,6216	0,1736	0,4540	0,0486	0,3393	0,0113
0,6928	0,7390	0,2771	0,5565	0,0887	0,4151	0,0238
0,9086	0,8473	0,4089	0,6650	0,1472	0,4995	0,0442
1,1338	0,9393	0,5669	0,7757	0,2268	0,5919	0,0756
1,3591	0,9174	0,7475	0,8437	0,3289	0,6760	0,1206
1,5759	1,0517	0,9456	0,9848	0,4539	0,7939	0,1815
1,7772	1,0637	1,1552	1,0731	0,6007	0,8983	0,2603
1,9571	1,0485	1,3700	1,1472	0,7672	1,0020	0,3580
2,1115	0,9925	1,5836	1,1955	0,9502	1,0978	0,4751
2,2380	0,9101	1,7904	1,2275	1,1459	1,1853	0,6111
2,3351	0,7981	1,9848	1,2227	1,3497	1,2595	0,7648
2,4030	0,6611	2,1627	1,1960	1,5571	1,3176	0,9343
2,4424	0,5020	2,3202	1,1415	1,7634	1,3563	1,1168
2,4550	0,3263	2,4550	1,0610	1,9640	1,3740	1,3094
2,4432	0,1363	2,5653	0,9545	2,1549	1,3681	1,5084
2,4094	−0,0616	2,6503	0,8258	2,3323	1,3389	1,7104
2,3564	−0,2662	2,7099	0,6757	2,4931	1,2843	1,9114
2,2870	−0,4700	2,7444	0,5088	2,6346	1,2071	2,1077
2,2084	−0,6719	2,7605	0,3281	2,7605	1,1068	2,3004
1,6820	−1,6960	2,5231	−0,8292	3,0277	0,1708	3,0277
1,1465	−2,1253	2,0056	−2,5754	2,8088	−0,6712	3,2769
0,8252	−2,3674	1,6523	−2,1878	2,6437	−1,5837	3,5249
0,4243	−2,3995	0,9548	−2,5190	1,7184	−2,2786	2,5775
0,2377	−2,2742	0,5943	−2,5483	1,1885	−2,5806	1,9808
0,1280	−2,1317	0,3520	−2,4898	0,7744	−2,7312	1,4198
0,0653	−1,9940	0,1958	−2,3856	0,4700	[illegible]	0,9101
0,0338	−1,8154	0,1099	−2,1199	0,2859	[illegible]	0,6192
0,0169	−1,7242	0,0594	−2,0278	0,1662	[illegible]	[illegible]
0,0083	−1,6782	0,0309	−1,9176	0,0927	[illegible]	[illegible]
0,0000	−1,0000	0,0000	−1,0000	0,0000	−1,0000	[illegible]

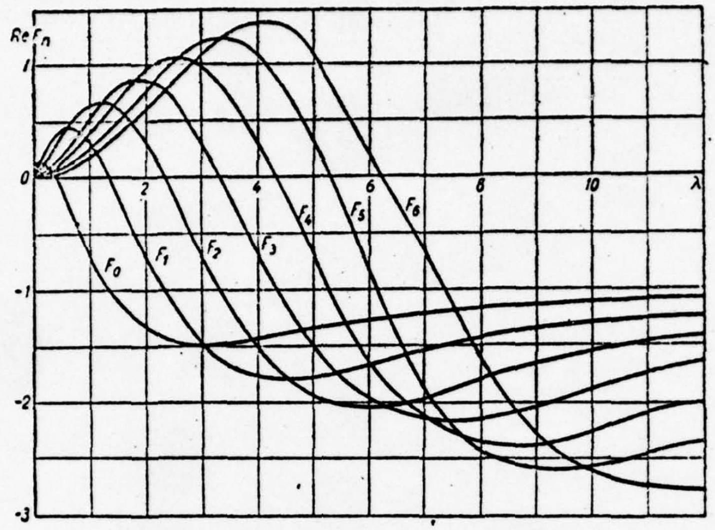

Fig. 1 [30

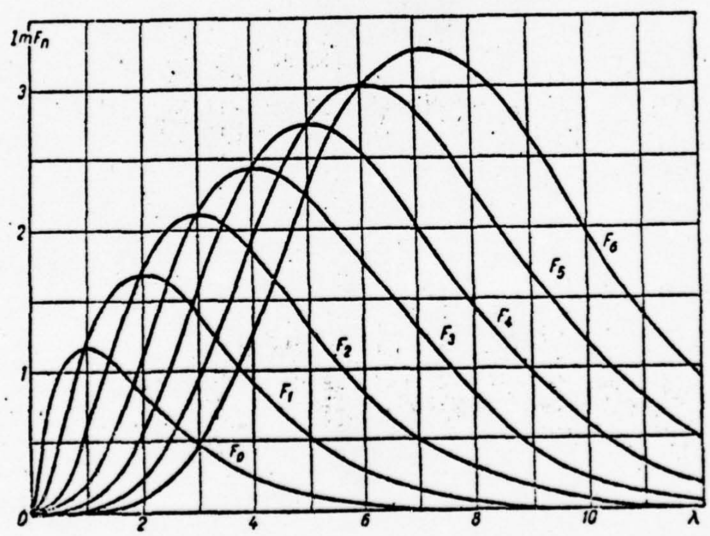

Fig. 2

If function $\bar{f}'(x)$ is given in the polynomial form, the computations become quite simple. The integrals, which are dealt with in computations, are reduced to three types, determining them in a closed form [147]: [31

$$T'_n = \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, x^n dx; \qquad T^{*}_n = \int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}}\, x^n dx;$$

$$T_n = \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, \frac{x^n}{x-x_0}\, dx; \qquad -1 < x_0 < 1;$$

$$T'_{2k} = -T'_{2k-1} = \frac{1\cdot 3\cdot 5\ldots(2k-1)}{2^k k!}\pi; \qquad T'_0 = \pi;$$

$$T^{*}_{2k} = T^{*}_{2k-1} = \frac{1\cdot 3\cdot 5\ldots(2k-1)}{2^k k!}\pi; \qquad T^{*}_0 = \pi;$$

$$T_n = x_0 T_{n-1} + T'_n; \quad T_0 = -\pi.$$

Let us examine the motion of a thin hydrofoil section with a central angle β. In this case $\bar{f}'(x) = \alpha - \bar{\beta}\bar{x}$.

After performing simple but cumbersome computations we obtain:

$$\bar{\gamma}_0(\bar{x}) = 2[(\alpha+\beta) - \bar{\beta}\bar{x}]\sqrt{\frac{1+x}{1-x}}$$

$$\gamma_1[\bar{x}] = -2\Bigg\{\Big[\Big(\alpha+\frac{1}{2}\beta\Big)\Big(\frac{3}{2}-\bar{x}\Big) - \frac{1}{4}\beta\Big]\Big[1 + 2\,\mathrm{Re}\,F_1\Big(\frac{\omega}{2\tau}\Big)\Big] +$$

$$+ \frac{\pi\omega e^{-2h\omega}}{\tau^2}\Big\{\Big(\alpha+\frac{1}{2}\beta\Big) - \frac{1}{2}\omega^2\Big[(\bar{x}^2 - 2\bar{x} + 2)\Big(\alpha + \frac{1}{2}\beta\Big) -$$

$$-\Big(\frac{5}{8} - \frac{1}{2}\bar{x}\Big)\beta\Big] + \frac{1}{384}\omega^4\Big[(\bar{x}^4 - 3\bar{x}^3 + \frac{11}{2}\bar{x}^2 - 6\bar{x} +$$

$$+\frac{19}{4}\Big)\Big(\alpha + \frac{1}{2}\beta\Big)\Big] - \frac{1}{64720}\omega^6\Big[\Big(\alpha+\frac{1}{2}\beta\Big)\Big(\bar{x}^6 + 4\bar{x}^5 +$$

$$\text{[illegible]}\,\bar{x}^4 - 17\bar{x}^3 + \frac{75}{4}\bar{x}^2 - \frac{66}{4}\bar{x} + \frac{52}{4}\Big) - \Big(-\frac{9}{2}\bar{x}^5 + \frac{27}{7}\bar{x}^4 -$$

$$-\frac{51}{8}\bar{x}^3 + \frac{117}{16}\bar{x}^2 - \frac{99}{16}\bar{x} + \frac{325}{64}\Big)\beta\Big]\Big\}\sqrt{\frac{1+x}{1-x}}$$

$$\gamma_2[\bar{x}] = 2\Bigg\{\Big[\Big(\alpha+\frac{1}{2}\beta\Big)\Big(\frac{5}{4}-\bar{x}\Big) - \frac{1}{4}\beta\Big(\frac{3}{2}-\bar{x}\Big)\Big]\Big[1 +$$ [32

$$+ 2\,\mathrm{Re}\,F_1\Big(\frac{\omega}{2\tau}\Big)\Big]^2 - 2\Big[\Big(\alpha + \frac{1}{2}\beta\Big)\Big(\frac{3}{2} - \bar{x}\Big) -$$

$$
\begin{aligned}
&-\frac{1}{4}\beta\Big]\Big[1+2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\Big]+\Big[\left(\alpha+\frac{1}{2}\beta\right)\times \\
&\times\left(-\bar{x}^3+\frac{5}{2}\bar{x}^2-\frac{7}{2}\bar{x}+\frac{25}{8}\right)-\beta\left(\frac{3}{4}\bar{x}^2-\frac{9}{8}\bar{x}-\frac{15}{16}\right)\Big]\times \\
&\times\Big[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\pi}\right)\Big]\Big\}\sqrt{\frac{1+\bar{x}}{1-\bar{x}}}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_3(\bar{x}) = &-2\Big\{\Big[\left(\alpha+\frac{1}{2}\beta\right)\left(\frac{7}{8}-\frac{3}{4}x\right)-\frac{1}{4}\beta\left(\frac{5}{4}-\bar{x}\right)\Big]\times \\
&\times\Big[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\Big]^3-4\Big[\left(\alpha-\frac{1}{2}\beta\right)\left(\frac{5}{4}-\bar{x}\right)- \\
&-\frac{1}{2}\beta\left(\frac{3}{2}-\bar{x}\right)\Big]\times\Big[1+2\operatorname{Re}F_2\left(\frac{\omega}{2\tau}\right)\Big]\Big[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\Big]+ \\
&+\Big[\left(\alpha+\frac{1}{2}\beta\right)\left(-\bar{x}^3+\frac{7}{4}\bar{x}^2-\frac{37}{8}\bar{x}+\frac{77}{16}\right)-\frac{\beta}{4}\Big(-\bar{x}^3+ \\
&+\frac{5}{2}\bar{x}^2-\frac{13}{2}\bar{x}+\frac{55}{8}\Big)\Big]\times\Big[1+2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\Big]\Big[1+ \\
&+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\Big]+3\Big[\left(\alpha+\frac{1}{2}\beta\right)\times\left(\frac{3}{2}-\bar{x}\right)- \\
&-\frac{1}{4}\beta\Big]\Big[1+2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\Big]-4\Big[\left(\alpha+\frac{1}{2}\beta\right)\times \\
&\times\left(-\bar{x}^3+\frac{5}{2}\bar{x}^2-\frac{7}{2}\bar{x}+\frac{25}{8}\right)-\beta\left(\frac{3}{4}\bar{x}^2-\frac{9}{8}\bar{x}+\frac{15}{16}\right)\Big]\times \\
&\times\Big[1+2\operatorname{Re}F_4\left(\frac{\omega}{2\tau}\right)\Big]+\Big[\left(\alpha+\frac{1}{2}\beta\right)\Big(-\bar{x}^5+\frac{7}{2}\bar{x}^4-8\bar{x}^3+ \\
&+\frac{21}{2}\bar{x}^2-\frac{39}{4}\bar{x}+\frac{63}{8}\Big)-\beta\Big(\frac{5}{4}\bar{x}^4-\frac{5}{2}\bar{x}^3+\frac{15}{4}\bar{x}^2- \\
&-\frac{15}{4}\bar{x}+\frac{45}{16}\Big)\Big]\Big[1+2\operatorname{Re}F_5\left(\frac{\omega}{2\tau}\right)\Big]\Big\}\sqrt{\frac{1+\bar{x}}{1-\bar{x}}}
\end{aligned}
\tag{I.49}
$$

The hydromechanical characteristics of the submerged hydrofoil are determined by the strength of the vortex layer $\gamma(s)$ from formulas (I.29), (I.30) and (I.31). Specifically, the following expression was obtained for the $\gamma = \frac{P_n}{P_\infty}$ function: [33

$$
\begin{aligned}
\gamma = 1 &- \left(1 - \frac{1}{2}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\left[1 + 2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]\tau_1^2 + \left\{\left(\frac{3}{4} - \right.\right. \\
&- \left.\frac{1}{2}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\left[1 + 2\operatorname{Re}F_1\left(\frac{\omega}{2\tau_1}\right)\right]^2 - \left(2 - \frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\left[1 + \right. \\
&+ \left.\operatorname{Re}F_2\left(\frac{\omega}{2\tau_1}\right)\right] + \left(\frac{9}{4} - \frac{3}{2}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\left.\left[1 + 2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\right]\right\}\tau_1^4 - \\
&- \left\{\left(\frac{1}{2} - \frac{3}{8}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\times\left[1 + 2\operatorname{Re}F_1\left(\frac{\omega}{2\tau_1}\right)\right]^3 - \right. \\
&- 2\left(\frac{3}{2} - \frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\left[1 + 2\operatorname{Re}F_2\left(\frac{\omega}{2\tau_1}\right)\right]\times\left[1 + 2\operatorname{Re}F_3\left(\frac{\omega}{2\tau_1}\right)\right] + \\
&+ \left(3 - \frac{3}{2}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\left[1 + 2\operatorname{Re}F_3\left(\frac{\omega}{2\tau_1}\right)\right] - \\
&- \left(9 - 6\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\left[1 + 2\operatorname{Re}F_4\left(\frac{\omega}{2\tau_1}\right)\right] + \left(\frac{25}{4} - \frac{76}{16}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\times \\
&\times \left.\left[1 + 2\operatorname{Re}F_5\left(\frac{\omega}{2\tau_1}\right)\right]\right\}\tau_1^6 - \pi\omega e^{-2h\omega}\left[1 - \frac{1}{8}\,\omega^2\left(\frac{3}{2} - \right.\right. \\
&- \left.\frac{3}{4}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right) + \frac{1}{384}\,\omega^4\left(\frac{15}{14} - \frac{7}{2}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right) - \\
&- \left.\frac{1}{64\cdot 720}\,\omega^6\left(\frac{171}{16} - \frac{259}{32}\,\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right) + \ldots\right],
\end{aligned}
\tag{I.50}
$$

where $\alpha_0 = \frac{1}{2}\beta$ is the angle of zero lift.

For $a_k = 0$, the formula will have the following form:

$$
\begin{aligned}
\gamma = 1 &- \frac{1}{2}\left[1 + 2\operatorname{Re}F_1\left(\frac{\omega}{2\tau_1}\right)\right]\tau_1^2 + \left\{\frac{1}{4}\left[1 + 2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]^2 - \right. \\
&- \left[1 + 2\operatorname{Re}F_2\left(\frac{\omega}{2\tau_1}\right)\right] + \frac{3}{4}\left.\left[1 + 2\operatorname{Re}F_3\left(\frac{\omega}{2\tau_1}\right)\right]\right\}\tau_1^4 - \\
&- \left[\frac{1}{8}\left[1 + 2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]^3 - \left[1 + 2\operatorname{Re}F_1\left(\frac{\omega}{2\tau_1}\right)\right]\left[1 + \right.\right. \\
&+ \left.2\operatorname{Re}F_2\left(\frac{\omega}{2\tau_1}\right)\right] + \frac{13}{16}\left[1 + 2\operatorname{Re}F_4\left(\frac{\omega}{2\tau}\right)\right]\times\left[1 + 2\operatorname{Re}F_3\left(\frac{\omega}{2\tau_1}\right)\right] + \\
&+ \frac{3}{2}\left[1 + 2\operatorname{Re}F_3\left(\frac{\omega}{2\tau_1}\right)\right] - 3\left[1 + 2\operatorname{Re}F_4\left(\frac{\omega}{2\tau_1}\right)\right] +
\end{aligned}
$$

[34

$$+\frac{25}{16}\left[1+2\operatorname{Re}F_5\left(\frac{\omega}{2\tau}\right)\right]\Big\}\tau_1^6-\pi\omega^{-2h\omega}\left[1-\frac{3}{32}\omega^2-\frac{1}{4}\times\right.$$

$$\left.\times\frac{1}{384}\omega^4-\frac{1}{64}\cdot\frac{1}{720}\cdot\frac{83}{32}\omega^6+\ldots\right]. \qquad \text{(I.51)}$$

Of interest is the submerged hydrofoil motion at high speeds ($\omega \to 0$). In this case formulas for γ_0, γ_1 and γ_2 will be in the form

$$\left.\begin{aligned}
\gamma_0(x) &= 2[(\alpha+\beta)-\beta\bar{x}]\sqrt{\frac{1+\bar{x}}{1-x}}\\
\gamma_1(\bar{x}) &= -2\left[\left(\alpha+\frac{1}{2}\beta\right)\left(\frac{3}{2}-\bar{x}\right)-\frac{1}{4}\beta\right]\sqrt{\frac{1+\bar{x}}{1-\bar{x}}}\\
\gamma_2(\bar{x}) &= 2\left[\left(\alpha+\frac{1}{2}\beta\right)\left(\frac{11}{8}-\frac{5}{2}\bar{x}+\frac{5}{2}\bar{x}^2-\bar{x}^3\right)-\right.\\
&\quad\left.-\frac{1}{4}\beta\left(\frac{13}{4}-\frac{11}{2}\bar{x}-3\bar{x}^2\right)\right]\sqrt{\frac{1+\bar{x}}{1-\bar{x}}}\\
\gamma_3(\bar{x}) &= -2\left[\left(\alpha+\frac{1}{2}\beta\right)\left(\frac{9}{16}-\frac{1}{8}\bar{x}+\frac{9}{4}\bar{x}^2-5\bar{x}^3+\frac{7}{2}\bar{x}^4-\bar{x}^5\right)-\right.\\
&\quad\left.-\frac{1}{4}\beta\left(\frac{11}{8}-\frac{1}{2}\bar{x}+\frac{11}{2}\bar{x}^2-11\bar{x}^3+5\bar{x}^4\right)\right]\sqrt{\frac{1+\bar{x}}{1-\bar{x}}}
\end{aligned}\right|. \qquad \text{(I.52)}$$

Accordingly, formula (I.50) will change into the following formula

$$\bar{\gamma}=1-\left(1-\frac{1}{2}\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\tau_1^2+\left(1-\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\tau_1^4-$$

$$-\frac{3}{4}\left(1-\frac{13\alpha_0}{12\alpha_0+\alpha_\kappa}\right)\tau^6+\ldots. \qquad \text{(I.53)}$$

which, for $\alpha_k = 0$, may be written as

$$\bar{\gamma}=1-\frac{1}{2}\tau_1^2+Q(\tau_1^3). \qquad \text{(I.54)}$$

Formula (I.54) produces results which agree closely with those obtained experimentally. By way of example, Figure 3 shows three curves: 1 - obtained from formula (I.54); 2 - the experimental curve obtained by S. D. Chudinov [167]; 3 - curve plotted from the results derived by M. V. Keldysh and M. A. Lavrent'yev. [35

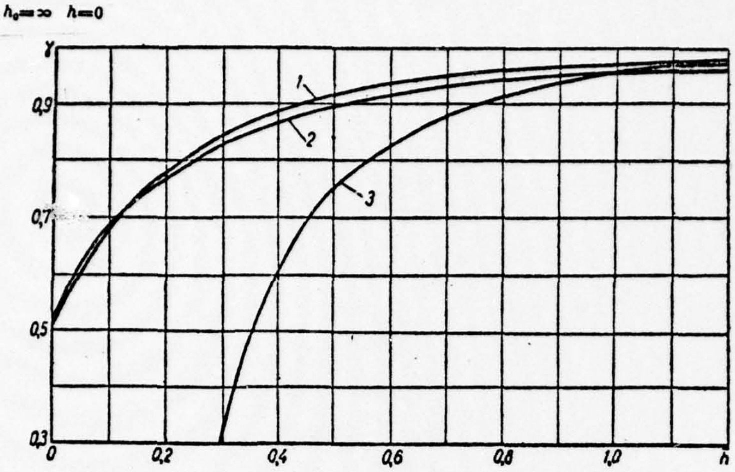

Fig. 3

It should be noted that the angle of the zero lifting force for the hydrofoil section depends on the relative submergence of the hydrofoil, with $\alpha_0 = \alpha_0$. This fact was originally noted by A. B. Lukashevich. The experiments carried out by him comfirmed this dependence. The experimental methods offered by V. T. Sokolov and S. D. Chudinov [167], and M. T. Kulayev [10], missed this point and, strictly speaking, these methods are wrong. However, considering the motion of actual hydrofoils with small edge angles of attack (when α_k is exactly zero), we obtain formula (I.54), which, in respect to the assumptions made, is similar to the experimental formulas obtained by the above authors. It is not difficult to demonstrate that the difference between the results obtained by these methods and formulas and those obtained by the methods based on the consideration of the above factor will be negligible for small edge angles of attack, while the amount of effort required for computations is lower when using the former approach. Based on these considerations a theoretical method of analyzing hydrofoils is given in [101, 102, 107 and 109]. It is easy to isolate terms in formulas (I.50) and (I.53) which determine the effect of a free surface on the zero lift angle of the hydrofoil section.

By writing function $\bar{\gamma}$ in the form of

$$\bar{\gamma} = \psi + \frac{\varkappa \alpha_0}{\alpha_0 + \alpha_\kappa}, \qquad \text{(I.55)}$$

we obtain [36

$$\psi = 1 - \left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]\tau_1^2 + \left\{\frac{3}{4}\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]^2 - \right.$$

$$- 2\left[1 + 2\,\mathrm{Re}\,F_2\left(\frac{\omega}{2\tau}\right)\right] + \frac{9}{4}\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right]\Bigg\}\tau_1^4 -$$

$$-\left\{\frac{1}{2}\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]^3 - 3\left[1 + 2\,\mathrm{Re}\,F_2\left(\frac{\omega}{2\tau}\right)\right]\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right] + \right.$$

$$+ 3\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right]\times\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right] +$$

$$+ 3\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right] - 9\left[1 + 2\,\mathrm{Re}\,F_4\left(\frac{\omega}{2\tau}\right)\right] +$$

$$+ \frac{25}{4}\left[1 + 2\,\mathrm{Re}\,F_5\left(\frac{\omega}{2\tau}\right)\right]\Bigg\}\tau_1^6 - \pi\omega e^{-2\bar{h}\omega}\left[1 - \frac{3}{16}\omega^2 + \right.$$

$$\left. + \frac{15}{5376}\omega^4 - \frac{171}{737280}\omega^6 + \ldots\right]; \qquad (\mathrm{I}.56)$$

$$\varkappa = \frac{1}{2}\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]\tau_1^2 + \left\{-\frac{1}{2}\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]^2 + \right.$$

$$+ \left[1 + 2\,\mathrm{Re}\,F_2\left(\frac{\omega}{2\tau}\right)\right] - \frac{3}{2}\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right]\Bigg\}\tau_1^4 -$$

$$-\left\{-\frac{3}{8}\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]^3 + 2\left[1 + 2\,\mathrm{Re}\,F_2\left(\frac{\omega}{2\tau}\right)\right]\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right] - \right.$$

$$- \frac{9}{4}\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right]\times\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right] -$$

$$- \frac{3}{2}\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right] + 6\left[1 + 2\,\mathrm{Re}\,F_4\left(\frac{\omega}{2\tau}\right)\right] -$$

$$- \frac{75}{16}\left[1 + 2\,\mathrm{Re}\,F_5\left(\frac{\omega}{2\tau}\right)\right]\Bigg\}\tau^6 - \pi\omega e^{-2\bar{h}\omega}\left[\frac{3}{32}\omega^2 - \right.$$

$$\left. - \frac{7}{768}\omega^4 + \frac{259}{1474360}\omega^6 + \ldots\right]. \qquad (\mathrm{I}.57)$$

Function $\varkappa$ determines the effect of the free surface on α_0. For a thin plate $\bar{\gamma} = \psi$. With $\omega \to 0$ functions ψ and $\varkappa$ are determined by the following formulas: [37

$$\psi = 1 - \tau^2 + \tau^4 - \frac{3}{4}\tau^6 + \frac{5}{8}\tau^8; \qquad (\mathrm{I}.58)$$

$$\varkappa = \frac{1}{2}\tau^2 - \tau^4 + \frac{13}{16}\tau^6 - \frac{5}{8}\tau^8. \qquad (\mathrm{I}.59)$$

Graphs of ψ and $\varkappa$ plotted from formulas (I.58) and (I.59) are shown in Fig. 4.

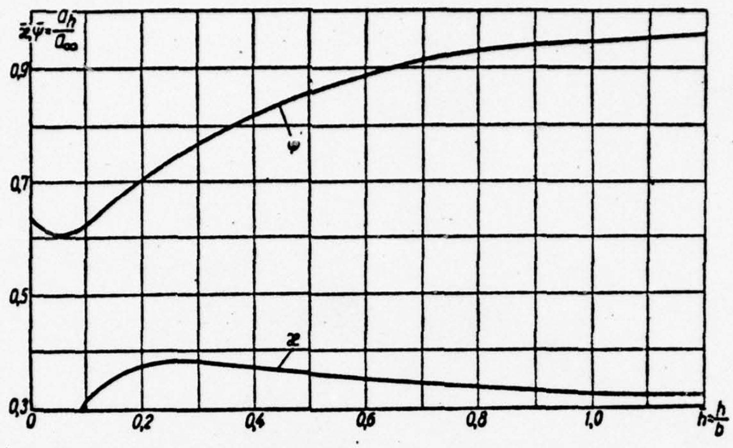

Fig. 4

The solution of this problem is given in [113] in the form of a power series of parameter $\tau_1 = \sqrt{2\bar{h}^2+1} - \sqrt{2}\,\bar{h}$. With $\bar{\omega} \to 0$ the formula for γ of a thin hydrofoil section is obtained in [113] in the form of

$$\gamma = 1 - \frac{1}{2}\left(1 - \frac{1}{2}\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\tau_1^2 - \frac{1}{4}\tau_1^4 + \left(\frac{21}{64} - \frac{49}{128}\frac{\alpha_0}{\alpha_0+\alpha_\kappa}\right)\tau^6. \quad (I.61)$$

With the submersion to a considerable depth the τ and τ_1 parameters will be determined by the asymptotic formulas

$$\tau = \frac{1}{4\bar{h}}, \quad \tau_1 = \frac{1}{2\sqrt{2}\,\bar{h}}.$$

By retaining in formulas (I.58) and (I.60) the first terms only, for a deep submersion we obtain the formula $\psi = 1 - \frac{1}{16\bar{h}^2}$, which agrees with an accuracy to the first order terms with the asymptotic formula (I.41) obtained by M. V. Keldysh and M. A. Lavrent'yev.

1.5. Regularization of the Singular Integral Equation. Solution of the Regular Integral Equation [38

By regularizing the singular integral equation (I.32)

it becomes possible to analyze and solve the new regular equation by means of the Fredholm equation methods. Equation (I.32) can be reduced to a regular form most easily by using I. N. Vekua's method [91].

Let us examine the equation

$$\frac{1}{2\pi}\int_{-1}^{+1}\gamma(s)\left[\frac{1}{x-s}+G(x-s)\right]ds=-\bar{f}'(s),$$

where G(x - s) is the regular function along the line of intersection. Let us write the equation in the form of

$$\int_{-1}^{+1}\frac{\gamma(s)}{x-s}d\bar{s}=2\pi\left[-\bar{f}'(s)-\frac{1}{2\pi}\int_{-1}^{+1}\gamma(s)G(\bar{x}-\bar{s})\,d\bar{s}\right].$$

The solution of this equation, limited at a point, is given by formula (I.37):

$$\gamma(x)=-\frac{2}{\pi}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{+1}\sqrt{\frac{1-\bar{s}}{1+\bar{s}}}\cdot\frac{\bar{f}'(s)}{\bar{s}-\bar{x}}ds-$$

$$-\frac{1}{\pi^2}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{+1}\sqrt{\frac{1-\bar{s}}{1+\bar{s}}}\int_{-1}^{+1}\frac{\gamma(p)G(\bar{s}-\bar{p})}{\bar{s}-\bar{x}}dsdp.$$

Since $-\frac{2}{\pi}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{+1}\sqrt{\frac{1-\bar{s}}{1+\bar{s}}}\cdot\frac{\bar{f}'(s)}{\bar{s}-x}ds=\gamma_0(x)$ is the solution of the equation for a thin hydrofoil in an infinite flow, the regular equation may be written as

$$\gamma(x)=\gamma_0(x)-\frac{1}{\pi^2}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{+1}\gamma(s)K(\bar{x},\bar{s})\,ds,$$

$$K(\bar{x},\bar{s})=\int_{-1}^{+1}\sqrt{\frac{1-s}{1+s}}\frac{G(\bar{s}-\bar{p})}{\bar{s}-x}ds \qquad \text{(I.62)}$$

or, by changing to function $\varphi(x)=\gamma(x)\sqrt{\frac{1-x}{1+x}}$: [39

$$\varphi(x) = \varphi_0(x) - \frac{1}{\pi^2}\int_{-1}^{+1} \varphi(p)\, k(\bar{p}, \bar{x})\, dp \tag{I.63}$$

$$k(p, x) = \sqrt{\frac{1+\bar{p}}{1-\bar{p}}} \int_{-1}^{+1} \sqrt{\frac{1-\bar{s}}{1+\bar{s}}} \frac{G(s-p)}{\bar{s}-\bar{x}}\, ds.$$

Evidently, if G(s - p) is assigned by the expansion (I.47) and a finite number of terms is retained, then the nucleus $k(\overline{x}, \overline{p})$ will be singular and, in solving equation (I.63), we may treat it as an equation with a singular nucleus.

Let us write the G(s - p) function in the form of

$$G(\bar{s} - \bar{p}) = \sum_{n=0}^{k} G_n(p)\, \psi_n(s).$$

Then

$$k(p, x) = \sqrt{\frac{1+p}{1-p}} \sum_{n=0}^{k} G_n(\varrho)\, \psi_n'(x),$$

$$\psi_n'(x) = \int_{-1}^{+1} \sqrt{\frac{1-\bar{s}}{1+s}} \frac{\psi_n(s)}{\bar{s}-\bar{x}}\, ds, \tag{I.64}$$

and equation (I.63) will be equivalent to a system of algebraic equations

$$\varphi_j\left(1 + \frac{1}{\pi^2} A_{jj}\right) + \sum_{n=0}^{k}{}' \varphi_n A_{jn} = B_j \quad j = 0, 1 \ldots k; \tag{I.65}$$

$$Ajn = \frac{1}{\pi^2}\int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, G_j(x)\, \psi_n'(x)\, dx,$$

$$B_j = \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, \varphi_0(x)\, G_j(x)\, dx,$$

while the function will be determined by the formula

$$\varphi(x) = \varphi_0(x) - \frac{1}{\pi^2} \sum_{n=0}^{k} \psi_n'(x)\, \varphi_n. \tag{I.66}$$

Equation (I.63) may also be solved by the iteration method. [40

Let us examine equation

$$\varphi(x) = \varphi_0(x) + \lambda k\varphi, \tag{I.67}$$

where $k\varphi = \int\limits_{-1}^{+1} \varphi(s)\, k(\bar{x}, \bar{s})\, ds$ is the Fredholm's operator.

By introducing a resolvent, let us write the solution of the equation in the form

$$\varphi(x) = \varphi_0(x) + \lambda \int\limits_{-1}^{+1} \Gamma(x, s, \lambda)\, \varphi_0(s)\, ds,$$

$$\Gamma(x, s, \lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} k_m(\bar{x}, \bar{s}),$$

where $k_m(\bar{x}, \bar{s})$ is the iterated nucleus [82].

Let us determine $\gamma(x)$ in the formulas for forces (I.29) and (I.30) from equation (I.61). After transformations the formulas for P and M will be written as

$$P = \varrho a v_0^2 \int\limits_{-1}^{+1} [\gamma_0(x) - \gamma(x) N_0(x)]\, dx, \tag{I.69}$$

$$M = -\varrho a^2 v_0^2 \int\limits_{-1}^{+1} [x\gamma_0(x) - \gamma(x) M_0(x)]\, dx, \tag{I.70}$$

where

$$N_0(x) = -\frac{1}{\pi} \int\limits_{-1}^{+1} \sqrt{\frac{1-\sigma}{1+\sigma}}\, G(\sigma - \bar{x})\, d\sigma;$$

$$M_0(x) = +\frac{1}{\pi} \int\limits_{-1}^{+1} \sqrt{1-\sigma^2}\, G(\sigma - \bar{x})\, d\sigma.$$

Formulas (I.69) and (I.70) have an advantage over those given in (I.29) and (I.30) because in the approximate determination of $\gamma(s)$ they make the values of P and M more accurate by one additional approximation step.

However, if the value of $\gamma(x)$ is found in terms of the resolvent, then for determining forces one may obtain the following expressions:

$$P = \varrho a v_0^2 \int_{-1}^{+1} \gamma_0(s) \left[1 - \sum_{n=0}^{\infty} N_n(s)(-1)^n \right] dx, \quad (I.71)$$

$$M = -\varrho a^2 v_0^2 \int_{-1}^{+1} \gamma_0(s) \left[s - \sum_{n=0}^{\infty} M_n(s)(-1)^n \right] dx, \quad (I.72)$$

where functions $N_n(s)$ and $M_n(s)$ are determined by the re- [41
currence formulas

$$N_n(s) = \frac{1}{\pi^2} \int_{-1}^{+1} \sqrt{\frac{1-\sigma}{1+\sigma}} G(\sigma - s) \int_{-1}^{+1} \sqrt{\frac{1+t}{1-t}} \frac{N_{n-1}(t)}{\sigma - t} dt; \quad (I.73)$$

$$M_n(x) = \frac{1}{\pi^2} \int_{-1}^{+1} \sqrt{\frac{1-\sigma}{1+\sigma}} G(\sigma - s) \int_{-1}^{+1} \sqrt{\frac{1+t}{1-t}} \frac{M_{n-1}(t)}{\sigma - t} dt. \quad (I.74)$$

Thus, if there is a resolvent of the equation (I.67), then the quantities in the problem which present the greatest interest will be determined from formulas (I.69)-(I.73).

Let us write $G(\sigma - s)$ in the form of a series

$$G(\sigma - s) = \xi\varphi_1(\sigma, -s) + \xi^2\varphi_2(\sigma, -s) + \xi^3\varphi_3(\sigma, -s) + \ldots, \quad (I.75)$$

where ξ is a certain small parameter.

In representing $G(\sigma - s)$ in the form of a series of functions with small parameters, $G(\sigma - s)$ may also be presented in the form of (I.75)

$$G(\sigma - s) = \xi_1\varphi_1(\sigma - s) + \xi_2\varphi_2'(\sigma; s) + \xi_3\varphi_3'(\sigma; s) + \ldots.$$

With $\xi_1 \sim \xi,\ \xi_2 \sim \xi^2,\ \xi_n \sim \xi^n$

$$\varphi_n(\sigma - s) = k_n\varphi_n'(\sigma; s),\ k_n = \frac{\xi_n}{\xi^n},\ k_1 = 1.$$

As a result, the function is presented in the form of series (I.75).

Let us find the solution with an accuracy of up to ξ^n. Then

$$N_0(s) = \xi\psi_{01}(s) + \xi^2\psi_{02}(s) + \ldots + \xi^n\psi_{0n}(s) + \ldots$$

$$\psi_{0n}(s) = -\frac{1}{\pi} \int_{-1}^{+1} \sqrt{\frac{1-\sigma}{1+\sigma}} \varphi_n(\sigma; s)\, d\sigma$$

$$N_1(s) = \xi^2\psi_{12}(s) + \xi^3\psi_{13}(s) + \ldots + \xi^4\psi_{14}(s)$$

$$\psi_{1s}(s) = \frac{1}{\pi^2}\int_{-1}^{+1}\sqrt{\frac{1-\sigma}{1+\sigma}}[\varphi_1(\sigma;\ s)\psi_{0,s-1}(\sigma) + \varphi_2(\sigma,\ s)\bar{\psi}_{0,s-2}(\sigma) + \ldots + \varphi_{s-1}(\sigma,\ s)\bar{\psi}_{0,1}(\sigma)]\,d\sigma$$

$$\psi_{0,n}(\sigma) = \int_{-1}^{+1}\sqrt{\frac{1+t}{1-t}}\,\frac{\psi_{0;n}(t)}{\sigma - t}\,dt$$

$$N_k(s) = \xi^{k+1}\psi_{k,k+1}(\sigma) + \xi^{k+2}_{k,k+2}\psi_{k,k+2}(\sigma) + \ldots + \xi^n\psi_{k,n}(\sigma)$$ [42

$$\psi_{k,s}(s) = \frac{1}{\pi^2}\int_{-1}^{+1}\sqrt{\frac{1-\sigma}{1+\sigma}}[\varphi_1(\sigma;\ s)\psi_{k-1,s-1}(\sigma) + \varphi_2(\sigma;\ s)\psi_{k-1,s-1}(\sigma) + \ldots + \varphi_{s-k}(\sigma;\ s)\psi_{k-1,k}(\sigma)]\,d\sigma$$

$$\bar{\psi}_{n,p}(\sigma) = \int_{-1}^{+1}\sqrt{\frac{1+t}{1-t}}\,\frac{\psi_{n,p}(t)}{\sigma - t}\,dt$$

$$M_k(\sigma) = \xi^{k+1}f_{k,k+1}(\sigma) + \xi^{k+2}f_{k,k+2}(\sigma) + \ldots + \xi^n f_{k,n}(\sigma),$$

$$f_{k,s}(s) = \frac{1}{\pi^2}\int_{-1}^{+1}\sqrt{\frac{1-\sigma}{1+\sigma}}[\varphi_1(\sigma;\ s)\bar{f}_{k-1,s-1}(\sigma) + \varphi_2(\sigma;\ s)\bar{f}_{k-1,s-2}(\sigma) + \ldots + \varphi_{s-k}(\sigma,\ s)\bar{f}_{k-1,s}(\sigma)]\,d\sigma$$

$$\bar{f}_{n,p}(\sigma) = \int_{-1}^{+1}\sqrt{\frac{1+t}{1-t}}\,\frac{f_{n,p}(t)}{p - t}\,dt$$ (I.76)

(I.77)

$$p = \varrho a v_0 \int_{-1}^{+1}\gamma_0(s)\left[1 - \sum_{l=0}^{n-1}N_l(s)(-1)^l\right]ds, \tag{I.78}$$

$$M = -\varrho a^2 v_0^2 \int_{-1}^{+1}\gamma_0(s)\left[s - \sum_{l=0}^{n-1}M_l(s)(-1)^l\right]ds. \tag{I.79}$$

It is seen that the given ordered iteration for the function in the form of (I.75) shortened computations considerably as compared with the direct computation of the resolvent with the aid of the iterated nuclei. It may be shown that the method discussed is, to some extent, equivalent to the small parameter method (p. 11).

Let us examine a case in which function $\varphi_i(\sigma,\ s)$ is given in the form of a polynomial in s:

$$\left.\begin{aligned}
&\varphi_1(\sigma, s) = B_{010}(\sigma)\\
&\varphi_2(\sigma, s) = B_{020}(\sigma) + B_{021}(\sigma)s\\
&\varphi_3(\sigma, s) = B_{030}(\sigma) + B_{031}(\sigma)s + B_{032}(\sigma)s^2\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\varphi_n(\sigma, s) = B_{0n0}(\sigma) + B_{0n1}(\sigma)s + B_{0n2}(\sigma)s^2 + B_{0n,n-1}(\sigma)s^{n-1}
\end{aligned}\right\}. \tag{I.80}$$

Then [43

$$\left.\begin{aligned}
&\psi_{01} = C_{010}\\
&\psi_{02} = C_{020} + sC_{021}\\
&\psi_{03} = C_{030} + sC_{031} + s^2C_{032}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\psi_{0n} = C_{0n0} + sC_{0n1} + s^2C_{0n2} + \dots + s^{n-1}s_{0n,n-1}\\
&C_{0nm} = -\frac{1}{\pi}\int_{-1}^{+1} B_{0nm}(\sigma)\sqrt{\frac{1-\sigma}{1+\sigma}}\,d\sigma
\end{aligned}\right\}, \tag{I.81}$$

$$\left.\begin{aligned}
&\psi_{p,k} = C_{p,k,0}\\
&\psi_{p,k+1} = C_{p,k+1,0} + sC_{p,k+1}\\
&\psi_{p,k+2} = C_{p,k+2,0} + sC_{p,k+2,1} + s^2C_{p,k+2,2}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\psi_{p,k+n} = C_{p,k+n,0} + sC_{p,k+n,1} + s^2C_{p,k+n,2} + \dots + s^{n-1}C_{p,k+n,n-1}
\end{aligned}\right\}, \tag{I.82}$$

$$\left.\begin{aligned}
&C_{p,k,m} = \frac{1}{\pi}\int_{-1}^{+1}\sqrt{\frac{1-\sigma}{1+\sigma}}\,B_{p,k,m}(\sigma)\,d\sigma\\
&B_{p,k,m}(\sigma) = B_{0,m+1,m}d_{p-1,k-n-1} + B_{0,m+2,m}d_{p-1,k-m-2} + \dots +\\
&\qquad\qquad + B_{0,k-p,m}d_{p-1,p}\\
&d_{p,k} = \frac{1}{\pi}\int_{-1}^{+1}\sqrt{\frac{1+t}{1-t}}\,\frac{1}{\sigma-t}\,[C_{p,k,0} + C_{p,k,1}t + \dots +\\
&\qquad\qquad + C_{p,k,k-1}t^{k-1}]\,dt
\end{aligned}\right\}, \tag{I.83}$$

$$\left.\begin{aligned}
&f_{p,k} = \bar{C}_{p,k,0}\\
&f_{p,k+1} = \bar{C}_{p,k+1,0} + s\bar{C}_{p,k+1,1}\\
&f_{p,k+2} = \bar{C}_{p,k+2,0} + s\bar{C}_{p,k+2,1} + s^2\bar{C}_{p,k+2,2}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&f_{p,k+n} = \bar{C}_{p,k+n,0} + s\bar{C}_{p,k+n,1} + s^2C_{p,k+n,2} + \dots + s^{n-1}\bar{C}_{p,k+n,n-1}
\end{aligned}\right\}, \tag{I.84}$$

[44

$$\left.\begin{aligned}
\bar{C}_{0,n,m} &= \frac{1}{\pi}\int_{-1}^{+1}\sqrt{1-\sigma^2}\,B_{0,n,m}(\sigma)\,d\sigma \\
\bar{C}_{p,k,m} &= \frac{1}{\pi}\int_{-1}^{+1}\sqrt{\frac{1-\sigma}{1+\sigma}}\,\bar{B}_{p,k,m}(\sigma)\,d\sigma \\
B_{p,k,m}(\sigma) &= B_{0,n+1,m}\bar{d}_{p-1,k-m-1} + \bar{B}_{0,m+2,m}\bar{d}_{p-1,k-m-2} + \\
&\quad + \ldots + \bar{B}_{0,k-p,m}d_{p-1,1} \\
\bar{d}_{p,k} &= \frac{1}{\pi}\int_{-1}^{+1}\sqrt{\frac{1+t}{1-t}}\,\frac{1}{\sigma-t}\,[\bar{C}_{p,k,0} + \bar{C}_{p,k,1}t + \ldots + \\
&\quad + \bar{C}_{p,n,k-1}t^{k-1}]\,dt
\end{aligned}\right| \tag{I.85}$$

Thus, when function $\varphi_i(\sigma, s)$ is given in the form of (I.80) the problem is reduced to the determination of coefficients C_{nmp} and $\bar{C}_{nmp}$ from formulas (I.80)-(I.85).

The method discussed may be extended to other types of nuclei k(x, s) as well. In Chapter VI it will be used for studying the unsteady motion of a submerged hydrofoil.

Let us examine a special case of motion of a thin plate under the surface of a fluid. In the preceding section the expansion of the regular part of the nucleus was used in the form of expression (I.47). The expansion of the same function may be written in the form

$$\bar{K}(\bar{x}) = \sum_{n=1,2,3}^{\infty}\tau^n\sum_{p=1,2,3}^{n}\bar{x}^{p-1}Q_{n,p}; \tag{I.86}$$

$$Q_{n,p} = \frac{(-1)^{\frac{p}{2}-1}\left(\frac{n+p}{2}-1\right)!}{(p-1)!\left(\frac{n-p}{2}\right)!}[1 + 2\,\mathrm{Re}\,F_{p-1}(\lambda)];$$

for even n and p

$$Q_{n,p} = \frac{2(-1)^{\frac{p-1}{2}}\left(\frac{n+p}{2}-1\right)!}{(p-1)!\left(\frac{n-p}{2}\right)!}\,\mathrm{Im}\,F_{p-1}(\lambda);$$

for odd n and p

$$\lambda = \frac{2\bar{h}}{\mathrm{Fr}_b^2}.$$

By calculating function $\psi = \frac{p_n}{p_\infty}$, we obtain [45

$$\begin{aligned}\psi = 1 - \tau C_{010} - \tau^2\left[C_{020} - \frac{C_{021}}{2} - C_{120}\right] - \\ - \tau^3\left[C_{030} - C_{130} + C_{230} - \frac{C_{031}}{2} + \frac{C_{131}}{2} + \frac{C_{032}}{2}\right] - \\ - \tau^4\left[C_{040} - C_{140} + C_{240} - C_{340} - \frac{C_{041} - C_{141} + C_{241}}{2} + \right. \\ \left. + \frac{C_{042} - C_{142}}{2} - \frac{3}{8} C_{043}\right].\end{aligned} \quad \text{(I.87)}$$

After determining the coefficients from formulas (I.80) and (I.81), formula (I.87) may be written in the following form:

$$\begin{aligned}\psi = 1 - \tau Q_{1,1} - \tau^2 (Q_{2,2} - Q_{1,1}^2) - \tau^3\left(Q_{1,1} + \frac{3}{2} Q_{3,3} - 2Q_{1,1}Q_{2,2} + Q_{1,1}^3\right) - \\ - \tau^4\left(\frac{9}{4} Q_{4,4} + 2Q_{2,2} - \frac{3}{4} Q_{2,2}^2 - 3Q_{1,1}Q_{3,3} - 2Q_{1,1}^2 + 3Q_{2,2}Q_{1,1}^2 - Q_{1,1}^4\right) + \\ + \ldots\end{aligned} \quad \text{(I.88)}$$

Here

$$Q_{1,1} = 2\,\mathrm{Im}\,F_0(\lambda); \quad Q_{2,2} = 1 + 2\,\mathrm{Re}\,F_1(\lambda);$$
$$Q_{3,3} = -2\,\mathrm{Im}\,F_2(\lambda); \quad Q_{4,4} = -[1 + 2\,\mathrm{Re}\,F_3(\lambda)].$$

With $Fr_b \to \infty$, equation (I.83) gives the first three terms in formula (I.58).

Curves [formula (I.88)] in Fig. 5 illustrate the effect of the Fr_b number on the Ψ function.

Fig. 5

CHAPTER II. THE THEORY OF THE SUBMERGED HYDROFOIL OF AN ARBITRARY PROFILE IN A PLANE-PARALLEL FLOW [46

2.1. Basic Relationships. Formulas for Determining Forces

The two-dimensional problem dealing with the steady motion of a submerged hydrofoil may also be solved for the case of a solid profile using the boundary conditions [2]. The method of solving this problem was suggested by N. Ye. Kochin [56]. This is a general method which may be used for solving a number of problems in the hydrodynamics of the submerged hydrofoil.

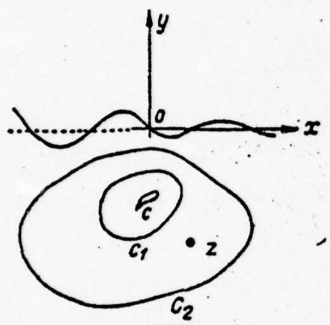

Fig. 6

Let us examine a two-dimensional profile of an arbitrary shape, submerged to a depth h under a free surface and moving at a constant velocity V_0.

Let us consider a point z in the lower half-plane and draw two contours C_1 and C_2 so that the point z would be located outside of the C_1 but within the C_2 contour (Fig. 6). The C_2 contour can be changed in such a way that it will fully envelop the lower half-plane.

Let us introduce a complex velocity of the flow being considered which is calculated as follows:

$$v(z) = W_z(z) = v_x - iv_y.$$

Let us use the Cauchy integral for the two-link area obtained in which v(z) will be an analytical function.

$$v(z) = \frac{1}{2\pi i}\int\limits_{C_1}\frac{v(\zeta)\,d\zeta}{z-\zeta} - \frac{1}{2\pi i}\int\limits_{C_2}\frac{v(\zeta)\,d\zeta}{z-\zeta}. \qquad \text{(II.1)}$$

Let us denote [47

$$\begin{aligned} v_1(z) &= \frac{1}{2\pi i}\int\limits_{C_1}\frac{v(\zeta)\,d\zeta}{z-\zeta}; \\ v_2(z) &= -\frac{1}{2\pi i}\int\limits_{C_2}\frac{v(\zeta)\,d\zeta}{z-\zeta}. \end{aligned} \qquad \text{(II.2)}$$

The $v_1(z)$ function is an analytical function on the outside of the C_1 contour and its magnitude at infinity

is of the order of $1/z$, while the $v_2(z)$ function is analytical within the C_2 contour.

The (I.1) condition indicates that the single-valued function

$$\Omega_1(z) = iv_z(z) - vv(z) \tag{II.3}$$

assumes real values along the Ox axis. According to the Schwarz symmetry principle it can be extended into area D', which is symmetrical with respect to D. Then $\Omega_1(z)$ will be an analytical function within the area confined between the C and C' contours, with the latter being symmetrical with respect to the Ox axis. The integral in equation (II.3), which converges to zero when $x \to \infty$, is in the following form:

$$v(z) = -ie^{-ivz}\int_{\infty}^{z} e^{ivz}\Omega_1(z)\,dz. \tag{II.4}$$

Let us examine the function

$$\int_0^z \Omega_1(z)\,dz = i[v(z) - v(0)] - v\int_0^z v(z)\,dz.$$

This function may become a many-valued function if the velocity circulation is different from zero. However, the $\int_0^z F(z)dz,\ F(z) = \Omega_1(z) - \Omega_2(z)$ function will be single-valued even within the area between the C and C' contours.

Since, according to the condition, the $v(z)$ and $v_1(z)$ functions are limited in modulus in the lower half-plane when $/z/ \to \infty$, it is clear that when $/z/ \to \infty$ the $\int_0^z F(z)dz$ function cannot increase in modulus at a greater rate than $k/z/$ does, where k is a constant. This means that $\int_0^z F(z)dz$ may have a pole only of the first order when $z = \infty$ and, therefore, the $F(z)$ and $\Omega_1(z)$ functions are regular functions at the point $z = \infty$. Moreover, at this point they vanish, since otherwise it would follow from equation (II.3) that when $y = 0$ and $x \to +\infty$, $v_z(z)$ tends to a

certain limit which is different from zero and, therefore, $v(z)$ increases in modulus without limits. The $v_1(z)$ function consists of terms of type $\frac{B}{z-\zeta}$ (where B is a complex [48

constant and ζ is a point in the lower half-plane). However, for the singularity of such type the $\omega(z)$ function, which satisfies the condition (I.1), may be determined from formulas (I.15) and (I.17):

$$\omega(z)=\frac{B}{z-\zeta}-\frac{\bar{B}}{z-\bar{\zeta}}+2i\nu\bar{B}e^{-i\nu z}\int\limits_{\infty}^{z}\frac{e^{i\nu t}}{t-\bar{\zeta}}dt. \qquad \text{(II.5)}$$

But assuming $B=\frac{1}{2\pi i}v(\zeta)\,d\zeta$, we will arrive at the following formula

$$v'(z)=\frac{1}{2\pi i}\int\limits_{C_1}\frac{v(\zeta)}{z-\zeta}d\zeta+\frac{1}{2\pi i}\int\limits_{C_1}\bar{v}(\zeta)\left[\frac{1}{z-\bar{\zeta}}-2i\nu e^{-i\nu z}\int\limits_{\infty}^{z}\frac{e^{i\nu t}}{t-\bar{\zeta}}dt\right]d\zeta. \qquad \text{(II.6)}$$

If the contour of integration with respect to t is assumed to be located entirely in the lower half-plane, then the $v'(z)$ function will be analytical in the region located between the C_1 contour and the OX axis, limited in the lower half-plane when $/z/\to\infty$, and will tend to zero when $x\to+\infty$. In such a case the function $v'(z) - v(z)$ will become analytical in the entire lower half-plane, will satisfy the boundary condition (I.1), be limited in the lower half-plane when $/z/\to\infty$, and will tend to zero when $x\to+\infty$.

Then, by using formula (II.4), in which function $\Omega_1(z)$ should be assumed to equal zero because it has no singularities in the entire plane of the complex variable and is vanishing when $z=\infty$, we will obtain an identity

$$v'(z) - v(z) = 0.$$

Thus, we have obtained another representation for the function $v_2(z)$:

$$v_2(z)=\frac{1}{2\pi i}\int\limits_{C_1}\bar{v}(\zeta)\left[\frac{1}{z-\bar{\zeta}}-2i\nu e^{-i\nu z}\int\limits_{+\infty}^{z}\frac{e^{i\nu t}}{t-\bar{\zeta}}dt\right]d\bar{\zeta}. \qquad \text{(II.7)}$$

Formula (II.7) may be reduced to the form

$$v_2(z) = \frac{1}{2\pi i} \int_{C_1} \bar{v}(\zeta) \left[i \int_0^\infty e^{i\lambda(z-\bar{\zeta})} \frac{\lambda+\nu}{\lambda-\nu} d\lambda - 2\pi\nu e^{-i\nu(z-\bar{\zeta})} \right] d\zeta. \qquad \text{(II.8)}$$

In order to determine the flow forces acting on a moving contour we will use the first formula derived by S. A. Chaplygin:

$$P - iQ = -\frac{\varrho}{2} \int_{C_1} v_0^2(z)\, dz,$$

where $v_0(z) = -v_0 + v_1(z) + v_2(z)$.

Calculating the force, we obtain [49

$$P - iQ = \varrho v_0 \Gamma - \varrho \int_{C'} v(z)\, v_2(z)\, dz, \qquad \text{(II.9)}$$

where $\Gamma = \int_{C'} v(z)\, dz$ is circulation along the contour which includes the contour C'.

Let us introduce N. Ye. Kochin's function $H(\lambda)$:

$$H(\lambda) = \int_C e^{-i\lambda z} v(z)\, dz. \qquad \text{(II.10)}$$

Then, the expression (II.8) will look as follows

$$v_2(z) = \frac{1}{2\pi} \int_0^\infty e^{-i\lambda\bar{z}}\, \overline{H(\lambda)}\, \frac{\lambda+\nu}{\lambda-\nu} d\lambda + i\nu e^{-i\nu z}\, \overline{H(\nu)}. \qquad \text{(II.11)}$$

Determining $V_2(z)$ in formula (II.9) by using formula (II.11) we obtain:

$$P = \varrho v_0 \Gamma - \frac{\varrho}{2\pi} \int_0^\infty |H(\lambda)|^2 \frac{\lambda+\nu}{\lambda-\nu} d\lambda + g\varrho s, \qquad \text{(II.12)}$$

where $g\varrho s$ is the archimedean lifting force;

$$Q = \nu\varrho\, |H(\nu)|^2. \qquad \text{(II.13)}$$

In determining the $v_2(z)$ function from formula (II.7) N. Ye. Kochin obtained another formula for the lifting force [56]:

$$P = \varrho v\Gamma - \frac{\varrho}{2\pi}\int_0^\infty |H(\lambda)|^2 d\lambda + \frac{\varrho v}{\pi}\int_{-\infty}^{1} |H(\nu - \lambda\nu)|^2 \frac{d\lambda}{\lambda} + g\varrho s, \qquad (II.14)$$

which, after simple transformations, may be written in the form of (II.12).

In applying formula (II.7) the following integral representation is utilized

$$\frac{1}{z-\bar{\zeta}} = i\int_0^\infty e^{-i\lambda(z-\bar{\zeta})} d\lambda, \qquad (II.15)$$

if the z and ζ points are in the lower half-plane.

We will determine the hydrodynamic moment from the second formula by S. A. Chaplygin:

$$M = \operatorname{Re}\frac{\varrho}{2}\int_{C'} z v_0^2 dz,$$

which, after transformations, will become [50

$$M = -\varrho v_0 \operatorname{Re}\int_{C'} z v(z)\, dz + \varrho \operatorname{Re}\int_{C'} z v(z) v_2(z)\, dz. \qquad (II.16)$$

Taking into account that $\frac{dH}{d\lambda} = -i\int_{C'} e^{-i\lambda z} v(z)\, dz$, we obtain the following formula for the total moment:

$$M_1 = -\varrho v_0 \operatorname{Re}[iH'(0)] - \varrho \operatorname{Re}\left[\frac{1}{2\pi i}\int_0^\infty H'(\lambda)\overline{H}(\lambda)\frac{\lambda+\nu}{\lambda-\nu} d\lambda + \right.$$
$$\left. + \nu H'(\nu)\overline{H}(\nu)\right] - g\varrho s x_c, \qquad (II.17)$$

where $g\varrho s x_c$ is the moment which maintains the archimedean lifting forces.

Formula (II.17) may be transformed into a form given by N. Ye. Kochin:

$$M_1 = -\varrho v_0 \operatorname{Re}(iH'(0)) - \varrho \operatorname{Re}\left[\frac{1}{2\pi i}\int_0^\infty H'(\lambda)\overline{H}(\lambda)\, d\lambda + \right.$$

$$+\frac{\nu i}{\pi}\int_{-\infty}^{1} H'(\nu-\lambda\nu)\overline{H(\nu-\lambda\nu)}\frac{d\lambda}{\lambda}+\nu H'(\nu)\overline{H(\nu)}\Bigg]-g\varrho s x_c. \qquad \text{(II.18)}$$

The integrals in formulas (II.12) and (II.17), and the second integrals in formulas (II.14) and (II.18) are determined as the main Cauchy values. Let us determine the shape of the free surface. The profile of the waves will be determined from formula (II.20).

We have

$$\lim_{|z|\to\infty} v_1(z)=0.$$

From formula (II.7)

$$\lim_{x\to+\infty} v_2(x)=0,$$

$$\lim_{x\to-\infty}\left(v_2(x)+2i\nu\int_{C_1}\bar{v}(\zeta)\,e^{-i\nu x}e^{i\nu\zeta}d\bar{\zeta}\right)=0,$$

hence it is seen that waves are not formed in front of a moving body and that the sinusoidal waves are formed far behind that body:

$$\eta=\frac{2v_0\nu}{g}\operatorname{Im}\left(\overline{H(\nu)}e^{-i\nu x}\right). \qquad \text{(II.19)}$$

The amplitude of the waves formed is [51

$$a=\frac{2}{v_0}|H(\nu)|. \qquad \text{(II.20)}$$

Thus, all the unknown values of the problem are determined through the H(λ) function. Since the $e^{-i\lambda z}v_2(z)$ function is analytical within the C_1 contour, then the contour integral containing this function will be equal to zero. The H(λ) function will be determined by the complex velocity $v_1(z)$ only:

$$H(\lambda)=\int_{C_1} e^{-i\lambda z}v_1(z)\,dz. \qquad \text{(II.21)}$$

Let us examine a problem of motion of a circular cylinder with a radius R, a depth h and a circulation velocity along the contour Γ. Let us determine the H(λ) function in the first approximation by the complex velocity of the moving cylinder in an infinite fluid.

$$v_\infty(z) = \frac{4R^2}{(z+hi)^2} + \frac{\Gamma}{2\pi i\,(z+hi)};$$

$$H(\lambda) = e^{-\lambda h}(\Gamma + 2\pi v_0 \lambda R^2). \qquad (II.22)$$

From formula (II.13) we derive the expression for the wave drag of the cylinder:

$$Q = \varrho\nu\left(\Gamma + \frac{2\pi g R^2}{v_0}\right)e^{-\nu h}. \qquad (II.23)$$

From equation (II.12) the expression for the lifting force is in the form [56]

$$P = \varrho v_0 \Gamma - 4\pi R^4 v_0^2 \varrho\left[\left(\frac{1}{2h}\right)^3 + \left(\frac{1}{2h}\right)^2 \nu + \frac{1}{2h}\nu^2 - \nu^3 e^{-2\nu h} E_{i1}(2\nu h)\right] -$$
$$- \varrho\Gamma^2\left[\frac{1}{4\pi h} - \frac{\nu}{\pi}e^{-2\nu h}E_{i1}(2\nu h)\right] - \qquad (II.24)$$
$$- \varrho v_0 \Gamma\left[\frac{1}{8}\left(\frac{2R}{h}\right)^2 + \frac{2\nu R^2}{h} - 4\nu^2 R^2 e^{-2\nu h}E_{i1}(2\nu h)\right] + \pi g \varrho R^2.$$

The formula for the cylinder moment will be

$$M = -\varrho\nu H(\nu) H'(\nu).$$

Since

$$H'(\lambda) = -hH(\lambda) + 2\pi v_0 R^2 e^{-\lambda h},$$

then

$$M = hQ - 2\pi\varrho v_0 R^2 \nu e^{-h\nu} H(\nu). \qquad (II.25)$$

2.2. The Effect of the Free Surface on Circulation of the Submerged Hydrofoil [52

Function Γ, which appears in formulas (II.12), (II.13), (II.17) and (II.18), determines both explicitly and implicitly the value of circulation along the contour through the function $H(\lambda)$. The value of circulation Γ is determined from the N. Ye. Zhukovskiy and S. A. Chaplygin postulate and, naturally, the free surface causing changes in velocities of fluid particles, leads to changes in the circulation value along the hydrofoil contour. Let us examine the problem of determining the effect of the free surface on circulation along the hydrofoil contour. Let us represent conformally the shape of the contour C on the circle with radius R in the plane of the complex variable so that the infinitely distant point in the z plane would

show as an infinitely distant point in the u plane and

$$\left(\frac{du}{dz}\right)_{z=\infty} = 1.$$

Function $v_1(z)$ at the infinitely distant point may be expanded as follows:

$$v_1(z) = \frac{\Gamma}{2\pi iz} + \frac{\alpha_1}{z^2} + \frac{\alpha_3}{z^3} + \ldots$$

In the u plane, the $v_1(z)$ function will be analytical outside the circle with radius R and will be determined, in this entire area, by the expansion

$$v_1(z) = \frac{\Gamma}{2\pi iu} + \frac{\beta_2}{u^2} + \frac{\beta_3}{u^3} + \ldots$$

Since the relative velocity has the same direction as that of the contour element, the boundary condition along the C contour [56] may be written in the form

$$\mathrm{Im}\,[v_1(z) + v_2(z) - v_0]\,dz = 0 \text{ на } C. \qquad \text{(II.26)}$$

For the entire area, $|u| \geqslant R\ \frac{dz}{du}$ may be expressed by the expansion

$$\frac{dz}{du} = 1 - \frac{\gamma_1}{u^2} + \frac{\gamma_2}{u^3} - \ldots,$$

since the expansion of z is in the form

$$z = u + \gamma_0 + \frac{\gamma_1}{u} + \frac{\gamma_2}{u^2} + \ldots$$

The function $v_2(z)$ is a single-valued analytic function of z in the entire lower half-plane. As a function of u, it will be single-valued and analytical at least in the area of a circle with $R < u < R_1$. The function [53

$f(u) = [v_2(z) - v_0]\frac{dz}{du}u$ will also be single-valued and analytical in a certain circle, while the function $g(u) = v_1(z)\frac{dz}{du}u$ will be analytical outside the circle $/u/ = R$ and can be expanded

into a series

$$g(u)=\frac{\Gamma}{2\pi i}+\sum_{n=1}^{\infty}\frac{b_n}{u^n}.$$

In the plane u, in which the C contour corresponds to the circle k, conditions along k will be

$$u=\mathrm{R}e^{i\Theta},\quad du=ire^{i\Theta}d\Theta=iud\Theta,$$

and the boundary condition (II.22) may be written in the following form:

$$\operatorname{Re} v_1(z)\frac{dz}{d\sigma}\sigma=-\operatorname{Re}\left[v_2(z)-v_0\right]\frac{dz}{d\sigma}\sigma \quad \text{on k.} \tag{II.27}$$

The condition (II.27) is a boundary condition of the problem of determining the function which is analytical outside the circle in terms of its real part in the circle.

N. Ye. Kochin solves then the problem by considering the Yu. V. Sokhotskiy formulas for the limit values of the Cauchy type integral.

Let us present a solution with the aid of the Schwarz formula. This method was used by N. Ye. Kochin in study-hydrodynamic grids [61]. According to Schwarz's formula which determines the analytical function F(u) outside the circle in terms of its real part k in the circle

$$F(u)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\operatorname{Re}F(\sigma)\frac{u+\sigma}{u-\sigma}d\theta+Ci$$

we obtain

$$v_1(z)\frac{dz}{du}u=-\frac{1}{4\pi i}\int_k\left\{\left[v_2(z)-v_0\right]\frac{dz}{d\sigma}\sigma+\overline{\left[v_2(z)-v_0\right]\frac{dz}{d\sigma}\sigma}\right\}\times$$
$$\times\frac{u+\sigma}{u-\sigma}\cdot\frac{d\sigma}{\sigma}+Ci.$$

Let us transform this expression:

$$\int_k\left[v_2(z)-v_0\right]\frac{u+\sigma}{u-\sigma}dz=\int_k\left[v_2(z)-v_0\right]\frac{u+\sigma}{u-\sigma}dz+$$
$$+\int_k\left[v_2(z)-v_0\right]dz=2u\int_k\frac{v_2(z)-v_0}{u-\sigma}dz;$$

$$\int_k \overline{[v_2(z)-v_0]}\,\frac{d\bar{z}}{d\sigma}\,\bar{\sigma}\,\frac{u+\sigma}{u-\sigma}\cdot\frac{d\sigma}{\sigma} = -\int_k \overline{[v_2(z)-v_0]}\,\frac{u+\sigma}{u-\sigma}\,d\bar{z} =$$ [54

$$= \int_k \overline{[v_2(z)-v_0]}\,d\bar{z} - \int_k \overline{[v_2(z)-v_0]}\,\frac{u+\sigma}{u-\sigma}\,d\bar{z} =$$

$$= -2\int_k \frac{\sigma}{u-\sigma}\,\overline{[v_2(z)-v_0]}\,dz = -2\,\frac{R^2}{u}\int_k \frac{\overline{[v_2(z)-v_0]}}{\bar{\sigma}-\frac{R^2}{u}}\,d\bar{z}.$$

Then

$$v_1(z)\,\frac{dz}{du}\,u = \frac{u}{2\pi i}\int_k \frac{v_2(z)-v_0}{\sigma-u}\,dz + \frac{R^2}{2\pi i u}\,\frac{\overline{v_2(z)-v_0}}{\bar{\sigma}-\frac{R^2}{u}}\,dz + Ci. \qquad \text{(II.29)}$$

By expressing $v_2(z)$ through functions $H(\lambda)$ and (II.11) and introducing functions $G(\lambda, u)$

$$G(\lambda, u) = \int_k \frac{e^{-i\lambda z}}{\zeta(z)-u}\,dz, \qquad \text{(II.30)}$$

we obtain the following from formula (II.25) [114]:

$$v_1(z) = \frac{du}{dz}\Bigg\{-v_0 + \frac{v_0 R^2}{u^2} + \frac{Ci}{u} + \frac{1}{4\pi^2 i}\int_0^\infty \Bigg[\overline{H(\lambda)}\,G(\lambda, u) +$$

$$+ \frac{R^2}{u^2}\,H(\lambda)\,\overline{G\left(\lambda, \frac{R^2}{u}\right)}\Bigg]\frac{\lambda+\nu}{\lambda-\nu}\,d\lambda + \frac{\nu}{2\pi}\Bigg[\overline{H(\nu)}\,G(\nu, u) -$$

$$- \frac{R^2}{u^2}\,H(\nu)\,\overline{G\left(\nu, \frac{R^2}{u}\right)}\Bigg]\Bigg\}. \qquad \text{(II.31)}$$

For the $v(z)$ function N. Ye. Kochin obtained the following expression:

$$v(z) = \frac{\Gamma}{2\pi i}\,\frac{1}{u}\,\frac{du}{dz} - v_0\,\frac{du}{dz} + v_0 + \frac{v_0 R^2}{u^2}\,\frac{du}{dz} +$$

$$+ \frac{1}{2\pi}\int_0^\infty \Bigg[\left(e^{-i\lambda z} + \frac{1}{2\pi i}\,\frac{du}{dz}\,G(\lambda, u)\right)\overline{H}(\lambda) +$$

$$+ \frac{1}{2\pi i}\,\frac{R^2}{u^2}\,\frac{du}{dz}\,\overline{G\left(\lambda, \frac{R^2}{u}\right)}\,H(\lambda)\Bigg]\,d\lambda +$$ [55

$$+ i\nu\left\{\left[e^{-i\nu z} + \frac{1}{2\pi i}\frac{du}{dz}G(\nu, u)\right]\overline{H(\nu)} -\right.$$

$$\left. - \frac{1}{2\pi i}\frac{R^2}{u^2}\frac{du}{dz}\overline{G\left(\nu; \frac{R^2}{\bar{u}}\right)}H(\nu)\right\} -$$

$$-\frac{\nu}{\pi}\int_{-\infty}^{1}\left\{\left[e^{-i\nu z(1-\lambda)} + \frac{1}{2\pi i}\frac{du}{dz}G(\nu - \lambda\nu; u)\right]\overline{H(\nu - \lambda\nu)} +\right.$$

$$\left. + \frac{1}{2\pi i}\frac{R^2}{u^2}\frac{du}{dz}\overline{G\left(\nu - \lambda\nu; \frac{R^2}{\bar{u}}\right)}H(\nu - \lambda\nu)\right\}d\lambda. \qquad \text{(II.32)}$$

Formula (II.31) contains an arbitrary, purely imaginary constant Ci, which is determined from the condition at the trailing edge.

Let us assume that point $u_0 = Re^{i\theta_0}$ corresponds to the angular point of contour C. The finite velocity requirements at this point may be satisfied by the condition

$$v(z)\,u\frac{dz}{du} = 0. \qquad \text{(II.33)}$$

Moreover, let us consider that $\frac{dz}{du}$ at this point also vanishes. The arbitrary constant C is not equal to the value of circulation Γ along the contour, since the integral terms in formula (II.31) may be expanded into Laurent series and these expansions will contain terms with the first power of u. Therefore, it is not necessary to determine the values of the purely imaginary quantity Ci, but one should isolate the terms containing u^{-1} in the expression (II.31) and determine the value of the new constant Γ_0. For this reason it is not yet possible to apply condition (II.33) to the expression (II.31). In order to use this condition formula (II.31) should be presented in the form of the expansion

$$v_1(z) = \frac{du}{dz}\left[C_0 + \frac{\Gamma}{2\pi i u} + \frac{C_2}{u^2} + \frac{C_3}{u^3} + \dots\right]. \qquad \text{(II.34)}$$

A further study of the problem may be performed only for a certain type of transforming function, since for obtaining the expansion (II.34) it is necessary to find the $G(\lambda, u)$ function. For determining the $G(\lambda, u)$ function the transforming function must be assigned.

2.3. Motion of a Thin Hydrofoil [56

Let us consider motion of a hydrofoil as obtained with the aid of the N. Ye. Zhukovskiy transforming function

$$z = u + \frac{R^2}{u^2}.$$

We obtain the following expression for functions $G(\lambda, u)$ and $\overline{G\left(\lambda, \frac{R^2}{u}\right)}$:

$$G(\lambda, u) = e^{-\lambda h} \int\limits_{|u|=R} \frac{e^{-i\lambda\left(\zeta + \frac{R^2}{\zeta}\right)}\left(1 - \frac{R^2}{\zeta^2}\right)}{\zeta - u}\, d\zeta.$$

Let us use the following identity

$$\frac{1}{\zeta^2(\zeta - u)} = \frac{1}{u^2(\zeta - u)} - \frac{1}{u^2\zeta} - \frac{1}{u\zeta^2}.$$

Then

$$G(\lambda, u) = e^{-\lambda h} \int\limits_{|u|=R} e^{-\lambda i\left(\zeta + \frac{R^2}{\zeta}\right)}\left[\frac{1}{\zeta - u}\left(1 - \frac{R^2}{u^2}\right) + \frac{R^2}{\zeta^2 u} + \frac{R^2}{\zeta u^2}\right] d\zeta,$$

from which

$$G(\lambda, u) = e^{-\lambda h}\left[B(\lambda, u)\frac{dz}{du} + 2\pi i J_0(2\lambda R)\frac{R^2}{u^2} + 2\pi J_2(2\lambda R)\frac{R}{u}\right],$$

$$B(\lambda, u) = e^{-\lambda h} \int\limits_{|u|=R} \frac{e^{-i\lambda\left(\zeta + \frac{R^2}{\zeta}\right)}}{\zeta - u}\, d\zeta, \tag{II.35}$$

$$\overline{G\left(\lambda, \frac{R^2}{u}\right)} = e^{-\lambda h}\left[\overline{B\left(\lambda, \frac{R^2}{u}\right)}\frac{dz}{d\frac{R^2}{u}} - 2\pi i J_0(2\lambda R)\frac{u^2}{R^2} + 2\pi J_1(2\lambda R)\frac{u}{R}\right]. \tag{II.36}$$

Here, in determining the $G(\lambda, u)$ function an integral representation of the Bessel functions was used in the following form:

$$J_{k-1}(2\lambda R) = \frac{1}{2\pi i} \int\limits_{|u|=R} e^{\lambda R\left(t - \frac{1}{t}\right)} \frac{dt}{t^k}. \tag{II.37}$$

After determining function G(λ, u) from formulas (II.35) and (II.36) the formula for $v_1(z)$ may be written in the form [57

$$v_1(z) = \frac{du}{dz}\left\{-\left[v_0 + \frac{1}{2\pi}\int_0^\infty e^{-\lambda h} H(\lambda) J_0(2\lambda R)\frac{\lambda+\nu}{\lambda-\nu}d\lambda - \right.\right.$$

$$\left. - i\nu e^{-\nu h} J_0(2\nu R) H(\nu)\right] + \frac{R^2}{u^2}\left[\bar{v}_0 + \frac{1}{2\pi}\int_0^\infty e^{-\lambda h}\overline{H}(\lambda) J_0(2\lambda R)\frac{\lambda+\nu}{\lambda-\nu}d\lambda + \right.$$

$$\left.\left. + \nu i e^{-\nu h} J_0(2\nu R)\overline{H}(\nu)\right] + \frac{\Gamma}{2\pi i u}\right\} +$$

$$+ \frac{1}{4\pi^2 i}\int_0^\infty e^{-\lambda h}\left[B_1(\lambda, u)\overline{H}(\lambda) + \frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u}}\overline{B\left(\lambda, \frac{R^2}{u}\right)} H(\lambda)\right]\frac{\lambda+\nu}{\lambda-\nu}d\lambda +$$

$$+ \frac{\nu e^{-\nu h}}{2\pi}\left[B(\nu, u)\overline{H}(\nu) + \frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u}}\overline{B\left(\nu, \frac{R^2}{u}\right)} H(\nu)\right]. \qquad \text{(II.38)}$$

Let us assume that the condition (II.33) is satisfied at a point u_0 = -R; then, from formula (II.38), we obtain [114]

$$\Gamma = 4\pi R \operatorname{Im}\left\{v_0 + \frac{1}{2\pi}\int_0^\infty e^{-\lambda h} H(\lambda) J_0(2\lambda R)\frac{\lambda+\nu}{\lambda-\nu}d\lambda - \right.$$

$$\left. - \nu i e^{-\nu h} J_0(2\nu R) H(\nu)\right\}. \qquad \text{(II.39)}$$

To determine the H(λ) function let us write a functional equation. Let us multiply both parts of the equation (II.34) by $e^{i\mu z}$ and integrate along the G_1 contour. Then the functional equation will acquire the form

$$H(\mu) = e^{-\mu h}\left\{\Gamma J_0(2\mu R) - 4i\pi R \operatorname{Im}\left[v_0 + \right.\right.$$

$$+ \frac{1}{2\pi}\int_0^\infty e^{-\lambda h} H(\lambda) J_0(2\lambda R)\frac{\lambda+\nu}{\lambda-\nu}d\lambda -$$

$$- i\nu e^{-\nu h} J_0(2\nu R)\, H(\nu) \Bigg] J_1(2\mu R) +$$ [58

$$+ \frac{1}{4\pi^2 i} \int_0^\infty e^{-\lambda h} [\overline{H}(\lambda) N_0(\lambda, \mu) + H(\lambda) N_1(\lambda, \mu)] \frac{\lambda + \nu}{\lambda - \nu} d\lambda +$$

$$+ \frac{\nu e^{-\nu h}}{2\pi} [\overline{H}(\nu) N_0(\nu, \mu) + H(\nu) N_1(\nu, \mu)] \Bigg\}, \qquad \text{(II.40)}$$

where

$$N_0(\lambda, \mu) = e^{\mu h} \int_C e^{-i\mu z} B(\lambda, u)\, dz;$$

$$N_1(\lambda, \mu) = e^{\mu h} \int_C e^{-i\mu z} \frac{R^2}{u^2} \overline{B\left(\lambda, \frac{R^2}{\overline{u}}\right)}\, dz.$$

By comparing expressions (3.39) and (3.40) [sic] we determine that the co-factor at $J_1(2\mu R)$ is exactly equal to Γ. Taking this into consideration, let us rewrite equation (II.40) in the form

$$H(\mu) = e^{-\mu h} \Bigg\{ \Gamma [J_0(2\mu R) - iJ_1(2\mu R)] +$$

$$+ \frac{1}{4\pi^2 i} \int_0^\infty [\overline{H}(\lambda) N_0(\lambda, \mu) + H(\lambda) N_1(\lambda, \mu)] \frac{\lambda + \nu}{\lambda - \nu} d\lambda +$$

$$+ \frac{\nu}{2\pi} [\overline{H}(\nu) N_0(\nu, \mu) - H(\nu) N_1(\nu, \mu)] \Bigg\}. \qquad \text{(II.41)}$$

Function $H(\mu)$ may be determined, in the first approximation, by the first two terms only:

$$H(\mu) = e^{-\mu h} \Gamma [J_0(2\mu R) - iJ_1(2\mu R)]. \qquad \text{(II.42)}$$

Then, the value of circulation Γ will be determined from equation (II.39):

$$\Gamma = \frac{\Gamma_\infty}{\left[1 + 2R \int_0^\infty e^{-2\lambda h} J_0(2\lambda R) J_1(2\lambda R) \frac{\lambda + \nu}{\lambda - \nu} d\lambda + 4\pi R \nu e^{-2\nu h} J_0(2\nu R)\right]}. \qquad \text{(II.43)}$$

Here Γ_∞ is the circulation of a hydrofoil in an infinite flow.

However, N. Ye. Kochin [56] showed that one may obtain a satisfactory approximation if, instead of the complex velocity $v_1(z)$, a complex velocity corresponding to the motion of a hydrofoil in an infinite flow is used in determining the $H(\lambda)$ function. Calculations show that the $H(\lambda)$ function determined in such a way will, at high velocities, provide results which are closer to the experimental results than those calculated by formula (II.42). The $H(\lambda)$ and Γ functions, which correspond to this approximation, will be determined with the aid of the following formulas: [59

$$H(\lambda)=e^{-\lambda h}\Gamma_\infty[J_0(2\lambda R)-iJ_1(2\lambda R)]; \tag{II.44}$$

$$\Gamma=\Gamma_\infty\left[1-2R\int\limits_0^\infty e^{-2\lambda h}J_0(2\lambda R)\,J_1(2\lambda R)\frac{\lambda+\nu}{\lambda-\nu}d\lambda-4\pi R\nu e^{-2\nu h}J_0(2\nu R)\right]. \tag{II.45}$$

Let us consider the following functions:

$$A_{nm}=R\int\limits_0^\infty e^{-2\lambda h}J_n(2\lambda R)\,J_m(2\lambda R)\,d\lambda;$$

$$B_{nm}=\int\limits_{-\infty}^{1} e^{-2\nu(1-\lambda)h}J_n[2\nu(1-\lambda)R]J_m[2\nu(1-\lambda)R]\frac{d\lambda}{\lambda}. \tag{II.46}$$

After transformations, the expressions for P and Γ may be written in terms of functions $A_{nm}B_{nm}$:

$$P_n=\varrho v_0\Gamma-\frac{\varrho\Gamma_\infty^2}{2\pi R}\left(A_{00}-\frac{\omega B_{00}}{2}+A_{11}-\frac{\omega}{2}B_{11}\right), \tag{II.47}$$

$$\Gamma=\frac{\Gamma_\infty}{1+2\left(A_{01}-\frac{\omega}{2}B_{01}\right)+4\pi\omega e^{-2\omega h}J_0^2\left(\frac{\omega}{2}\right)}, \tag{II.48}$$

$$\Gamma=\Gamma_\infty\left[1-2\left(A_{01}-\frac{\omega}{2}B_{01}\right)-4\pi\omega e^{-2\omega\bar{h}}J_0^2\left(\frac{\omega}{2}\right)\right], \tag{II.49}$$

where

$$\omega=4\nu R.$$

For hydrofoil motion with large Fr_b numbers, formulas (II.47)-(II.49) will be in the form

$$P_n=\varrho v_0\Gamma-\frac{\varrho\Gamma_\infty^2}{2\pi R}(A_{00}+A_{01}); \tag{II.50}$$

$$\Gamma=\frac{\Gamma_\infty}{1+2A_{01}}; \tag{II.51}$$

$$\Gamma=\Gamma_\infty(1-2A_{01}). \tag{II.52}$$

For hydrofoil motion with small Fr_b numbers, the terms containing A_{nm} functions in formulas (II.50)-(II.52) will reverse their signs. This follows directly from the type of boundary conditions for which Fr = 0 (the hydrofoil is near a solid wall). It will be formally shown by considering the extreme values of the $A_{nm}-\frac{\omega}{2}B_{nm}$ combination. [60

The A_{00}, A_{01} and A_{11} functions may be expressed directly through the Legendre functions of the second kind.

From the general formula [8]

$$\int_0^\infty e^{-dt}J_n(\mathit{в}t)J_n(ct)\,dt=\frac{1}{\pi\sqrt{\mathit{в}c}}Q^{n-\frac{1}{2}}\left(\frac{a^2+\mathit{в}^2+c^2}{2\mathit{в}c}\right)$$

it follows

$$A_{01}=\frac{1}{2\pi}Q_{-\frac{1}{2}}(1+8\bar{h}^2),\quad A_{11}=\frac{1}{2\pi}Q_{\frac{1}{2}}(1+8\bar{h}^2).$$

By using the relation $J_0'(t) = J_1(t)$ and integrating by parts, we have

$$A_{01}=\frac{1}{4}\left[1-\frac{4\bar{h}}{\pi}Q_{-\frac{1}{2}}(1+8\bar{h}^2)\right].$$

For $Re(n + 1) > 0$ and any arbitrary n, with the exception of negative integers in the region $/z/ > 1$ and $/\arg z/ < \pi$, the expansion given below is valid

$$Q_n(z)=\frac{\sqrt{\pi}\,\Gamma_{(n+1)}}{2^{n+1}\Gamma_{\left(n+\frac{3}{2}\right)}z^{n+1}}F\left(\frac{1}{2}n+\frac{1}{2};\ \frac{1}{2}n+1;\ n+\frac{3}{2};\ \frac{1}{z^2}\right).$$

In our case $/z/ = 1 + 8\bar{h}^2 > 1$ and, therefore, the expansion is valid in the entire lower half-plane:

$$Q_{-\frac{1}{2}}(z)=\frac{\pi}{\sqrt{2}}\frac{1}{\sqrt{8\bar{h}^2+1}}F_1,$$

$$Q_{\frac{1}{2}}(z)=\frac{\pi}{4\sqrt{2}}\frac{1}{(8\bar{h}^2+1)^{3/2}}F_2,$$

where $F_1=F\left(\frac{1}{4};\ \frac{3}{4};\ 1;\ \frac{1}{(8\bar{h}^2+1)^2}\right)$ and $F_2=F\left(\frac{3}{4};\ \frac{5}{2};\ 2;\ \frac{1}{(8\bar{h}^2+1)^2}\right)$

are the hypergeometrical functions whose values may be determined by the expansion into a hypergeometrical series [73]

$$F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha \cdot \beta}{1!\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)} z^2 + \\ + \dots + \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\gamma(\gamma+1)\dots(\gamma+n-1)} z^n.$$

The final expressions for the A_{00}, A_{01} and A_{11} functions may be written in the form [61

$$\left.\begin{aligned} A_{00} &= \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{8\bar{h}^2+1}} F_1 \\ A_{01} &= \frac{1}{4}\left(1 - \frac{4\bar{h}}{\sqrt{2}} \frac{1}{\sqrt{8\bar{h}^2+1}} F_1\right) \\ A_{11} &= \frac{1}{8\sqrt{2}} \frac{1}{(8\bar{h}^2+1)^{3/2}} F_2 \end{aligned}\right\}. \tag{II.53}$$

Taking into account expressions (II.53), let us rewrite formula (II.48) in the form

$$\Gamma = \Gamma_\infty \left[1 - \frac{1}{2}\left(1 - \frac{4\bar{h}}{\sqrt{2}\sqrt{8\bar{h}^2+1}} F_1\right)\right]. \tag{II.54}$$

The results obtained from formula (II.54) are close to those produced by the linear theory of a thin hydrofoil and experimentally; in the limit, when $\bar{h} \to 0$, they provide the circulation values for a gliding plate.

It follows from formula (II.50) that its second term, containing Γ_∞^2, gives a value of α of the second order of smallness. The main effect of a free surface on the lifting force is determined by the variation of circulation along the contour and, in the linear approximation, the lifting force will be calculated from Zhukovskiy's formula

$$P = \rho v_0 \Gamma. \tag{II.55}$$

The B_{nm} function cannot be expressed through hypergeometrical functions. However, they can be expressed effectively by an expansion with respect to τ-parameter. These expansions are more effective than the hypergeometric series and, therefore, it is practical to determine A_{nm} also in the same way [113].

The function $f(x) = J_n(x)$; $J_m(x)$ may be represented

by a convergent series [23]

$$J_n(x); J_m(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \pi(m+n+2s)}{\pi(m+s)\,\pi(n+s)\,\pi(s)\,\pi(m+n+s)} \left(\frac{x}{2}\right)^{m+n+2s}.$$

After performing the computations for the combination $A_{nm} - \frac{\omega}{2}B_{nm}$ we obtain

$$\left.\begin{aligned}
A_{00} - \frac{\omega}{2}B_{00} &= \sum_{n=0}^{\infty} \tau^{2n+1} \sum_{s=0}^{n} \frac{(-1)^s 2s!\,(s+n)!}{s!s!s!s!\,(n-s)!\,2^{2s+1}} \times \\
&\quad \times \left[1 + 2\,\mathrm{Re}\,F_{n+s}\left(\frac{\omega}{2\tau}\right)\right] \\
A_{10} - \frac{\omega}{2}B_{10} &= \\
= \sum_{n=0}^{\infty} \tau^{2n+1} &\sum_{s=0}^{\infty} \frac{(-1)^s (2s+1)!\,(s+n+1)!\left[1 + 2\,\mathrm{Re}\,F_{n+s+1}\left(\frac{\omega}{2\tau}\right)\right]}{s!s!\,(s+1)!\,(s+1)!\,(n-s)!\,2^{2s+2}}, \\
A_{11} - \frac{\omega}{2}B_{11} &= \\
= \sum_{n=0}^{\infty} \tau^{2n+2} &\sum_{s=0}^{\infty} \frac{(-1)^s (2s+2)!\,(s+n+2)!\left[1 + 2\,\mathrm{Re}\,F_{n+s+3}\left(\frac{\omega}{2\tau}\right)\right]}{s!\,(s+1)!\,(s+1)!\,(s+2)!\,(n-s)!\,2^{2s+3}}
\end{aligned}\right\} \quad \text{(II.56)}$$

[62

where $F_n(\lambda)$ is a function introduced in Chapter I.

With $\lambda \to 0$, $\mathrm{Re}F_n \to 0$, and with $\lambda \to \infty$, $F_n \to -\infty$. It follows then that the combination $A_{nm} - \frac{\omega}{2}B_{nm}$ in limiting cases changes signs:

$$\lim_{\omega \to 0}\left[A_{nm} - \frac{\omega}{2}B_{00}\right] = A_{nm};$$

$$\lim_{\omega \to \infty}\left[A_{nm} - \frac{\omega}{2}B_{00}\right] = -A_{nm}.$$

Let us write another expression for the function A_{nm} and combination $A_{01} - \frac{\omega}{2}B_{01}$, retaining several terms in each of the (II.56) expansions:

$$\left.\begin{aligned} A_{00} &= \frac{1}{2}\tau + \frac{1}{8}\tau^5 + \frac{9}{128}\tau^9 + \ldots \\ A_{01} &= \frac{1}{4}\tau^2 - \frac{1}{16}\tau^4 + \frac{1}{16}\tau^6 + \frac{1}{256}\tau^8 + \frac{9}{256}\tau^{10} - \frac{25}{1024}\tau^{12} + \ldots \\ A_{11} &= \frac{1}{4}\tau^3 + \frac{3}{32}\tau^7 + \frac{15}{256}\tau^{11} + \ldots \end{aligned}\right\} \qquad \text{(II.57)}$$

$$\begin{aligned} A_{01} - \frac{\omega}{2}B_{01} = \frac{1}{4}\tau^2\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right] + \tau^4\left\{\frac{1}{2}\left[1 + 2\,\mathrm{Re}\,F_2\left(\frac{\omega}{2\tau}\right) -\right.\right. \\ \left.\left. - \frac{9}{16}\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right]\right\} + \left\{\frac{3}{4}\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right] -\right. \\ \left. - \frac{9}{4}\left[1 + 2\,\mathrm{Re}\,F_4\left(\frac{\omega}{2\tau}\right)\right] + \frac{25}{16}\left[1 + 2\,\mathrm{Re}\,F_5\left(\frac{\omega}{2\tau}\right)\right]\right\}\tau^6. \end{aligned} \qquad \text{(II.58)}$$

If only the first term is retained in the expansion [63
A_{01}, then from formula (II.52) we obtain $\gamma = 1 - \frac{1}{2}\tau^2$, which agrees with the equation (I.54) in the linear theory of a thin hydrofoil.

To evaluate the obtained expansions quantitatively, calculations of the function F_1 were carried out by using 20 terms in the hypergeometric series and taking into account the results of calculation of the A_{nm} function, based on three terms in the expansion (II.57). Calculations with an accuracy to the fifth digit after the period produced very close results.

One may write two new formulas for the F_1 and F_2 functions which, in terms of convergence, have an advantage over the hypergeometric series [112]:

$$\begin{aligned} F\left(\frac{1}{4}; \frac{3}{4}; 1; z\right) = \frac{2\sqrt{2}}{z^{\frac{1}{4}}}\sum_{n=0}^{\infty}\left(\sqrt{\frac{z^{-\frac{1}{2}} - 1}{2} + 1} -\right. \\ \left. + \frac{1}{\sqrt{2}}\sqrt{z^{-\frac{1}{2}} - 1}\right)^{2n+1}\sum_{s=0}^{\infty}\frac{(-1)^s 2s!\,(s+h)!}{s!\,s!\,s!\,s!\,(n-s)!\,2^{2s+1}}, \end{aligned} \qquad \text{(II.59)}$$

$$F\left(\frac{3}{3};\frac{5}{4};2;z\right)=\frac{8\sqrt{2}}{z^{\frac{3}{4}}}\sum_{n=0}^{\infty}\left(\sqrt{\frac{z^{-\frac{1}{2}}-1}{2}+1}-\right.$$

$$\left.-\frac{1}{\sqrt{2}}\sqrt{z^{-\frac{1}{2}}-1}\right)^{2n+2}\sum_{s=0}^{\infty}\frac{(-1)^s(2s+2)!\,(s+n+2)!}{s!(s+1)!(s+1)!\,(s+2)!\,(n-s)!\,2^{2s+3}}.$$

N. Ye. Kochin determines the A_{nm} and B_{nm} functions in the form of a series in ascending powers of $\frac{1}{2h}$:

$$A_{00}=\sum_{m=0}^{\infty}\frac{(-1)^m(2m)!\left(\frac{1}{2h}\right)^{2m}}{2^{4m+1}}\sum_{k=0}^{m}\frac{1}{k!\,k!\,(m-k)!(m-k)!},$$

$$A_{10}=\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)!\left(\frac{1}{2h}\right)^{2m+2}}{2^{4m+3}}\sum_{k=0}^{m}\frac{1}{k!(k+1)!(m-k)!(m-k)!},$$

$$A_{11}=\sum_{m=0}^{\infty}\frac{(-1)^m(2m+2)!\left(\frac{1}{2h}\right)^{2m+2}}{2^{4m+5}}\times$$ [64

$$\times\sum_{k=0}^{m}\frac{1}{k!\,(k+1)\,!(m-k)\,!(m-k+1)!},$$

$$B_{00}=\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{4m}}\left(\frac{1}{2h}\right)^{2m}f_{2m}(2b)\sum_{k=0}^{m}\frac{1}{k!\,k!\,(m-k)!\,(m-k)!},$$

$$B_{10}=\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{4m+2}}\left(\frac{1}{2h}\right)^{2m+1}f_{2m+1}(2b)\sum_{k=0}^{m}\frac{1}{k!\,(k+1)!\,(m-k)!\,(m-k)!},$$

$$B_{11}=\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{4m+4}}\left(\frac{1}{2h}\right)^{2m+2}f_{2m+2}(2b)\times$$

$$\times\sum_{k=0}^{m}\frac{1}{k!\,(k+1)!\,(m-k)!\,(m-k+1)!},$$

$$b=\nu h, \tag{II.60}$$

where $f_n(\lambda)=e^{-\lambda}\int\limits_{-\infty}^{\lambda}e^u(\lambda-u)^n\frac{du}{u}.$

Function $f_m(x)$ is related to $\operatorname{Re}F_n(\lambda)$ as follows:

$$\operatorname{Re} F_n(\lambda) = -\frac{2\lambda}{n!} f_n(\lambda).$$

In deriving the functional equation for the $H(\lambda)$ function and determining circulation Γ, N. Ye. Kochin assumed the purely imaginary constant $Ci = \Gamma i$, did not isolate terms of the $1/u$ type from the integral terms in the expression (I.32), and obtained different results concerning the effect of a free surface on the circulation of a hydrofoil. In particular, the formula for Γ obtained by him may be written, in the linear approximation when $Fr_b \to \infty$, in the form

$$\gamma = \frac{1 - 2A_{01}}{1 + 2A_{01}}. \qquad \text{(II.61)}$$

This formula produces considerably lower results than those obtained experimentally and by means of formulas (II.51) and (II.52). [65

The wave drag of the hydrofoil can be determined easily with the aid of the $H(\lambda)$ function using formula (II.13). For a thin hydrofoil we determine the wave drag from the formula

$$Q = 16\pi^2 R^2 \varrho g \sin^2 \alpha e^{-2\nu h} \gamma^2 [J_0^2(2\nu R) + J_1^2(2\gamma R)]. \qquad \text{(II.62)}$$

Let us introduce coefficient c_{xB}:

$$c_{xB} = \frac{Q}{2\varrho V_0^2 R}.$$

Then

$$c_{xB} = \frac{8\pi^2 R g \sin^2 \alpha e^{-2\nu h} \gamma^2}{V_0^2} [J_0^2(2\nu R) + J_1^2(2\nu R)], \qquad \text{(II.63)}$$

or

$$c_{xB} = \frac{c_{y\infty}^2 \gamma^2}{2} \omega e^{-2\bar{h}\bar{\omega}} \left[J_0^2\left(\frac{\omega}{2}\right) + J_1^2\left(\frac{\omega}{2}\right)\right].$$

By determining function $J_0(x)$, for small values of $\frac{\omega}{2}$, using asymptotic formulas

$$J_0(x) = 1 - \frac{x^2}{4};\ J_1(x) = \frac{x}{2},$$

we obtain

$$c_{xB} = \frac{c_{y\infty}^2}{2} \gamma^2 \omega e^{-2\bar{h}\omega} \left[1 - \frac{\omega^2}{16} + \frac{\omega^4}{256}\right] \qquad \text{(II.64)}$$

and for $\omega \ll 1$

$$c_{xB} = \frac{c^2_{y\infty}}{2}\omega\gamma^2 e^{-2h\omega}.$$

2.4. The Approximate Method of Solving Problems of Motion of a Hydrofoil with an Arbitrary Profile

The results, which are close to those obtained in the first approximation by N. Ye. Kochin's method, may be obtained more easily by using a method based on the introduction of the first approximation assumptions directly into the formulation of the problem [113].

Let us first note one point related to the N. Ye. Kochin function. If the function $H(\lambda)$ is determined from formula (II.10)

$$H(\lambda) = \int_C e^{-i\lambda z} v(z)\, dz,$$

then the value of the function remains the same for the complex velocity of the absolute and the relative motions, because $\int_C e^{-i\lambda z}\, dz = 0$. For practical purposes it is convenient [66
to determine the $H(\lambda)$ function using another formula.

By introducing a function $z = f(u)$, which maps conformally the plane of a cylinder (u) on the plane of a hydrofoil (z), we obtain

$$\frac{dw(z)}{dz} = \frac{dw[f(u)]}{du}\cdot\frac{du}{dz}$$

or

$$v(z)\, dz = v(u)\, du.$$

Then, for the hydrofoil function $H(\lambda)$ we obtain

$$H(\lambda) = \int_C e^{-i\lambda f(u)} v(u)\, du. \qquad \text{(II.65)}$$

This expression makes it possible to simplify the problem of determining the function, since the complex velocity of the cylinder's motion is represented by the Laurent series and finding the expansion of $e^{-i\lambda(u)}$ into the series does not present any difficulties. The $H(\lambda)$ function, determined from formula (II.65), is different for the complex velocity of the absolute and relative motions of the cylinder. This follows from the condition that

$$\int_C e^{-i\lambda f(u)}\,du \neq 0,$$

since the transforming function f(u) for the hydrofoil of an arbitrary profile may be represented by the expansion

$$f(u) = u + \frac{a_1}{u} + \frac{a_2}{u^2} + \ldots$$

Substitution into expression (II.65) of the complex velocity of the relative motion of a cylinder in the first approximation resulted in formula (II.44). For the complex velocity of the absolute motion the result is different:

$$H(\lambda) = e^{-i\lambda h}[J_0(2\lambda R) + 2\pi R v e^{-i\alpha} J_1(2\lambda R)]. \qquad \text{(II.66)}$$

Let us assume that the unknown density of the layer $v(\zeta)$ in the integral expressions $v_1(z)$ and $v_2(z)$ is equal to the density of the layer during motion in an infinite flow. Then

$$v_1(z) = \frac{1}{2\pi i}\int_{C_1} \frac{v_\infty(\zeta)}{z-\zeta}\,d\zeta = \bar{v}_\infty(z);$$

$$v(z) = v_\infty(z) + v_2(z); \qquad \text{(II.67)}$$

$$v_2(z) = \frac{1}{2\pi i}\int_{C_1} \bar{v}_\infty(\zeta)\left[J\int_0^\infty e^{-i\lambda(z-\bar{\zeta})}\frac{\lambda+\nu}{\lambda-\nu}\,d\lambda - 2\pi\nu e^{-i\nu(z-\bar{\zeta})}\right]d\zeta \qquad \text{(II.68)}$$

By performing computations using the formula by S. A. Chaplygin we obtain [67

$$P - iQ = \varrho v_0 \Gamma_\infty - \varrho\int_C v_\infty(z)\,v_2(z)\,dz. \qquad \text{(II.69)}$$

We then obtain formulas for the lifting force and wave drag. Let us determine the $H(\lambda)$ function, which is part of the formula for the lifting force, in terms of the complex velocity of the absolute motion:

$$P = \varrho v_0 \Gamma_\infty - \frac{\varrho}{2\pi}\int_0^\infty |H_1(\lambda)|^2 \frac{\lambda+\nu}{\lambda-\nu}\,d\lambda + g\varrho s,$$

$$Q = \nu\varrho\,|H(\nu)|^2. \qquad \text{(II.70)}$$

Let us examine the motion of a thin hydrofoil.

In this case function $H_1(\lambda)$ will be determined by

formula (II.66) and the formula for P will be as follows:

$$P = \varrho v_0 \Gamma_\infty - \frac{\varrho}{2\pi}\int_0^\infty e^{-2\lambda h}[\Gamma^2 J_0^2(2\lambda R) + 4\pi R J_0(2\lambda R) J_1(2\lambda R) + $$
$$+ 4\pi^2 R^2 J_1^2(2\lambda R)]\frac{\lambda+\nu}{\lambda-\nu}d\lambda + g\varrho s. \qquad (II.71)$$

The third term in brackets is determined by the mutual effect of dipoles. The moment of the cylinder's dipole with the axis directed along the Ox axis may be written in the form

$$M = m\Delta\varepsilon,$$

where $\Delta\varepsilon$ is an infinitely small distance between the source and the run-off which, in combination, produce a dipole. In the conformal representation the value of $\Delta\xi$ will change and will be equal to $\Delta\gamma$, while the power of the source and of the run-off will not change. Then the moment of the hydrofoil dipole will be

$$M_k = m\frac{\Delta\gamma}{\Delta\xi}.$$

Let us determine the ratio $\frac{\Delta x}{\Delta\xi}$ in terms of finite values:

$$\frac{\Delta x}{\Delta\xi} = \frac{\delta}{R} = \frac{4\delta_1}{B} = \bar{\delta}_1,$$

where δ_1 is the thickness of the profile above the chord.

Consequently, the third term for a thin hydrofoil is assumed to be equal to zero, while for a hydrofoil with a solid profile we will multiply it by δ. The above reasoning is not rigid. However, the formulas obtained in this way provide results close to those obtained experimentally. Particularly good congruence was observed for large Froude numbers [102]. [68

Let us determine the final results for certain types of hydrofoils.

Thin plate:

$$\frac{P_n}{P_\infty} = \gamma = 1 - \frac{\sin\alpha_k}{\sqrt{2}}\frac{1}{\sqrt{8\bar{h}^2+1}}F_1 - \frac{\cos\alpha_k}{2}\left[1 - \frac{4\bar{h}}{\sqrt{2}}\frac{1}{\sqrt{8\bar{h}^2+1}}F_1\right]. \qquad (II.72)$$

This formula up to the terms of the first order agrees with formula (II.54).

Thin hydrofoil section:

$$\gamma = 1 - \frac{\sin(\alpha_0 + \alpha_\kappa)}{\sqrt{2}\sqrt{8\bar{h}^2 + 1}\cos\alpha_0} F_1 - \frac{\cos\alpha_\kappa}{2\cos 2\alpha_0}\left[1 - \frac{4\bar{h}}{\sqrt{2}}\frac{1}{\sqrt{8\bar{h}^2 + 1}} F_1\right]. \quad \text{(II.73)}$$

The N. Ye. Zhukovskiy hydrofoil:

$$\gamma = 1 - \frac{\sin(\alpha_0 + \alpha_\kappa)}{\sqrt{2}\sqrt{8\bar{h}^2 + 1}\cos\alpha_0} F_1 - \frac{\cos\alpha_\kappa(1+\mu)^2}{2\cos 2\alpha_0}\Bigg[1 -$$

$$- \frac{4\bar{h}}{\sqrt{2}}\frac{1}{\sqrt{8\bar{h}^2 + 1}} F_1\Bigg] - \frac{K\delta}{4\sqrt{2}(8\bar{h}^2 + 1)^{3/2}(\alpha_0 + \alpha_\kappa)} F_2, \quad \text{(II.74)}$$

where $\mu = \frac{0{,}77\delta}{1 - 0{,}6\delta}$;

K - the ratio of the curvature of the suction wall to the total thickness of the profile;
δ - relative thickness of the profile.

Hydrofoil consisting of two intersecting circular arcs:

$$\gamma = 1 - \frac{\sin(\alpha_0 + \alpha_\kappa)}{\sqrt{2}\sqrt{8\bar{h}^2 + 1}\cos\alpha_0\sqrt{\frac{\varkappa^2 - 1}{3}}} F_1 - \frac{\cos\alpha_\kappa}{2\cos 2\alpha_0\left(\frac{\varkappa^2 - 1}{3}\right)} \times$$

$$\times \left(1 - \frac{4\bar{h}F_1}{\sqrt{2}\sqrt{8\bar{h}^2 + 1}}\right) - \frac{K\delta}{4\sqrt{2}(8\bar{h}^2 + 1)^{3/2}(\alpha_0 + \alpha_\kappa)} F_2, \quad \text{(II.75)}$$

where $\varkappa = 2 - \frac{\beta}{\pi}$;

β - angle between the tangentials to the circular arcs at the extreme points of the hydrofoil.

If we consider the effect of the free surface as both the change in the angular coefficient in the expression P = f(a) and as the variation in the zero lift angle, as was done in Chapter I, then for the Zhukovskiy hydrofoil [69
functions $\Delta\alpha_0$ and ψ will be determined by formulas

$$\psi = 1 - \frac{\sin(\alpha_0 + \alpha_\kappa)}{\sqrt{2}\sqrt{8\bar{h}^2 + 1}\cos\alpha_0} F_1 c - \frac{(1+\mu)^2\cos\alpha_\kappa}{2\cos 2\alpha_0}\Bigg[1 -$$

$$- \frac{4\bar{h}}{\sqrt{2}\sqrt{8\bar{h}^2 + 1}} F_1\Bigg],$$

$$\Delta\alpha_h = \frac{k\delta F_2}{4\sqrt{2}(8\bar{h}^2+1)^{3/2}\psi}. \qquad \text{(II.76)}$$

These formulas can be expanded in $\tau = \sqrt{4\bar{h}^2+1} - 2\bar{h}$:

$$\psi = 1 - \sin(\alpha_0 + \alpha_\kappa)\tau + \frac{(1+\mu)^2 \cos\alpha_\kappa}{2\cos 2\alpha_0}\tau^2;$$

$$\Delta\alpha_n = \frac{1}{2}\frac{K\delta\tau^3}{\psi}. \qquad \text{(II.77)}$$

For the arbitrary Froude numbers the formula for $\Delta\alpha_n$ was obtained in the following form:

$$\Delta\alpha_n = \frac{1}{2}\frac{K\delta}{\psi}\left[\tau^3 - \frac{1}{2}\omega^2 e^{-\frac{1}{2}\frac{\omega}{\tau}} E_{i1}\left(\frac{1}{2}\frac{\omega}{\tau}\right)\right]. \qquad \text{(II.78)}$$

The computed results with the use of formulas (II.76) and (II.77) agree well with the experimental data*.

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*Some of the material in this section was published in other studies by the author [102, 104-108, 113, 114] in which the practical problems were also examined.

2.5. Solution of a Two-Dimensional Problem with the Aid of Integral Equations

N. Ye. Kochin also suggested a method for solving a two-dimensional problem with the aid of integral equations. Below are given some results of his studies.

Let us assume that a contour C was formed bya simple contour which envelops some region s and which has a continuous curvature. Let us place a vortex of intensity Γ at some point ζ_0 in region s. Let us distribute the sources with a density $\gamma(\sigma)$ along contour C, where σ is the length of contour C between a fixed point and a variable point on the contour. Accordingly, let us attempt to represent function W(z) in the following way

$$W(z) = \frac{\Gamma}{2\pi i}\ln(z-\zeta_0) + \frac{1}{2\pi}\int_C \gamma(\sigma)\ln(z-\zeta)\,d\sigma + F(z). \qquad \text{(II.79)}$$

where F(z) is the regular function of z in the lower half-plane. Function F(z) is selected from conditions (II.1) and (II.2). [70

It is easy to show that functions

$$\frac{\Gamma}{2\pi i}\left[\ln(z-\zeta_0)-\ln(z-\bar{\zeta}_0)+2e^{-ivz}\int\limits_{+\infty}^{z}\frac{e^{ivz}}{z-\bar{\zeta}_0}\,dz\right],$$

$$\frac{\gamma(\sigma)\,d\sigma}{2\pi}\left[\ln(z-\zeta)+\ln(z-\bar{\zeta})-2e^{-ivz}\int\limits_{+\infty}^{z}\frac{e^{ivz}}{z-\bar{\zeta}}\,dz\right]$$

satisfy both the conditions on the free surface as well as the conditions at infinity.

Let us seek W(z) in the form

$$W(z)=\frac{\Gamma}{2\pi i}\left[\ln(z-\zeta_0)-\ln(z-\bar{\zeta}_0)+2e^{-ivz}\int\limits_{\infty}^{z}\frac{e^{ivz}}{z-\zeta_0}\,dz\right]+$$

$$+\frac{1}{2\pi}\int\limits_{C}\gamma(\sigma)\left[\ln(z-\zeta)+\ln(z-\bar{\zeta})-2e^{-ivz}\int\limits_{+\infty}^{z}\frac{\cdot e^{ivz}}{z-\bar{\zeta}}\,dz\right]. \qquad \text{(II.80)}$$

Let us assign $\zeta=\xi+i\eta$, $\zeta_0=\xi_0+i\eta_0$ and isolate the real part in (s, u):

$$\varphi(x,y)=\frac{\Gamma}{2\pi}\left(\theta-\theta'+2\int\limits_{\infty}^{x}\frac{(t-\zeta_0)\sin v(t-x)-(y-\eta_0)\cos v(t-x)}{(t-\zeta)^2+(y+\eta_0)^2}dt\right)+$$

$$+\frac{1}{2\pi}\int\limits_{C}\gamma(\sigma)\left(\ln z+\ln r'-\right.$$

$$\left.-2\int\limits_{+\infty}^{x}\frac{(t-\zeta)\cos v(t-x)+(y+\eta)\sin v(t-x)}{(t-\zeta)^2+(y+\eta)^2}\,dt\right), \qquad \text{(II.81)}$$

where

$$\theta=\operatorname{arc\,tg}\frac{y-\eta_0}{x-\zeta_0};\ \theta'=\operatorname{arc\,tg}\frac{y-\eta_0}{x-\zeta_0};$$

$$r=\sqrt{(x-\xi)^2+(y-\eta)^2};\quad r'=\sqrt{(x-\xi)^2+(y+\eta)^2}.$$

The boundary values of φ_n from the outside at a point on the contour are determined by formula

$$\varphi_n=\frac{1}{2}\gamma(s)+\frac{1}{2}\int\limits_{C}\gamma(\sigma)K(\sigma,s)\,d\sigma+\frac{\Gamma}{2}h(s), \qquad \text{(II.82)}$$

where [71

$$K(\sigma,s)=\frac{1}{\pi r}\cos(n,r)-\frac{1}{\pi r'}\cos(n,r')-$$

$$- 2\nu \int_{+\infty}^{x} \frac{\sin[\nu(t-x) - (\widehat{r_t n})]}{\pi r_t} dt;$$

$$r_t = \sqrt{(t-\zeta)^2 + (y-\eta)^2}. \qquad \text{(II.83)}$$

Vectors r and r' are directed from points $\xi + i\eta$ and $\xi - i\eta$ to point x + iy, while vector r_t is directed from point $\zeta - i\eta$ to point t - iy, with $(\widehat{r_t; n})$ measured in the positive direction from the direction of the perpendicular to vector r_t:

$$h(s) = -\frac{1}{\pi r_0} \sin(r_0 n) - \frac{1}{\pi r_0'} \sin(r_0' n) -$$

$$- 2\nu \int_{-\infty}^{x} \frac{\cos[\nu(t-x) - (\widehat{r_t, n})]}{\pi r_t} dt, \qquad \text{(II.84)}$$

and vectors r_0 and r_0' are directed from points $\zeta_0 + i\eta_0$ and $\xi_0 + i\eta_0$ to point x + iy, while vector τ_{t_0} is a vector going from point $\zeta_0 - i\eta_0$ to point t - iy.

Thus, for determining function $\gamma(\sigma)$ we obtain the integral equation

$$\gamma(s) = -\int_C K(s, \sigma) \gamma(\sigma) ds + g(s). \qquad \text{(II.85)}$$

It is easy to show that $\int_C K(s, \sigma) ds = 1$, $\int_C g(s) ds = 0$ and that under such conditions $\int_C \gamma(s) ds = 0$.

With both large and small values of the parameter ν the solution of equation (II.85) may be obtained by the method of iteration. Let us examine the more general integral equation

$$\gamma(s) = \lambda \int_C \gamma(\sigma) K(\sigma s) d\sigma + g(s). \qquad \text{(II.86)}$$

The nucleus of the equation depends on parameter ν, and therefore the characteristic values λ of this equation will also depend on this parameter. With the large values of the parameter ν the characteristic values will differ little from those in the equation with a nucleus $K_\infty(\sigma, s)$. [72

With $\nu \to \infty$ the nucleus is determined by the formula

$$K_\infty(\sigma, s) = \frac{\cos(n, r)}{\pi r} + \frac{\cos(n, r')}{\pi r'}. \qquad \text{(II.87)}$$

With $\nu \to \infty$ the integral equation may be written in the form

$$\gamma(s) = \lambda \int\limits_{C+C'} \gamma(\sigma) \frac{\cos(n, r)}{\pi r} d\sigma + g(s), \qquad \text{(II.88)}$$

where C' is a contour which is symmetrical with C with respect to the OX axis.

The characteristic values in equation (II.88) are all real, simple, and with respect to the modulus are not less than unity. In this case $\lambda = -1$ is not a characteristic value of the equation, while $\lambda = 1$ is a characteristic quantity in this equation. For $\lambda = 1$, $\gamma(\sigma)$ is the only independent solution for the conjugate homogeneous integral equation. Under the condition of $\int\limits_C g(s)\,ds = 0$, the solution $\gamma(s)$ is a meromorphic function of λ, with $\lambda = 1$ not a pole of this function.

For the nucleus $K(\sigma, s)$ it can be asserted that with sufficiently large values of parameter ν a circle for which $/\lambda/ < R$ and $R > 1$ will contain only one characteristic value. For the conjugate homogeneous integral equation

$$\gamma(s) = \lambda \int\limits_C \gamma(\sigma) K(s, \sigma)\,d\sigma. \qquad \text{(II.89)}$$

$\lambda = 1$ is a characteristic value with its respective solution being $\gamma(\sigma) = 1$.

A necessary and sufficient condition for solving equation (II.86) when $\lambda = 1$ is

$$\int\limits_C g(s)\,ds = 0, \qquad \text{(II.90)}$$

which in this case is satisfied. Therefore, the solution $\gamma(s)$ of the equation (II.86) is a meromorphic function of λ, for which $\lambda = 1$ is not a pole and the solution of the equation may be obtained by the method of iteration:

$$\gamma(s) = g(s) + \sum^{\infty} \lambda^n \gamma_n(s),$$

$$\gamma_n(s) = \int\limits_C \gamma_{n-1}(\sigma) K(\sigma, s)\, d\sigma. \qquad \text{(II.91)}$$

Since the radius of convergence of this series with sufficiently large values of the parameter γ is larger than 1, we may assume the value of $\lambda = -1$. Function $H(\lambda)$ is expressed simply through the intensity of sources. We have [73

$$H(\lambda) = \int\limits_{C_1} e^{-i\lambda z} \left(\frac{\Gamma}{2\pi i} \frac{1}{z - \zeta_0} + \frac{1}{2\pi} \int\limits_C \frac{\gamma(\sigma)\, d\sigma}{z - \zeta} \right) dz,$$

and simple calculations give us

$$H(\lambda) = \Gamma e^{-i\lambda} + i \int\limits_C \gamma(\sigma) e^{-i\lambda} d\sigma. \qquad \text{(II.92)}$$

For small values of the ν parameter the nucleus of equation (II.86) is in the following form:

$$K_0(\sigma, s) = \frac{\cos(n, r)}{\pi r} - \frac{\cos(n, r')}{\pi r'}, \qquad \text{(II.93)}$$

and the equation corresponding to this nucleus may be written in the form

$$\gamma(s) = \lambda \int\limits_{C+C'} \gamma(\sigma) \frac{\cos(n, r)}{\pi r} d\sigma + g(s), \qquad \text{(II.94)}$$

if we assume that the values of function $\gamma(s)$ and $g(s)$ at the points on the C' contour are different in sign from the values of these functions at points on the contour C symmetrical with respect to the OX axis. Repeating these arguments we again come to the conclusion that series (II.91) also provides a solution to the problem when the values of ν are small.

CHAPTER III. THEORY OF THE HYDROFOIL IN A PLANE-PARALLEL FLOW OF FLUID OF FINITE DEPTH [74

In problems investigating motion of a solid body in a fluid of finite depth the potential of velocities should satisfy an additional boundary condition which equates to zero the expression for the velocities perpendicular to the surface of the channel. If the fluid has a constant depth h_0, this condition will be in the form

$$\varphi_y = 0 \quad \text{at} \quad y = -h_0. \tag{III.1}$$

Let us examine a two-dimensional problem dealing with the steady forward motion of an arbitrary-shape hydrofoil submerged under a free surface in a channel of finite depth. This problem can be formulated mathematically in the following way: in an area limited by straight lines $y = 0$ and $y = -h_0$ and a contour C_1 (C_1 is the contour enveloping hydrofoil profile C) it is necessary to determine the analytical function $W(z) = \varphi + i\psi$ which satisfies the following conditions:

1) with $0 > y > -h_0$ in the region occupied by the fluid the derivative $W_z(z)$ is limited:

$$\lim_{x \to +\infty} W_z(z) = 0; \tag{III.2}$$

2) on the free surface

$$\operatorname{Im}(iW_{zz}(z) - \nu W_z(z)) = 0; \tag{III.3}$$

3) on the hydrofoil profile

$$\varphi_n = v_0 \cos(n, x); \tag{III.4}$$

4) on the bottom of the channel

$$\operatorname{Im} W_z(z) = 0 \quad (y = -h_0). \tag{III.5}$$

3.1. Motion of the Vortex and the Source Under a Free Surface of a Fluid of Finite Depth [75

Solution of the problem dealing with the motion of the vortex in a fluid of finite depth was given by A. I. Tikhonov [147], and of that of the source by M. D. Khaskind [156]. Let us examine a problem of the vortex motion in a fluid of finite depth more thoroughly. The complex potential of velocities caused by the motion of a vortex

located at point y = -h will be sought in the form

$$W(z) = \frac{\Gamma}{2\pi i}\ln(z+ih) - \frac{\Gamma}{2\pi i}\ln[z+i(2h_0-h)] + F(z), \qquad \text{(III.6)}$$

where F(z) is an analytical function within the whole band.

According to condition (III.3), when y = 0

$$\operatorname{Re}(F_{zz} + i\nu F_z) = -\operatorname{Re}\Big[-\frac{1}{(z+ih)^2}\frac{\Gamma}{2\pi i} + \frac{\Gamma}{2\pi i}\frac{1}{[z+i(2h_0-h)]^2} + $$
$$+ \frac{\nu\Gamma}{2\pi(z+ih)} - \frac{\nu\Gamma}{2\pi}\frac{1}{[z+i(2h_0-h)]}\Big]. \qquad \text{(III.7)}$$

With y + h > 0 the following relations are valid:

$$\frac{1}{z+ih} = -i\int_0^\infty e^{i\lambda[z+ih]}d\lambda;$$

$$\frac{1}{z+i(2h_0-h)} = -i\int_0^\infty e^{i\lambda[z+i(2h_0-h)]}d\lambda;$$

$$\frac{1}{(z+ih)^2} = -\int_0^\infty \lambda e^{i\lambda(z+ih)}d\lambda;$$

$$\frac{1}{[z+i(2h_0-h)]^2} = -\int_0^\infty \lambda e^{i\lambda[z+i(2h_0-h]}d\lambda.$$

Then condition (III.7) may be written in the form

$$\operatorname{Re}(F_{zz} - i\nu F_z) = -\frac{\Gamma}{\pi}\int_0^\infty (\lambda+\nu)\,e^{-\lambda h_0}\operatorname{sh}\lambda(h_0-h)\sin\lambda x\,d\lambda. \qquad \text{(III.8)}$$

Conditions (III.5) and (III.8) define the problem of finding the analytical function F(z) according to conditions for its real part [76

$$F_{1x} + \nu F_{1y} = -\frac{\Gamma}{\pi}\int_0^\infty (\lambda+\nu)\,e^{-\lambda h_0}\operatorname{sh}\lambda(h_0-h)\sin\lambda x\,d\lambda \qquad \text{(III.9)}$$

$$F_{1y} = 0 \quad (y = -h_0). \quad (y = 0). \qquad \text{(III.10)}$$

Let us solve this problem by the Fourier method. Let us look for a solution in the form

$$F_1(x, y) = \int_0^\infty \operatorname{ch} \lambda (y + h_0) [C_1(\lambda) \cos \lambda x + C_2(\lambda) \sin \lambda x] \, d\lambda,$$

which satisfies condition (III.10).

According to condition (III.10)

$$C_1(\lambda) = 0,$$

$$C_2(\lambda) = -\frac{\Gamma}{\pi} (\nu + \lambda) e^{-\lambda h_0} \frac{\operatorname{sh} \lambda (h_0 - h)}{\lambda(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)}.$$

Then, function F(z) will be determined by the formula

$$F_1(x, y) = -\frac{\Gamma}{\pi} \int_0^\infty \frac{\nu + \lambda}{\lambda} e^{-\lambda h_0} \frac{\operatorname{sh} \lambda (h_0 - h) \sin \lambda x \operatorname{ch} \lambda (y + h_0)}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0}. \quad \text{(III.11)}$$

Let us write function F(z) in the form

$$F(z) = -\frac{\Gamma}{\pi} \int_0^\infty \frac{\nu + \lambda}{\lambda} e^{-\lambda h_0} \frac{\operatorname{sh} \lambda (h_0 - h) \sin \lambda (z + i h_0)}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} \, d\lambda. \quad \text{(III.12)}$$

The integrand function of the expression (III.12) is a meromorphic function of λ. The poles of the function are determined as the real and positive roots of the equation

$$\nu \operatorname{sh} \lambda h_0 = \lambda \operatorname{ch} \lambda h_0. \quad \text{(III.13)}$$

With $v_0^2 > g h_0$ this equation has only imaginary roots.

By adding to the solution the terms which determine the system of free waves on a free surface we can write

$$W(z) = \frac{\Gamma}{2\pi i} \ln \frac{z + ih}{z + i(2h_0 - h)} -$$

$$-\frac{\Gamma}{\pi} \int_0^\infty \frac{\nu + \lambda}{\lambda} e^{-\lambda h_0} \frac{\operatorname{sh} \lambda (h_0 - h) \sin \lambda (z + i h_0)}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} \, d\lambda +$$

$$+ A_1 \cos \lambda_0 (z + i h_0) + A_2 \sin \lambda_0 (z + h_0), \quad \text{(III.14)}$$

$$v(z) = \frac{\Gamma}{2\pi i} \frac{1}{z + ih} - \frac{\Gamma}{2\pi i} \frac{1}{z + i(2h_0 - h)} -$$

$$-\frac{\Gamma}{\pi}\int\limits_0^\infty (\nu+\lambda)\,\frac{e^{-\lambda h_0}\operatorname{sh}\lambda(h_0-h)\cos\lambda(z+ih_0)}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}\,d\lambda -$$

$$-A_1\lambda_0\sin_0(z+ih_0)+A_2\lambda_0\cos\lambda_0(z+ih_0), \qquad \text{(III.15)}$$

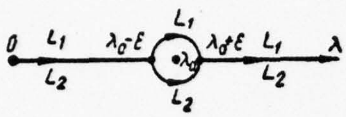

Fig. 7

where λ_0 is a root of the transcendental equation (III.13). Constants A_1 and A_2 are determined from condition (III.2).

Let us examine the sum of two line integrals

$$H(z)=\frac{1}{2}\int\limits_{L_1}(\nu+\lambda)\,e^{-\lambda h_0}\frac{\operatorname{sh}\lambda(h_0-h)\,e^{i\lambda(z+ih)}}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}\,d\lambda +$$

$$+\frac{1}{2\pi}\int\limits_{L_1}(\nu+\lambda)\,\frac{e^{-\lambda h_0}\operatorname{sh}\lambda(h_0-h)e^{-i\lambda(z+ih)}}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}\,d\lambda, \qquad \text{(III.16)}$$

where contour L_1 passes over the special point λ_0 and contour L_2 passes under that point (Fig. 7).

By determining the values of integrals along the half-circle of small radius ξ we obtain

$$H(z)=\int\limits_0^\infty(\nu+\lambda)\,e^{-\lambda h_0}\frac{\operatorname{sh}\lambda(h_0-h)\cos\lambda(z+ih_0)}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}\,d\lambda +$$

$$+\pi\nu\,\frac{\operatorname{sh}\lambda_0(h_0-h)\sin\lambda_0(z+ih_0)}{[\nu h_0-\operatorname{ch}^2\lambda_0 h_0]}. \qquad \text{(III.17)}$$

Let us integrate one of the integrals (III.16) by parts:

$$\int\limits_{L_1}(\nu+\lambda)\,\frac{e^{-\lambda h_0}\operatorname{sh}\lambda(h_0-h)\,e^{i\lambda(z+ih)}}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}\,d\lambda =$$

$$=\int\limits_{L_1}Q(\lambda)\,e^{i\lambda x}d\lambda = -\frac{Q'(\lambda)}{ix}\int\limits_0^\infty + \frac{i}{x}\int\limits_{L_1}e^{i\lambda x}\frac{d}{dx}Q(\lambda)d\lambda,$$

however, when $Q(\lambda) = 0$

$$\lim_{x\to\infty}\int\limits_{L}^{\lambda\to\infty}Q(\lambda)\,e^{i\lambda x}d\lambda = 0.$$

A similar result will also be obtained for the second integral: [78

$$\lim_{x\to+\infty} H(z) = 0. \quad \text{(III.18)}$$

By combining expressions (III.15), (III.17) and (III.18) we obtain

$$A_1 = \frac{\Gamma\nu}{\lambda_0}\frac{\operatorname{sh}\lambda_0(h_0-h)}{\nu h_0-\operatorname{ch}^2\lambda_0 h_0}, \quad A_2 = 0,$$

and the formula for the complex velocity of the vortex will be in the form

$$v(z) - \frac{\Gamma}{2\pi i}\frac{1}{z+ih} - \frac{\Gamma}{2\pi i}\frac{1}{z+i(2h_0-h)} -$$

$$-\frac{\Gamma\nu \operatorname{sh}\lambda_0(h_0-h)\sin\lambda_0(z+ih_0)}{\nu h_0-\operatorname{ch}^2\lambda_0 h_0} -$$

$$-\frac{\Gamma}{\pi}\int_0^\infty \frac{(\nu+\lambda)e^{-\lambda h_0}\operatorname{sh}\lambda(h_0-h)}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}\cos\lambda(z+ih_0)\,d\lambda. \quad \text{(III.19)}$$

When $v_0^2 > gh_0$, there are no free waves which satisfy conditions in the problem and there is no third term in expression (III.19). The forces of the flow acting on the vortex will be determined by the following formula:

$$P + iQ = \varrho v_{\text{отн}}\Gamma, \quad \text{(III.20)}$$

where

$$v_{\text{отн}} = v_0 - \left(\frac{dW}{dz} - \frac{\Gamma}{2\pi i}\frac{1}{z+ih}\right)_{z=-ih}.$$

We have

$$P = \varrho v\Gamma - \frac{\varrho\Gamma^2}{4\pi}\frac{1}{h_0-h} + \frac{\varrho\Gamma^2}{2\pi}\int_0^\infty(\nu+\lambda)e^{-\lambda h_0}\frac{\operatorname{sh}2\lambda(h_0-h)}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}d\lambda, \quad \text{(III.21)}$$

$$Q = \varrho\Gamma^2\nu\frac{\operatorname{sh}^2\lambda_0(h_0-h)}{\nu h_0-\operatorname{ch}^2\lambda_0 h}. \quad \text{(III.22)}$$

When $v_0^2 > gh_0$ the wave drag is equal to zero.

It was established [147] that the integral in expression (III.21) has a discontinuity at point $v_0^2 = gh_0$.

The asymptotic expression of the complex velocity

W(z) with x → -∞ is in the form

$$\underset{x \to -\infty}{W(z)} = -2\Gamma\nu \frac{\operatorname{sh} \lambda_0 (h_0 - h)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \sin \lambda_0 (z + ih_0). \tag{III.23}$$

For the source located at point $\zeta = \xi + i\eta$, the expression for the complex velocity may be obtained in the same way as for the case of the vortex. Below is the formula for the complex velocity of the source v(z) obtained by M. D. Khaskind: [79

$$v(z) = \frac{Q}{2\pi(z + \zeta)} + \frac{Q}{2\pi(z + \zeta + 2ih_0)} +$$
$$+ \frac{Q}{\pi} \int_0^\infty (\nu + \lambda) e^{-\lambda h_0} \frac{\operatorname{ch} \lambda (\eta + h_0) \sin \lambda (z - \zeta + ih_0)}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} d\lambda -$$
$$- Q\nu \frac{\operatorname{ch} \lambda_0 (\eta + h_0)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \cos \lambda (z - \zeta + ih_0). \tag{III.24}$$

The asymptotic expression of the complex velocity for the source is in the form

$$\underset{v \to -\infty}{v(z)} = -2Q\nu \frac{\operatorname{ch} \lambda_0 (\eta + h_0)}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0} \cos \lambda_0 (z - \zeta + ih_0). \tag{III.25}$$

3.2. Motion of a Thin Hydrofoil in a Fluid of Finite Depth

For a thin hydrofoil, instead of condition (III.4), we have a linear condition (I.4).

$$\varphi_y = -v_0 f'(x).$$

The principal general formulas discussed in Chapter I remain the same also for the hydrofoil in a fluid of finite depth. The complex potential W(z) for the hydrofoil in a fluid of finite depth may also be sought in the form (I.9).

$$W(z) = \frac{1}{2\pi i} \int_C \gamma(s) [\ln (z - s) + K(s, z)] ds, \tag{III.26}$$

where K(z, s) is an analytical function in the band 0 > > y > $-h_0$.

The integral equation of the problem is also determined by formulas (I.8):

$$\frac{1}{2\pi} \int_C \gamma(s) \left[\frac{1}{x - s} + G(s, x)\right] ds = F(x); \tag{II.27}$$

$$F(x) = -v_0 f'(x), \quad G(s, x) = \operatorname{Re} K_z(s, x).$$

Functions K(s, z) and G(s, z) will be determined by the complex potential and complex velocity of the vortex:

$$K(s, z) = -\ln[z - s + 2i(h_0 - h)] - \frac{1}{\pi}\int_0^\infty \frac{\nu + \lambda}{\lambda} e^{-\lambda h_0} \times$$

$$\times \frac{\operatorname{sh}\lambda(h_0 - h)\sin\lambda(z + i(h_0 - h) - s)\,d\lambda}{\nu\operatorname{sh}\lambda h_0 - \lambda\operatorname{ch}\lambda h_0} +$$ [80

$$+ \frac{\nu}{\lambda_0}\frac{\operatorname{sh}\lambda(h_0 - h)}{\nu h_0 - \operatorname{ch}^2\lambda_0 h_0}\cos\lambda_0[z - s + i(h_0 - h)], \qquad \text{(III.28)}$$

$$G(s, x) = \operatorname{Re}\left[-\frac{1}{x - s + 2i(h_0 - h)} -\right.$$

$$-2i\int_0^\infty \frac{(\nu+\lambda)e^{-\lambda h_0}\operatorname{sh}\lambda(h_0 - h)\cos\lambda[x - s + i(h_0 - h)]}{\nu\operatorname{sh}\lambda h_0 - \lambda\operatorname{ch}\lambda h_0}\,d\lambda -$$

$$\left.-2\pi i\nu\frac{\operatorname{sh}\lambda_0(h_0 - h)\sin\lambda_0[x + i(h_0 - h)]}{\nu h_0 - \operatorname{ch}^2\lambda_0 h_0}\right]. \qquad \text{(III.29)}$$

In a dimensionless form, function G(s, x) is determined by the expression

$$\overline{G}(s, x) = \operatorname{Re}\left[\frac{-\dfrac{2\beta}{\bar{h}}}{\dfrac{2\beta}{\bar{h}}(\bar{x} - \bar{s}) + 2i(1 - \beta)} -\right.$$

$$-\frac{4i\beta}{\bar{h}}\int_0^\infty \frac{(1 + \omega_n\mu)e^{-\mu}\operatorname{sh}\mu(1 - \beta)\cos\mu\left[\dfrac{2\beta}{\bar{h}}(\bar{x} - \bar{s}) + i(1 - \beta)\right]}{\operatorname{sh}\mu - \omega_n\mu\operatorname{ch}\mu}\,d\mu +$$

$$+ 4\pi i\frac{\beta}{\bar{h}}\frac{\operatorname{sh}\mu_0(1 - \beta)\sin\mu_0\left[\dfrac{2\beta}{\bar{h}}(\bar{x} - \bar{s}) + i(1 - \beta)\right]}{\omega_n\operatorname{ch}^2\mu_0 - 1}, \qquad \text{(III.30)}$$

where $\beta = \frac{h}{h_0}$;

$$\omega_n = \frac{v_0^2}{gh_0} = \frac{1}{\nu h_0}.$$

A. I. Tikhonov [146] solves the integral equation of the problem by the method of M. V. Keldysh and M. A. Lavrent'yev by solving the problem in the form of a series

in the positive powers of $\frac{2}{h}$. In expanding the regular part of the nucleus in the form (I.35) the system of recurrence equations for determining $\gamma_n(s)$ is also in the form (I.36). For a thin plate moving at a small angle of attack, A. I. Tikhonov obtained [81

$$\bar{\gamma}(\bar{x}) = \alpha\left[2 + \frac{4G_0}{\bar{h}} + (2G_0^2 - 3G_1 + 2G_1\bar{x})\frac{4}{\bar{h}_0^2}\right]\sqrt{\frac{1+\bar{x}}{1-\bar{x}}}, \qquad \text{(III.31)}$$

where

$$G_0 = -2\pi\beta\frac{\operatorname{sh}^2\mu_0(1-\beta)}{\omega_n\operatorname{ch}^2\mu_0 - 1};$$

$$G_1 = -\frac{1}{4}\frac{\beta}{(1-\beta)^2} - 2\beta^2\int_0^\infty \frac{\mu(1+\alpha\mu)e^{-\mu}\operatorname{sh}^2\mu(1-\beta)}{\operatorname{sh}\mu - \omega_n\mu\operatorname{ch}\mu}\,d\mu.$$

The wave drag of a thin hydrofoil may be determined by the amplitude of a plane wave far behind the moving hydrofoil from the formula [47]

$$Q = \frac{1}{4}\varrho g a^2\left(1 - \frac{2h_0}{\operatorname{sh}2\lambda_0 h_0}\right). \qquad \text{(III.32)}$$

The asymptotic expression for the complex velocity far behind the hydrofoil is determined by the formula

$$v(z) = -2\nu\int_{-a}^{+a}\gamma(s)\,\frac{\operatorname{sh}\lambda_0(h_0-h)\sin\lambda_0(z+i(h_0-h)-s)}{\nu h_0 - \operatorname{ch}^2\lambda_0 h_0}\,ds. \qquad \text{(III.33)}$$

Using the formula for raising the free surface we obtain

$$\eta = \frac{2v_0\nu}{g}\frac{\operatorname{ch}\lambda_0 h_0}{\nu h_0 - \operatorname{ch}^2\lambda_0 h_0}\int_{-a}^{+a}\operatorname{sh}\lambda(h_0-h)\sin\lambda_0(x-s)\,\gamma(s)\,ds.$$

Thus, sinusoidal waves with a length of $\frac{2\pi}{\lambda_0}$ and an amplitude of

$$a = \frac{2v_0\nu}{g}\frac{\operatorname{ch}\lambda_0 h_0}{\nu h_0 - \operatorname{ch}^2\lambda_0 h_0}\left|\int_{-a}^{+a}\operatorname{sh}\lambda_0(h_0-h)\,e^{i\lambda s}\,\gamma(s)\,ds\right|$$

are also formed in a fluid of finite depth behind a hydrofoil.

Then, the wave drag will be determined from the

formula

$$Q = \varrho\nu \frac{\mathrm{sh}^2\lambda_0 (h_0 - h)}{\mathrm{ch}^2 \lambda_0 h_0 - \nu h_0} \left| \int_{-a}^{+a} e^{i\lambda_0 s} \gamma(s)\, ds \right|^2. \qquad \text{(III.34)}$$

The lift and the moment will be determined by formulas (I.24) and (I.25). For determining forces, A. I. Tikhonov obtained the following formulas: [82

$$Q = \varrho \frac{a^2 v_0^2}{h} \alpha^2 \left(R_0 + \frac{2R_1}{\bar{h}} + \frac{4R^2}{\bar{h}^2} \right), \qquad \text{(III.35)}$$

$$P = \varrho a v_0^2 \alpha^2 \left(R_0' + \frac{2R_1'}{\bar{h}} + \frac{4R_2'}{\bar{h}^2} \right), \qquad \text{(III.36)}$$

$$M = -\varrho a^2 v_0^2 \alpha \left(B_0'' + \frac{2R_1''}{\bar{h}} + \frac{4R_2''}{\bar{h}^2} \right), \qquad \text{(III.37)}$$

where

$$R_0 = -2\pi G_0; \quad R_1 = -4\pi G_0^2; \quad R_2 = -2\pi G_0 (3G_0^2 - 2G_1);$$

$$R_0' = 2\pi; \quad R_1' = 2\pi G_0; \quad R_2' = 2\pi (G_0^2 - G_1);$$

$$R_0'' = \pi; \quad R_1'' = \pi G_0; \quad R_2'' = \pi \left(G_0^2 - \frac{G_1}{2} \right).$$

The curves for the coefficients for a number of values of β are illustrated in Figures 8 and 9 [146]:

$$GQ = R_0 + \frac{2R_1}{\bar{h}} + \frac{4R_2}{\bar{h}^2}, \quad G_P = R_0' + \frac{2R_1'}{\bar{h}} + \frac{4R_2'}{\bar{h}^2}.$$

The regular integral equation of the problem will be determined from formula (I.58). By assigning the corresponding functions $\varphi_j(\sigma, s)$ in the form given in (I.75) one may determine coefficients c_{nmp} and $\overline{c_{nmp}}$. These may then be used to determine the hydromechanical characteristics of the hydrofoil in a fluid of finite depth. Let

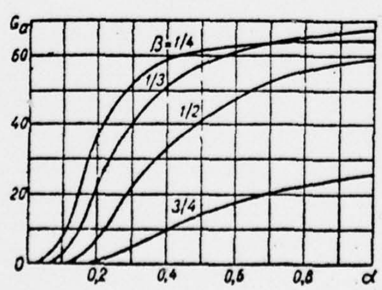

Fig. 8

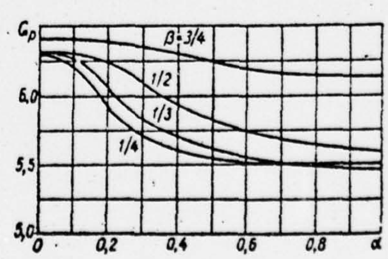

Fig. 9

us examine the motion of a hydrofoil with large Fr_b numbers. In this case the regular part of the nucleus may be transformed as follows: [83

$$G(\bar{x}) = \mathrm{Re}\sum_{k=0}^{\infty}\left[\frac{1}{\bar{x}+4i\left(\frac{k+1}{2}\right)\bar{t}}+\frac{1}{x-4i\left(\bar{h}+\frac{k}{2}\bar{t}\right)}-\right.$$

$$-\frac{1}{\bar{x}-4i\left(\bar{h}_1+\frac{k}{2}\bar{t}\right)}-\frac{1}{\bar{x}-4i\left(\frac{k+\frac{1}{2}}{2}\right)\bar{t}}-$$

$$-\frac{1}{\bar{x}+4i\left(\frac{k+\frac{1}{2}}{2}\right)\bar{t}}-\frac{1}{\bar{x}-4i\left(-\bar{h}_1+\frac{k+1}{2}\bar{t}\right)}+$$

$$\left.+\frac{1}{\bar{x}+4i\left(-\bar{h}+\frac{k+1}{2}\bar{t}\right)}+\frac{1}{\bar{x}-4i\left(\frac{k+1}{2}\right)\bar{t}}\right], \qquad (III.38)$$

where $\bar{t} = 4\bar{h}_0$;

$\bar{h}_1 = \frac{h_1}{2a}$ - relative distance between the hydrofoil and the bottom.

For the function $\mathrm{Re}\frac{1}{z \pm 4ix}$ the expansion in powers of the parameter $\tau_x = \sqrt{4x^2+1} - 2x$ is in the form

$$\mathrm{Re}\,\frac{1}{z \pm 4ix} = \sum_{n=2,4}^{\infty}\tau_x^n \sum_{p=0}^{\frac{n}{2}-1}\frac{(n-1-p)\ldots(p+1)(-1)^{\frac{n}{2}-p}}{(n-1-2p)!}z^{s-1-2p}. \qquad (III.39)$$

Then

$$G(\bar{x}) = \sum_{k=0}^{\infty}\sum_{n=2,4}^{\infty}\left(2\tau^n_{k_1}+\tau^n_{k_2}-\tau^n_{k_3}-2\tau^n_{k_4}-\tau^n_{k_5}+\tau^n_{k_6}\right)\times$$

$$\times\sum_{p=0}^{\frac{n}{2}-1}\frac{(n-1-p)\ldots(p+1)}{(n-1-2p)!}(-1)^{\frac{n}{2}-p}\bar{x}^{s-1-2p}, \qquad (III.40)$$

$$\tau_{k_1} = \sqrt{(k+1)^2\bar{t}^2+1}-(k+1)\bar{t},$$

$$\tau_{k_2} = \sqrt{4\left(\bar{h}+\frac{k}{2}\bar{t}\right)^2+1}-2\left(\bar{h}+\frac{k}{2}\bar{t}\right),$$

$$\tau_{k_3} = \sqrt{4\left(\bar{h}_1 + \frac{k}{2}\bar{t}\right)^2 + 1} - 2\left(\bar{h}_1 + \frac{k}{2}\bar{t}\right),$$ [84

$$\tau_{k_4} = \sqrt{\left(k + \frac{1}{2}\right)^2 \bar{t}^2 + 1} - \left(k + \frac{1}{2}\right)\bar{t},$$

$$\tau_{k_5} = \sqrt{4\left(\frac{k+1}{2}\bar{t} - \bar{h}_1\right)^2 + 1} - 2\left(\frac{k+1}{2}\bar{t} - \bar{h}_1\right),$$

$$\tau_{k_6} = \sqrt{4\left(\frac{k+1}{2}\bar{t} - \bar{h}\right)^2 + 1} - 2\left(\frac{k+1}{2}\bar{t} - \bar{h}\right)$$

and

$$\left.\begin{aligned}
&\varphi_1(\sigma - s) = 0\\
&\varphi_2(\sigma - s) = (\sigma - s)\\
&\varphi_3(\sigma - s) = 0\\
&\varphi_4(\sigma - s) = k_4[2(\sigma - s) - (\sigma - s)^3]\\
&\varphi_5(\sigma - s) = 0\\
&\varphi_6(\sigma - s) = k_6[3(\sigma - s) - 4(\sigma - s)^3 + (\sigma - s)^5]
\end{aligned}\right\}. \qquad \text{(III.41)}$$

For a thin hydrofoil section we obtain after computations

$$\begin{aligned}
\bar{\gamma} = 1 - &\left(1 - \frac{1}{2}\frac{\alpha_0}{\alpha_0 + \alpha_\kappa}\right)\xi_1 + \\
+ &\left[\left(\frac{3}{4} - \frac{1}{2}\frac{\alpha_0}{\alpha_0 + \alpha_\kappa}\right) + \left(\frac{1}{4} - \frac{1}{2}\frac{\alpha_0}{\alpha_0 + \alpha_\kappa}\right)k_4\right]\xi_1^2 + \\
+ &\left[\left(\frac{1}{2} - \frac{3}{8}\frac{\alpha_0}{\alpha_0 + \alpha_\kappa}\right) - \frac{1}{4}\frac{\alpha_0}{\alpha_0 + \alpha_\kappa}k_4 + \left(\frac{1}{4} - \frac{3}{16}\frac{\alpha_0}{\alpha_0 + \alpha_\kappa}\right)k_6\right]\xi_1^3,
\end{aligned} \qquad \text{(III.42)}$$

where

$$k_4 = \frac{\xi_2}{\xi_1^2}; \quad k_6 = \frac{\xi_3}{\xi_1^3};$$

$$\xi_l = \sum_{k=0}^{\infty}\left(2\tau_{k_1}^{2l} + \tau_{k_2}^{2l} - \tau_{k_3}^{2l} - 2\tau_{k_4}^{2l} - \tau_{k_4}^{2l} - \tau_{k_5}^{2l} + \tau_{k_6}^{2l}\right). \qquad \text{(III.43)}$$

In the same way as in Chapter II, by representing $\bar{\gamma}$ in the form

$$\bar{\gamma} = \psi + \frac{\varkappa\alpha_0}{\alpha_0 + \alpha_\kappa},$$

we obtain [85

$$\psi = 1 - \xi_1 + \left[\frac{3}{4} + \frac{1}{4}k_4\right]\xi_1^2 + \frac{1}{4}k_6\xi_1^3, \qquad \text{(III.44)}$$

$$\varkappa - \frac{1}{2}\xi_1^2 - \frac{1}{2}[1 + k_4]\xi_1^2 + \left[\frac{3}{8} + \frac{1}{4}k_4 + \frac{3}{16}k_6\right]\xi_1^3. \qquad \text{(III.45)}$$

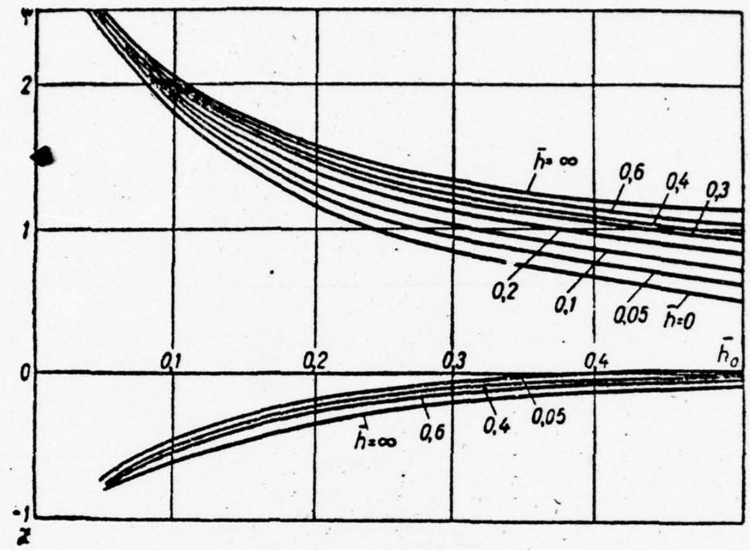

Fig. 10

The values of the function ξ_j are given in Table 2. Curves for ψ and $\varkappa$ obtained according to formulas (III.40) and (III.45) are illustrated in Fig. 10 and their values are given in Tables 3 and 4.

The problem for any arbitrary values of the Fr number may be investigated in the same way.

3.3. Motion of an Arbitrary-Profile Hydrofoil in a Fluid of Finite Depth

The problem of motion of a hydrofoil with an arbitrary profile may be solved by generalizing the results discussed in Chapter III for a fluid of finite depth. Formulas for determining forces acting on a hydrofoil for this problem were obtained by M. D. Khaskind [156] using the N. Ye. Kochin functions. Khaskind also examined certain simple cases of motion. The formulation of the problem for a hydrofoil of an arbitrary profile is given at the beginning of this chapter.

As was done in Chapter III, by using the Cauchy formula for the region between the C_1 and C_2 contours (contour [88

Table 2

$\bar{h}$	$\bar{h}_0$	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	h_0	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6
0	0	—1	—1	—1	—1	—1	—1	0,3	+0,0715	+0,6436	+0,8760	+0,9592	+0,9864	+0,9956
0,05		—1	—1	—1	—1	—1	—1		—0,1046	+0,4022	+0,4678	+0,4261	+0,2510	+0,2991
0,1		—1	—1	—1	—1	—1	—1		—0,0535	+0,2366	+0,2435	+0,1873	+0,1321	+0,0907
0,2		—1	—1	—1	—1	—1	—1		—0,1440	+0,0502	+0,0531	+0,0323	+0,0164	+0,0082
0,3		—1	—1	—1	—1	—1	—1		—0,2080	+0,0316	—0,0046	—0,0002	0	0
0,4		—1	—1	—1	—1	—1	—1		—0,2484	—0,0682	—0,0228	—0,0071	—0,0027	—0,0009
0,6		—1	—1	—1	—1	—1	—1		—0,2882	—0,0934	—0,0313	—0,0099	—0,0034	—0,0011
∞		—1	—1	—1	—1	—1	—1		—0,3206	—0,1030	—0,0330	—0,0100	—0,0034	—0,0011
0	0,05	—0,7944	—0,5926	—0,3996	—0,2364	+0,0456	+0,0780	0,4	+0,3045	+0,8089	+0,9532	+0,9884	+0,9870	+0,9992
0,05		—0,8062	—0,6722	—0,4652	—0,3370	—0,2392	+0,0164		+0,3450	+0,5245	+0,5171	+0,4427	+0,3666	+0,3008
0,1		—0,8020	—0,6308	—0,4920	—0,3943	—0,2655	—0,2392		+0,1226	+0,3329	+0,2798	+0,1992	+0,1360	+0,0920
0,2		—0,7999	—0,6520	—0,5295	—0,4382	—0,3507	—0,2959		—0,0031	+0,1218	+0,0789	+0,0410	+0,0195	+0,0091
0,3		—0,8153	—0,6634	—0,5422	—0,4468	—0,3670	—0,3009		—0,0892	+0,0290	+0,0184	+0,0077	+0,0027	+0,0009
0,4		—0,8144	—0,6680	—0,5471	—0,4487	—0,3681	—0,3012		—0,1334	—0,0128	—0,0012	0	0	0
0,6		—0,3158	—0,6696	—0,5454	—0,4495	—0,3681	—0,3012		—0,1866	—0,0418	—0,0104	—0,0028	—0,0007	—0,0002
∞		—0,8198	—0,6708	—0,5488	—0,4495	—0,3681	—0,3012		—0,2310	—0,0534	—0,0123	—0,0029	—0,0007	—0,0002
0	0,1	—0,7032	—0,2392	+0,0942	+0,3282	+0,5208	+0,6648	0,6	+0,8081	+0,9671	+0,9977	1	1	1
0,05		—0,6104	—0,2940	—0,0763	+0,0554	+0,1217	+0,1474		+0,6444	+0,6444	+0,5465	+0,4495	+0,3681	+0,3012
0,1		—0,6184	—0,3384	—0,1716	—0,0826	+0,0390	+0,0165		+0,5088	+0,4288	+0,3013	+0,2040	+0,1371	+0,0922
0,2		—0,6174	—0,3989	—0,2625	—0,1777	—0,1228	—0,0851		+0,3083	+0,1904	+0,0939	+0,0441	+0,0202	+0,0093
0,3		—0,6539	—0,4274	—0,2903	—0,1983	—0,1349	—0,0915		+0,1812	+0,0846	+0,0307	+0,0106	+0,0034	+0,0011
0,4		—0,6540	—0,4406	—0,2990	—0,2028	—0,1368	—0,0920		+0,0948	+0,0355	+0,0100	+0,0029	+0,0007	0
0,6		—0,6640	—0,4483	—0,3029	—0,2039	—0,1371	—0,0922		+0,0025	—0,0001	0	0	0	0
∞		—0,6721	—0,4517	—0,3036	—0,2040	—0,1371	—0,0922		—0,1310	—0,0172	0	0	0	0
0	0,2	—0,2367	+0,3232	+0,6566	+0,8348	+0,9220	+0,9636	∞	+1,0000	+1,0000	+1,0000	+1,0000	+1,0000	+1,0000
0,05		—0,3712	+0,1601	+0,3254	+0,3552	+0,3290	+0,2847		+0,8190	+0,6708	+0,5488	+0,4495	+0,3681	+0,3012
0,1		—0,2937	+0,0421	+0,1385	+0,1381	+0,1095	+0,0805		+0,6721	+0,4517	+0,3036	+0,2040	+0,1371	+0,0922
0,2		—0,3540	—0,0956	—0,0234	—0,0058	+0,0014	—0,0004		+0,4583	+0,2100	+0,0962	+0,0441	+0,0202	+0,0093
0,3		—0,3849	—0,1576	—0,0723	—0,0359	—0,0172	—0,0082		+0,3206	+0,1028	+0,0330	+0,0106	+0,0034	+0,0011
0,4		—0,4031	—0,1848	—0,0881	—0,0410	—0,0195	—0,0091		+0,2310	+0,0534	+0,0123	+0,0029	+0,0007	+0,0002
0,6		—0,4378	—0,2034	—0,0953	—0,0440	—0,0202	—0,0093		+0,1310	+0,0172	+0,0023	0	0	0
∞		—0,4583	—0,2100	—0,0962	—0,0441	—0,0202	—0,0093		0	0	0	0	0	0

[86

[87

Table 3 [86

h	Values of ψ for h_0					
	0,05	0,1	0,2	0,3	0,4	0,6
0	2,4442	2,1646	1,2019	0,8740	0,7149	0,4102
0,05	2,5040	1,9492	1,2949	1,0969	0,7253	0,5578
0,1	2,5076	1,9817	1,3469	1,0539	0,9010	0,6494
0,2	2,5051	1,9869	1,4405	1,1602	1,0138	0,7723
0,3	2,5543	2,0802	1,5032	1,2543	1,0980	0,8539
0.4	2,5516	2,0793	1,5335	1,2911	1,1451	0,9179
0,6	2,5587	2,1047	1,5965	1,3548	1,1882	0,9975
∞	2,5689	2,1257	1,6355	1,3967	1,2680	1,1407

Table 4 [87

$\bar{h}$	Values of $\varkappa$ for h_0					
	0,05	0,1	0,2	0,3	0,4	0,6
0	—0,7729	—0,6569	—0,1857	—0,1235	—0,0905	+0,1369
0,05	—0,7849	—0,4715	—0,1960	—0,1704	—0,0211	+0,0621
0,1	—0,7943	—0,4845	—0,1938	—0,1007	—0,0582	+0,0450
0,2	—0,7894	—0,4753	—0,2070	—0,0981	—0,0477	+0,0446
0,3	—0,8233	—0,5320	—0,2286	—0,1455	—0,0598	+0,0406
0,4	—0,8207	—0,5287	—0,2390	—0,1321	—0,0704	+0,0275
0,6	—0,8556	—0,5441	—0,2722	—0,1548	—0,0946	+0,0013
∞	—0,8328	—0,5577	—0,3795	—0,1814	—0,1236	—0,0604

C_1 about the hydrofoil profile can be as close to it as desired, while contour C_2 may be deformed toward the band $0 > y > -h_0$) we obtain an expression for the complex velocity of the flow [88

$$v(z) = v_1(z) + v_2(z), \qquad \text{(III.46)}$$

where $v_1(z) = \frac{1}{2\pi i}\int \frac{v(\zeta)\,d\zeta}{z-\zeta}$ - function which is analytical outside the contour;

$v_2(z)$ - function which is analytical within the band $0 > y > -h$.

Using the expressions (III.19) and (III.24) it is not difficult to obtain the expression for the complex velocity of the vortex source which is located at point h in the lower half-plane:

$$v(z) = \frac{B}{z-\zeta} - \frac{\bar{B}}{(z-\zeta+2ih_0)} -$$

$$-\int_0^\infty e^{-\lambda h_0}(\nu+\lambda)\frac{\bar{B}\sin\lambda(z-\bar{\zeta}+2ih_0)-B\sin\lambda(z-\zeta)}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\,d\lambda+$$

$$+\nu\pi\frac{\bar{B}\cos\lambda_0(z-\bar{\zeta}+2ih_0)-B\cos\lambda_0(z-\zeta)}{\nu h_0-\mathrm{ch}^2\lambda_0 h_0}. \qquad \text{(III.47)}$$

Assuming that $B=\nu(\zeta)d\zeta$ and integrating along the contour C_1 we can obtain

$$v_2(z)=\frac{1}{2\pi i}\int_{C_1}\bar{v}(\zeta)\Big[\frac{1}{z-\bar{\zeta}+2ih_0}+$$

$$+\int_0^\infty(\nu+\lambda)e^{-\lambda h_0}\frac{\sin\lambda(z-\bar{\zeta}+2ih_0)}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\,d\lambda-$$

$$-\pi\nu\frac{\cos\lambda_0(z-\zeta+2ih_0)}{\nu h_0-\mathrm{ch}^2\lambda_0 h_0}\Big]d\bar{\zeta}+$$

$$+\frac{1}{2\pi i}\int_{-C_1}v(\zeta)\Big[\int_0^\infty(\nu+\lambda)\frac{e^{-\lambda h_0}\sin\lambda(z-\zeta)}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\,d\zeta-$$

$$-\pi\nu\frac{\cos\lambda_0(z-\zeta)}{\nu h_0-\mathrm{ch}^2\lambda_0 h_0}\Big]d\zeta. \qquad \text{(III.48)}$$

The expression (III.48) satisfies the conditions in our problem. As was done in Chapter III, one may show that this is the only expression which satisfies these conditions. After simple transformations, function $v_2(z)$ may be written in terms of N. Ye. Kochin's functions: [89

$$v_2(z)=\frac{1}{2\pi}\Big\{\int_0^\infty\Big[\bar{H}(-\lambda)\,e^{i\lambda(z+2ih_0)}+$$

$$+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}(\bar{H}(-\lambda)\,e^{i\lambda(z+2ih)}-\bar{H}(\lambda)e^{-i\lambda(z+2ih_0)}-$$

$$-H(\lambda)\,e^{i\lambda z}+H(-\lambda)\,e^{-i\lambda z})\Big]d\lambda-$$

$$-\frac{\pi i\nu}{2(\nu h-\mathrm{ch}^2\lambda_0 h_0)}[H(-\lambda_0)]\,e^{i\lambda_0(z+2ih_0)}+$$

$$+\bar{H}(\lambda_0)\,e^{-i\lambda_0(z+2ih_0)}-H(\lambda_0)\,e^{i\lambda z}-H(-\lambda_0)\,e^{-i\lambda_0 z}\Big\}. \qquad \text{(III.49)}$$

Forces acting on the contour are also calculated by S. A. Chaplygin's formulas. The lift and wave drag will be determined from formula (II.9). Determining $\int\limits_{C_0} v_1(z)\, v_2(z)\, dz$ and separating the real and imaginary parts we obtain [156]:

$$P = \varrho v_0 \Gamma - \frac{\varrho}{2\pi}\int\limits_0^\infty \Big[|H(-\lambda)|^2 e^{-2\lambda h_0} + (\nu + \lambda)\, e^{\lambda h_0} \times$$

$$\times \frac{|H(-\lambda)|^2 e^{-2\lambda h_0} - |H(\lambda)|^2 e^{2\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} \Big] d\lambda +$$

$$+ \nu\varrho \frac{\operatorname{Im}[H(\lambda_0) H(-\lambda_0)]}{2(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} + g\varrho s; \qquad \text{(III.50)}$$

$$Q = -\frac{\varrho\nu}{4} \frac{|H(\lambda_0)|^2 e^{2\lambda_0 h_0} + |H(-\lambda_0)|^2 e^{-2\lambda_0 h_0} - 2\operatorname{Re}[H(\lambda_0) H(-\lambda_0)]}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0}. \qquad \text{(III.51)}$$

Formula (III.51) may also be written in the form

$$Q = -\frac{\varrho\nu}{4} \frac{|H(\lambda_0)\, e^{\lambda_0 h_0} - H(-\lambda_0)\, e^{-\lambda_0 h_0}|^2}{\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0}. \qquad \text{(III.52)}$$

Let us examine the type of free surface far behind the hydrofoil. The asymptotic expression of the v(z) function is in the form [90

$$\underset{x \to -\infty}{v(z)} = -\frac{i\nu}{2(\nu h - \operatorname{ch}^2 \lambda_0 h)} [H(-\lambda_0)\, e^{i\lambda_0(z + 2ih_0)} +$$

$$+ \overline{H}(\lambda_0)\, e^{-i\lambda_0(z + 2ih_0)}] - H(\lambda_0)\, e^{i\lambda_0 z} - H(-\lambda_0)\, e^{i\lambda_0 z}].$$

From formula $\eta = \frac{v}{g} \operatorname{Re} \underset{y=0}{v(z)}$ we will easily find that when $x \to -\infty$ sinusoidal waves $\frac{2\pi}{\lambda_0}$ in length are formed and their amplitude is

$$a = \frac{\operatorname{ch} \lambda_0 h_0}{v_0(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} [H(\lambda_0)\, e^{\lambda_0 h_0} - H(-\lambda_0)\, e^{-\lambda_0 h_0}]. \qquad \text{(III.53)}$$

It may be determined that the total resistance of the hydrofoil is actually the wave drag only. By transforming formulas (III.32) and (III.53) we obtain formula (III.52). Let us determine the moment of hydrodynamic forces from formula (II.16).

The total moment will be determined from the formula

$$M_1 = -\varrho v_0 \operatorname{Re}[iH'(0)] + \varrho \operatorname{Re}\left\{\frac{1}{2\pi i}\int_0^\infty \left\{H'(-\lambda)\overline{H}(-\lambda)e^{-2\lambda h_0} + \right.\right.$$

$$+ \frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu \operatorname{sh}\lambda h_0 - \lambda \operatorname{ch}\lambda h_0)}[H'(-\lambda)\overline{H}(-\lambda)e^{-2\lambda h_0} + H'(\lambda)\overline{H}(\lambda)e^{2\lambda h_0} -$$

$$\left.\left. - H'(-\lambda)H(+\lambda) - H(-\lambda)H(\lambda_0)]\right\}d\lambda\right\} -$$

$$-\frac{\nu}{4(\nu h_0 - \operatorname{ch}^2\lambda_0 h_0}[H'(-\lambda_0)\overline{H}(-\lambda_0)e^{-2\lambda_0 h_0} - H'(\lambda_0)\overline{H}(\lambda_0)e^{2\lambda_0 h_0} -$$

$$- H'(-\lambda_0)H(\lambda_0)]\, g\varrho s x_c. \qquad \text{(III.54)}$$

M. D. Khaskind considered special cases dealing with the problems of motion of a circular cylinder with circulation and elliptical cylinder without circulation.

In the first approximation function $H(\lambda)$ for the cylinder with radius R is determined from formula (II.22). Then we obtain

$$P = \varrho v_0 \Gamma - \frac{\varrho \Gamma^2}{4\pi(h_0 - h)} + \frac{\varrho v_0 R^2 \Gamma}{2(h_0-h)^2} - \frac{\pi\varrho v_0^2 R^2}{2(h_0-h)^3} +$$

$$+\frac{\varrho}{2\pi}\int_0^\infty \frac{(\nu+\lambda)e^{-\lambda h_0}(\Gamma^2 + 4\pi^2 v_0^2 R^4\lambda^2)\operatorname{sh}2\lambda(h_0-h) + 4\pi v_0 R^2\Gamma\lambda \operatorname{ch}2\lambda(h_0-h)}{\nu \operatorname{sh}\lambda h_0 - \lambda\operatorname{ch}\lambda h_0}d\lambda + \pi g\varrho R^2; \qquad \text{(III.55)}$$ [91

$$Q = \varrho\nu\frac{[\Gamma\operatorname{sh}\lambda_0(h_0-h) + 2\pi v_0\lambda_0 R^2 \operatorname{ch}\lambda_0(h_0-h)]^2}{\operatorname{ch}^2\lambda_0 h_0 - \nu h_0}; \qquad \text{(III.56)}$$

$$M = hQ - \frac{2\pi\varrho v_0 R^2 \nu\Gamma\operatorname{sh}^2\lambda_0(h_0-h) + \pi v_0 R^2\lambda_0 \operatorname{sh}2\lambda_0(h_0-h)}{\operatorname{ch}^2\lambda_0 h_0 - \nu h_0}. \qquad \text{(III.57)}$$

The point of intersection of the resultant of the flow forces acting on the body with the Oy axis is determined by the formula

$$y_0 = -\frac{M}{R} = -h + \frac{2\pi v_0 R^2}{\Gamma + 2\pi v R^2\lambda_0 ct\, h\lambda_0(h_0-h)}. \qquad \text{(III.58)}$$

For the elliptic cylinder without circulation M. D. Khaskind limited his calculations to determining the wave drag

$$Q = 4\pi^2\varrho g\beta^2\frac{\alpha+\beta}{\alpha-\beta}\frac{\operatorname{ch}^2\lambda_0 h_0(h_0-h)}{\operatorname{ch}^2\lambda_0 h_0 - \nu h_0}J_1^2(\lambda_0\sqrt{\alpha^2-\beta^2}), \qquad \text{(III.59)}$$

where α and β are semi-axes of the ellipse, which are directed parallel to axes Ox and Oy.

An interesting result follows from formula (III.59), i.e., for certain values of λ_0 and, therefore, for a certain velocity $v_0 < \sqrt{gh_0}$, the wave drag is equal to zero. This will be the case when

$$\alpha_0 \sqrt{\alpha^2 - \beta^2} = s_k, \quad k = 1, 2,$$

s_k - positive roots of the Bessel function $J_1(s)$. The first root of this function equals 3.832. Then, the first velocity at which the wave drag is equal to zero will be determined by the formula

$$v_0 = 0{,}51 \sqrt{g \sqrt{\alpha^2 - \beta^2} + h \frac{3{,}832 h_0}{\sqrt{\alpha^2 - \beta^2}}}\,.$$

3.4. Effect of the Free Surface on Circulation of the Hydrofoil in a Fluid of Finite Depth. Motion of a Thin Hydrofoil

As was the case with an infinite depth, the circulation Γ in the formulas for determining forces [(III.50), (III.51) and (III.54)] is an unknown quantity, the determination of which is connected with the solution of a boundary problem for the shape of a cylinder and with satisfying the N. Ye. Zhukovskiy-S. A. Chaplygin postulate. [92
Let us present the solution of this problem.

Generally, it will be analogous to the solution given in Chapter II. To determine function $V_1(z)$ let us use formula (II.29):

$$\int\limits_k \frac{v_2(z)}{\sigma - u}\, dz = \frac{1}{2\pi} \Big\{ \int\limits_0^\infty \Big[\bar{H}(-\lambda) G(-\lambda, u) e^{-2\lambda h_0} +$$

$$+ \frac{(\nu + \lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} (\bar{H}(-\lambda) G(-\lambda, u) e^{-2\lambda h_0} - \bar{H}(\lambda) G(\lambda, u) e^{2\lambda h_0} -$$

$$- H(\lambda) G(-\lambda, u) + H(-\lambda) G(\lambda, u)) \Big] d\lambda -$$

$$- \frac{\nu \pi i}{2(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} [\bar{H}(-\lambda_0) G(-\lambda_0, u) e^{-2\lambda_0 h_0} +$$

$$+ \bar{H}(\lambda_0) G(\lambda_0, u) e^{2\lambda_0 h_0} - H(\lambda_0) G(-\lambda_0, u) - H(-\lambda_0) G(\lambda_0 u)] \Big\}, \qquad \text{(III.60)}$$

$$\int_{k} \frac{\bar{v}_2(z)}{\bar{\sigma} - \frac{R^2}{\bar{u}}} d\bar{z} = \frac{1}{2\pi} \left\{ \int_0^\infty \left[H(-\lambda) \overline{G\left(-\lambda, \frac{R^2}{\bar{u}}\right)} e^{-2\lambda h_0} + \right.\right.$$

$$+ \frac{(\nu + \lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} \left(H(-\lambda) \overline{G\left(-\lambda, \frac{R^2}{\bar{u}}\right)} e^{-2\lambda h_0} - \right.$$

$$- H(\lambda) \overline{G\left(-\lambda, \frac{R^2}{\bar{u}}\right)} e^{2\lambda h_0} - \bar{H}(\lambda) \overline{G\left(-\lambda, \frac{R^2}{\bar{u}}\right)} +$$

$$\left.\left. + \overline{H(-\lambda)} \overline{G\left(-\lambda, \frac{R^2}{u}\right)}\right)\right] d\lambda + \frac{\nu\pi i}{2(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} \left[H(-\lambda_0) \overline{G\left(-\lambda_0, \frac{R^2}{\bar{u}}\right)} e^{-2\lambda_0 h_0} \times \right.$$

$$\left.\left. \times H(\lambda_0) \overline{G\left(\lambda_0, \frac{R^2}{u}\right)} e^{2\lambda_0 h_0} - \bar{H}(\lambda_0) \overline{G\left(-\lambda_0, \frac{R^2}{\bar{u}}\right)} - \bar{H}(-\lambda_0) \overline{G\left(\lambda_0, \frac{R^2}{\bar{u}}\right)} \right] \right\}. \quad \text{(III.61)}$$

Then from formula (II.29) we obtain

$$v_1(z) = \frac{du}{dz} \left\{ - v_0 + \frac{\bar{v}_0 R^2}{u^2} + \frac{c_1 i}{u} + \frac{1}{4\pi^2 i} \left\{ \int_0^\infty \left[\bar{H}(-\lambda) G(-\lambda, u) e^{-2\lambda h_0} + \right.\right.\right.$$

$$+ \frac{(\nu + \lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} (\bar{H}(-\lambda) G(-\lambda, u) e^{-2\lambda h_0} -$$

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$$\left. - \bar{H}(\lambda) G(\lambda, u) e^{2\lambda h_0} - H(\lambda) G(-\lambda, u) + H(-\lambda) G(\lambda, u)) \right| d\lambda +$$

$$+ \frac{R^2}{u^2} \int_0 \left[H(-\lambda) \overline{G\left(-\lambda, \frac{R^2}{\bar{u}}\right)} e^{-2\lambda h_0} + \frac{(\nu + \lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} \times \right.$$

$$\times \left(H(-\lambda) \overline{G\left(\lambda, \frac{R^2}{\bar{u}}\right)} e^{-2\lambda h_0} - H(\lambda) \overline{G\left(\lambda, \frac{R^2}{\bar{u}}\right)} e^{2\lambda h_0} - \right.$$

$$- \bar{H}(\lambda) \overline{G\left(-\lambda, \frac{R^2}{\bar{u}}\right)} -$$

$$\left.\left. - H(-\lambda) G\left(\lambda, \frac{R^2}{u}\right)\right)\right] d\lambda - \frac{\nu}{8\pi(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} \times$$

$$\times [\bar{H}(-\lambda_0) G(-\lambda_0, u) e^{-2\lambda_0 h_0} + \bar{H}(\lambda_0) G(\lambda_0) e^{2\lambda_0 h_0} - H(\lambda_0) G(-\lambda_0 u) -$$

$$- H(-\lambda_0) G(\lambda_0 u)] + \frac{\nu}{8\pi(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} \frac{R^2}{u^2} \times$$

$$\times\left[H(-\lambda_0)\overline{G\left(-\lambda_0, \frac{R^2}{u}\right)}e^{-2\lambda_0 h_0}+H(\lambda_0)G\left(\overline{\lambda_0, \frac{R^2}{\bar{u}}}\right)e^{2\lambda_0 h_0}-\right.$$

$$\left.\left.\left.-\overline{H}(\lambda_0)G\overline{\left(\lambda_0, \frac{R^2}{u}\right)}-\overline{H}(-\lambda)G\left(\overline{\lambda_0 \frac{R^2}{\bar{u}}}\right)\right]\right\}\right\}. \qquad \text{(III.62)}$$

In order to use condition (II.33) it is necessary to transform expression (III.62) for a given type of hydrofoil. Let us examine motion of the hydrofoil, obtained by means of the transforming function

$$z = u + \frac{R^2}{u}.$$

Functions G(λ, u) and $G\left(\lambda, \frac{R^2}{\bar{u}}\right)$ are determined from formulas (II.35) and (II.36). Then, from expression (III.62) we obtain [94

$$v_1(z) = \frac{du}{dz}\left\{\left(-v_0 - \frac{1}{2\pi}\int_0^\infty J_0(2\lambda R)\left[H(-\lambda)e^{-\lambda(2h_0-h)}+\right.\right.\right.$$

$$+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu \operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}(H(-\lambda)e^{-\lambda(2h_0-h)}-\overline{H}(\lambda)e^{\lambda h}-H(\lambda)e^{\lambda(2h_0-h)}+$$

$$\left.+\overline{H}(-\lambda)e^{-\lambda h})\right]d\lambda+\frac{i\nu}{4}\frac{J_0(2\lambda_0 R)}{(\nu h_0-\operatorname{ch}^2\lambda_0 h_0)}[H(-\lambda_0)e^{-\lambda_0(2h_0-h)}+$$

$$+H(\lambda_0)e^{\lambda_0(2h_0-h)}-\overline{H}(\lambda_0)e^{\lambda_0 h}+\overline{H}(-\lambda_0)e^{-\lambda_0 h}]+\frac{\Gamma}{2\pi iu}+$$

$$+\frac{R^2}{u^2}\left\{\overline{V}_0+\frac{1}{2\pi}\left(\int_0^\infty J_0(2\lambda R)\left[\overline{H}(-\lambda)e^{-\lambda(2h_0-h)}+\right.\right.\right.$$

$$+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\operatorname{sh}\lambda_0 h_0-\lambda\operatorname{ch}\lambda h_0)}(H(-\lambda)e^{-\lambda(2h_0-h)}\overline{H}(\lambda)e^{\lambda(2h_0-h)}-$$

$$-H(\lambda)e^{\lambda h}+H(-\lambda)e^{-\lambda h})]d\lambda-$$

$$-\frac{i\nu}{4}\frac{J_0(2\lambda_0 R)}{(\nu h_0-\operatorname{ch}^2\lambda_0 h_0)}[\overline{H}(-\lambda_0)e^{-\lambda_0(2h_0-h)}+\overline{H}(\lambda_0)e^{\lambda_0(2h_0-h)}-$$

$$-H(\lambda_0)e^{\lambda_0 h}+H(-\lambda_0)e^{-\lambda_0 h}]+\frac{dz}{du}\left\{\frac{1}{4\pi^2 i}\int_0^\infty\left[\overline{H}(-\lambda)B(-\lambda,u)e^{-\lambda(2h_0-h)}+\right.\right.$$

$$+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}(\overline{H}(-\lambda)B(-\lambda,u)e^{-\lambda(2h_0-h)}+$$

$$+\overline{H}(\lambda)B(\lambda,u)e^{\lambda(2h_0-h)}-H(\lambda)B(-\lambda,u)e^{\lambda h}+$$

$$+ H(-\lambda) B(\lambda, u) e^{-\lambda h}\Big)\Big] d\lambda - \frac{\nu}{8\pi(\nu h - \mathrm{ch}^2 \lambda_0 h_0)} \times$$

$$\times [\overline{H}(-\lambda_0) B(-\lambda, u) e^{-\lambda_0(2h_0-h)} + \overline{H}(\lambda_0) B(\lambda, u) e^{\lambda(2h_0-h)} -$$

$$- H(\lambda_0) B(-\lambda_0, u) e^{\lambda h} + H(-\lambda_0) B(\lambda_0, u) e^{-\lambda h} + \frac{dz}{d\frac{R^2}{\overline{u}}} \times$$

$$\times \frac{R^2}{u^2}\left\{\frac{1}{4\pi^2 i}\int_0^\infty \left[H(-\lambda)\overline{B\left(-\lambda, \frac{R^2}{u}\right)} e^{-\lambda(2h_0-h)} + \right.\right. \tag{95}$$

$$+ \frac{(\nu+\lambda) e^{-\lambda h_0}}{2(\nu \,\mathrm{sh}\, \lambda h_0 - \lambda \,\mathrm{ch}\, \lambda h_0)} \left(H(-\lambda)\overline{B\left(-\lambda, \frac{R^2}{\overline{u}}\right)}\right) e^{-\lambda(2h_0-h)} -$$

$$- H(\lambda)\overline{B\left(\lambda, \frac{R^2}{\overline{u}}\right)} e^{\lambda(2h_0-h)} -$$

$$- H(\lambda)\overline{B\left(-\lambda, \frac{R^2}{\overline{u}}\right)} e^{\lambda h} + \overline{H}(-\lambda)\overline{B\left(\lambda, \frac{R^2}{\overline{u}}\right)} e^{-\lambda h}\Big\} d\lambda +$$

$$+ \frac{\nu}{8\pi(\nu h - \mathrm{ch}^2 \lambda_0 h_0)} \Big[H(-\lambda_0)\overline{B\left(-\lambda_0, \frac{R^2}{\overline{u}}\right)} e^{-\lambda_0(2h_0-h)} +$$

$$+ H(\lambda_0)\overline{B\left(\lambda_0, \frac{R^2}{\overline{u}}\right)} e^{\lambda_0(2h_0-h)} - \overline{H}(\lambda_0)\overline{B\left(-\lambda_0, \frac{R^2}{\overline{u}}\right)} e^{-\lambda_0 h} +$$

$$+ H(-\lambda_0)\overline{B\left(\lambda_0, \frac{R^2}{\overline{u}}\right) e^{-\lambda_0 h}}\Big]\Big\}\Big\}. \tag{III.63}$$

Let us assume that $u_0 = -R$ corresponds to an angular point. Then, at this point, condition (II.33) should be satisfied and from expression (III.63) will will find

$$\Gamma = 4\pi R \,\mathrm{Im}\left\{v_0 + \frac{1}{2\pi}\int_0^\infty J_0(2\lambda R)\left[H(-\lambda) e^{-\lambda(2h_0-h)} + \right.\right.$$

$$+ \frac{(\nu+\lambda) e^{-\lambda h_0}}{2(\nu \,\mathrm{sh}\, \lambda h_0 - \lambda \,\mathrm{ch}\, \lambda h_0)} (H(-\lambda) e^{-\lambda(2h_0-h)} - H(\lambda) e^{\lambda(2h_0-h)} - \overline{H}(\lambda) e^{\lambda h} +$$

$$+ H(-\lambda) e^{-\lambda h})\Big] d\lambda + \frac{i\nu}{4} \cdot \frac{J_0(2\lambda_0 R)}{(\nu h_0 - \mathrm{ch}^2 \lambda_0 h_0)} [H(-\lambda_0) e^{\lambda_0(2h_0-h)} +$$

$$\left. + H(\lambda_0) e^{\lambda_0(2h_0-h)} - \overline{H}(\lambda_0) e^{\lambda h} - H(-\lambda) e^{-\lambda_0 h}]\right\}. \tag{III.64}$$

With $h_0 \to \infty$

$$\frac{(\nu+\lambda) e^{-\lambda h_0}}{2 \,\mathrm{ch}\, \lambda h_0 (\nu \,\mathrm{th}\, \lambda h - \lambda)} \to \frac{\nu+\lambda}{\nu-\lambda}, \quad \frac{e^{2\lambda_0 h_0}}{2\,\mathrm{ch}^2 \lambda_0 h_0 \left(\frac{\nu h_0}{\mathrm{ch}^2 \lambda_0 h_0} - 1\right)} \to -2,$$

and we obtain formula (II.39) for the circulation value during the motion of a hydrofoil in a fluid of infinite depth.

Let us obtain a functional equation for the function [96
$H(\lambda)$. Multiplying expression (III.63) by $e^{-i\mu z}$ and integrating along the contour C_1 we obtain:

$$H(\mu) = e^{-\mu h}\Big\{\Gamma J_0(2\mu R) - i4\pi R J_\mu\Big\{v_0 + \frac{1}{2\pi}\int_0^\infty J_0(2\lambda R)\times$$

$$\times\Big[H(-\lambda)e^{-\lambda(2h_0-h)} + \frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0 - \lambda\,\mathrm{ch}\,\lambda h_0)}(H(-\lambda)e^{-\lambda(2h_0-h)} -$$

$$- H(\lambda)e^{-\lambda(2h_0-h)} - H(\lambda)e^{\lambda h} + \overline{H}(-\lambda)e^{-\lambda h}\Big]d\lambda\Big\} +$$

$$+ \frac{i\nu}{4}\cdot\frac{J_0(2\lambda_0 R)}{\nu h_0 - \mathrm{ch}^2\lambda_0 h_0}[H(-\lambda_0)e^{-\lambda_0(2h_0-h)} + H(\lambda_0)e^{\lambda_0[2h_0-h]} -$$

$$- \overline{H}(\lambda_0)e^{\lambda_0 h} + \overline{H}(-\lambda_0)e^{-\lambda_0 h_0}\Big]\Big\}J_1(2\mu R) +$$

$$+ \frac{1}{4\pi^2 i}\int_0^\infty\Big[\overline{H}(-\lambda)N_0(-\lambda,\mu)e^{-\lambda(2h_0+h)} + \frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0 - \lambda\,\mathrm{ch}\,\lambda h_0)}\times$$

$$\times(\overline{H}(-\lambda)N_0(-\lambda,\mu)e^{-\lambda(2h_0-h)} - \overline{H}(\lambda)N_0(\lambda,\mu)e^{\lambda(2h_0-h)} -$$

$$- \overline{H}(\lambda)N_0(-\lambda,\mu)e^{\lambda h} + \overline{H}(-\lambda)N_0(\lambda,\mu)e^{-\lambda h})\Big]d\lambda\times$$

$$\times\frac{\nu}{8\pi(\nu h - \mathrm{ch}^2\lambda_0 h_0)}[\overline{H}(-\lambda_0)N_0(-\lambda_0,\mu)e^{-\lambda_0(2h_0-h)} +$$

$$+ \overline{H}(\lambda_0)N_0(\lambda_0,\mu)e^{\lambda_0(2h_0-h)} - H(\lambda_0)N_0(-\lambda_0,\mu)e^{\lambda_0 h} +$$

$$+ H(-\lambda_0)N_0(\lambda_0,\mu)e^{-\lambda_0 h}\Big] + \frac{1}{4\pi^2 i}\int_0^\infty\Big[H(-\lambda)N_1(-\lambda,\mu)e^{-\lambda(2h_0-h)} +$$

$$+ \frac{(\nu+\mu)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0 - \lambda\,\mathrm{ch}\,\lambda_1 h_0)}(H(-\lambda)N_1(-\lambda,\mu)e^{-\lambda(2h_0-h)} -$$

$$- H(\lambda)N_1(-\lambda,\mu)e^{-\lambda(2h_0-h)} - H(\lambda)N_1(\lambda,\mu)e^{\lambda(2h_0-h)} -$$

$$- \overline{H}(\lambda)N_1(-\lambda,\mu)e^{\lambda h} + \overline{H}(-\lambda)N_1(\lambda,\mu)e^{-\lambda h})\Big]d\lambda +$$

$$+ \frac{\nu}{8\pi(\nu h - \mathrm{ch}^2\lambda_0 h_0)}[H(-\lambda_0)N_1(-\lambda_0,\mu)e^{-\lambda_0(2h_0-h)} +$$

$$+ H(\lambda_0) N_1(\lambda_0, \mu) e^{\lambda_0(2h_0-h)} - \overline{H}(\lambda_0) N_1(-\lambda_0, \mu) e^{\lambda_0 h} +$$ [97

$$+ \overline{H}(-\lambda_0) N_1(\lambda_0, \mu) e^{-\lambda_0 h}]\Big\}. \qquad \text{(III.65)}$$

However, the co-factor $yJ_1(2\mu R)$ is equal to the value of circulation in terms of c. Then, the functional equation will be in the form

$$H(\mu) = e^{-\mu h}\Big\{\Gamma[J_0(2\mu R) - iJ_1(2\mu R)] +$$

$$+ \frac{1}{4\pi^2 i}\int_0^\infty \Big[\overline{H}(-\lambda) Ne(-\lambda, \mu) e^{-\lambda(2h_0+h)} + \frac{(\nu+\lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh}\lambda h_0 - \lambda \operatorname{ch}\lambda h_0)} \times$$

$$\times (\overline{H}(-\lambda) N_0(-\lambda, \mu) e^{-\lambda(2h_0-h)} - \overline{H}(\lambda) N_0(\lambda, \mu) e^{\lambda(2h_0-h)} -$$

$$- H(\lambda) N_0(-\lambda, \mu) e^{\lambda h} +$$

$$+ H(-\lambda) N_0(\lambda, \mu) e^{-\lambda h})\Big] d\lambda - \frac{\nu}{8\pi(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} \times$$

$$\times [\overline{H}(-\lambda_0) N_0(-\lambda_0, \mu) e^{-\lambda_0(2h_0-h)} + \overline{H}(-\lambda_0) N_0(\lambda_0, \mu) e^{\lambda(2h_0-h)} -$$

$$- H(\lambda_0) N_0(-\lambda_0, \mu) e^{\lambda_0 h} +$$

$$+ H(-\lambda_0) N_0(\lambda_0, \mu) e^{-\lambda_0 h}] + \frac{1}{4\pi^2 i}\int_0^\infty \Big[H(-\lambda) N_1(-\lambda, \mu) e^{-\lambda(2h_0-h)} +$$

$$+ \frac{(\nu+\lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh}\lambda h_0 - \lambda \operatorname{ch}\lambda h_0)} (H(-\lambda) N_1(-\lambda, \mu) e^{-\lambda(2h_0-h)} -$$

$$- H(\lambda) N_1(\lambda, \mu) e^{\lambda(2h_0-h)} - \overline{H}(\lambda) N_1(-\lambda, \mu) e^{\lambda h} +$$

$$+ \overline{H}(-\lambda) N_1(-\lambda, \mu) e^{-\lambda h})\Big] d\lambda + \frac{\nu}{8\pi(\nu h - \operatorname{ch}^2(\lambda_0, h_0))} \times$$

$$\times [H(-\lambda_0) N_1(-\lambda_0, \mu) e^{-\lambda_0(2h_0-h)} + H(\lambda_0) N_1(\lambda_0, \mu) e^{\lambda_0(2h_0-h)} -$$

$$- \overline{H}(\lambda_0) N_1(-\lambda_0, \mu) e^{\lambda_0 h} + \overline{H}(-\lambda_0, \mu) e^{-\lambda_0 h}]\Big\}. \qquad \text{(III.66)}$$

Solution of the functional equation gives the exact value of the $H(\mu)$ function. Let us determine the approximate value of function $H(\mu)$ by using the first two terms in equation (III.66). In this case function $H(\mu)$ will be determined in the following way:

$$\Gamma = \frac{\Gamma_\infty}{1+F}, \qquad \text{(III.67)}$$

where [98

$$F = -2R\int_0^\infty J_0(2\lambda R)J_1(2\lambda R)\left[e^{-2\lambda(h_0-h)} + \frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0 - \lambda\,\mathrm{ch}\,\lambda h_0)}\times\right.$$
$$\left.\times(e^{-2\lambda(h_0-h)} + e^{2\lambda(h_0-h)} - 2)\right]d\lambda - \frac{\pi R\nu J_0^2(2\lambda_0, R)}{\nu h_0 - \mathrm{ch}^2\lambda_0 h_0}(e^{-2\lambda_0(h_0-h)} +$$
$$+ e^{2\lambda_0(h_0-h)} - 2).$$

In determining function H(λ) by using the complex velocity of the hydrofoil motion in an infinite flow, circulation Γ will be determined from the formula

$$\Gamma = \Gamma_\infty(1-F). \tag{III.68}$$

For further computations, it is convenient to transform formulas. By taking into consideration the identity

$$\frac{1}{1+e^{-2\lambda h_0}} = \sum_{k=0}^{\infty}\left(e^{-4h_0 k\lambda} - e^{-4\left(k+\frac{1}{2}\right)h_0\lambda}\right), \tag{III.68a}$$

we can write

$$P = \varrho v\Gamma - \frac{\varrho\Gamma_\infty^2}{2\pi}\int_0^\infty\left[|H(-\nu)|^2 e^{-2\lambda h_0}\times\right.$$
$$\left(1 - \frac{e^{-2\lambda h_0}v(\mathrm{th}\,\lambda h_0 + 1)}{\lambda\left(1 - \frac{\nu}{\lambda}\mathrm{th}\,\lambda h_0\right)}\right) + |H(\lambda)|^2\left.\left(1 + \frac{\nu(\mathrm{th}\,\lambda h_0 + 1)}{\lambda\left(1 - \frac{\nu}{\lambda}\mathrm{th}\,\lambda h_0\right)}\right)\right]\times$$
$$\times\sum_{k=0}^{\infty}\left(e^{-4h_0\lambda k} - e^{-4h_0\lambda\left(k+\frac{1}{2}\right)}\right)d\lambda + \nu p\,\frac{\mathrm{Im}[H(\lambda_0)H(-\lambda_0)]}{2(\nu h_0 - \mathrm{ch}^2\lambda_0 h_0)}. \tag{III.69}$$

For the N. Ye. Zhukovskiy hydrofoil

$$P = \varrho v_0\Gamma - \frac{\varrho\Gamma_\infty^2}{2\pi}\int_0^\infty\left\{[J_0^2(2\lambda R) + J_1^2(2\lambda R)]\left[e^{-2\lambda(h_0-h)} + e^{-2\lambda h} +\right.\right.$$
$$\left.+\frac{\nu(\mathrm{th}\,\lambda h_0 + 1)}{\lambda\left(1 + \frac{\nu}{\lambda}\mathrm{th}\,\lambda h_0\right)}(e^{-2\lambda h} + e^{-2\lambda(2h_0-h)})\right]\times$$
$$\left.\times\sum_{k=0}^{\infty}\left(e^{-4h_0\lambda k} - e^{-4h_0\left(k+\frac{1}{2}\right)\lambda}\right)\right\}d\lambda. \tag{III.70}$$

At this stage, computations with the aid of these [99

formulas do not present any difficulties.

Introducing functions $A_{nm}(x)$ and $B_{nm}(x)$ in the form of expressions

$$A_{nm}(x) = R\int_0^\infty e^{-2\lambda x} J_n(2\lambda R)\, J_m(2\lambda R)\, d\lambda,$$

$$B_{nm}(x) = \int_0^\infty e^{-2\lambda x} \frac{(\operatorname{th}\lambda h_0 + 1)\, J_n(2\lambda R)\, J_m(2\lambda R)}{\lambda\left(1 - \frac{\nu}{\lambda}\operatorname{th}\lambda h_0\right)}\, d\lambda, \qquad \text{(III.71)}$$

we obtain:

$$P = \varrho v_0 \Gamma - \frac{\varrho\Gamma_\infty^2}{2\pi R}\Big[\sum_{k=0}^{\infty} A_{00}\Big[-\bar h + \Big(k+\frac{1}{2}\Big)2\bar h_0\Big] +$$
$$+ A_{11}\Big[-\bar h + \Big(k+\frac{1}{2}\Big)2\bar h_0\Big] - A_{00}[\bar h + 2(k+1)\bar h_0] -$$
$$- A_{11}[-\bar h + 2(k+1)\bar h_0] + A_{11}(\bar h + 2kh_0) - A_{00}\Big[\bar h + 2\Big(k+\frac{1}{2}\Big)h_0\Big] -$$
$$- A_{11}\Big[\bar h + 2\Big(k+\frac{1}{2}\Big)\bar h_0\Big] + \frac{\omega}{4}\sum_{k=0}^{\infty} - B_{00}\Big[-\bar h + 2\Big(k+\frac{1}{2}\Big)\times$$
$$\times \bar h_0\Big] - B_{11}[-\bar h + 2(k+1)\bar h_0] + B_{00}[-\bar h + 2(k+2)\bar h_0] +$$
$$+ B_{11}[-\bar h + 2(k+2)\bar h_0 + B_{00}(+\bar h + 2k\bar h_0) + B_{11}(\bar h + 2k\bar h_0) -$$
$$- B_{00}\Big[\bar h + 2\Big(k+\frac{1}{2}\Big)\bar h_0\Big] - B_{11}\Big[\bar h + 2\Big(k+\frac{1}{2}\Big)\bar h_0\Big]; \qquad \text{(III.72)}$$

$$F = -2\sum_{k=0}^{\infty}\Big\{A_{01}\Big[-\bar h + \Big(k+\frac{1}{2}\Big)\frac{t}{2}\Big] - A_{01}\Big[-\bar h + (k+1)\frac{t}{2}\Big] -$$
$$- A_{01}\Big(\bar h + k\frac{t}{2}\Big) + A_{01}\Big[\bar h + \Big(k+\frac{1}{2}\Big)\frac{t}{2}\Big] + 2A_{01}\Big[\Big(k+\frac{1}{2}\Big)\frac{t}{2}\Big] -$$
$$- 2A_{01}\Big[(k+1)\frac{t}{2}\Big] - \frac{\omega}{2}\sum_{k=0}^{\infty}\Big\{B_{01}\Big[-\bar h + (k+1)\frac{t}{2}\Big] -$$
$$- B_{01}\Big[-\bar h + \Big(k+\frac{3}{2}\Big)\frac{t}{2}\Big] + B_{01}(\bar h + 2k\bar h_0) -$$ [100
$$- B_{01}\Big[\bar h + \Big(k+\frac{1}{2}\Big)\frac{t}{2}\Big]\Big\} - 2B_{01}\Big[\Big(k+\frac{1}{2}\Big)\frac{t}{2}\Big] + 2B_0\Big[(k+1)\frac{t}{2}\Big] -$$

$$-\frac{\pi\omega J_0^2(\lambda_0)}{\omega\bar{h}_0-\mathrm{ch}^2\lambda_0\bar{h}_0}\left(e^{-2\lambda_0(h_0-\bar{h})}+e^{2\lambda_0(h_0-\bar{h})}\right). \qquad \text{(III.73)}$$

Functions $A_{nm}(x)$ may be represented by the expansions in powers of the parameter:

$$\tau_x=\sqrt{4\bar{x}^2+1}-2\bar{x} \quad \left(\bar{x}=\frac{x}{4R}\right),$$

$$A_{00}=\sum_{k=0}^{\infty}\tau_x^{2n+1}\sum_{s=0}^{n}\frac{(-1)^s 2s!\,(s+n)!}{s!\,s!\,s!\,s!\,(n-s)!\,2^{2s+1}}, \qquad \text{(III.74)}$$

$$A_{01}=\sum_{n=0}^{\infty}\tau_x^{2n+2}\sum_{s=0}^{n}\frac{(-1)^s(2s+1)!\,(s+n+1)!}{s!\,s!\,(s+1!)\,(s+1)!\,(n+s)!\,2^{2s+2}},$$

$$A_{11}=\sum_{n=0}^{\infty}\tau_x^{2n+3}\sum_{s=0}^{n}\frac{(-1)^s(2s+1)!\,(s+n+2)!}{s!\,(s+1)!\,(s+1)!\,(s+2)!\,(n-s)!\,2^{2s+3}}.$$

These expansions converge rapidly and for practical purposes it is sufficient to use a few terms only (no more than three):

$$A_{00}=\frac{1}{2}\tau_x+\frac{1}{8}\tau_x^5+\frac{9}{128}\tau_x^9+\ldots,$$

$$A_{01}=\frac{1}{4}\tau_x^2-\frac{1}{10}\tau_x^4+\frac{1}{16}\tau_x^6+\ldots,$$

$$A_{11}=\frac{1}{4}\tau_x^3+\frac{3}{32}\tau_x^7+\frac{15}{256}\tau_x^{11}+\ldots.$$

Functions $B_{nm}(x)$ may be represented in the form

$$B_{nm}=\sum_{k=0}^{\infty}\tau_x^{n+m+2k}\sum_{s=0}^{k}\frac{(-1)^k(m+n+2s)!\,F(n+m+k+s-1)}{(m+s)!\,(n+s)!\,(n+m+s)!\,(k-s)!\,2^{n+m+2s}},$$

where [101

где

$$F_p=\int_0^{\infty}e^{-u}\frac{(\mathrm{th}\,ua+1)}{1-\frac{\omega}{2}\tau\frac{\mathrm{th}\,ua}{u}}\,du;$$

$$a=2\tau_x(\bar{h}_0+\bar{h}),\quad \omega_\tau=\frac{1}{Fr_1^2\tau_x}=\frac{\omega_b}{\tau_x}.$$

Let us examine a motion at the supercritical velocity. Function F_p may be approximately determined by approximating th ua within the intervals $0 < 0u < 1{,}5$ $\operatorname{th} ua = ua - \frac{1}{3}(ua)^3 + \frac{2}{15}(ua)^5$ and $\operatorname{th} ua > 1{,}5$ $\operatorname{th} ua = 1$. We obtain:

$$
\begin{aligned}
F_p = \frac{1}{1-\omega_h}\Big\{ & p! - e^{-\frac{1,5}{a}} \sum_{m=0}^{p} \frac{p!}{m!}\left(\frac{1,5}{a}\right)^m + a(p+1)! - \\
& - a e^{-\frac{1,5}{a}} \sum_{m=0}^{p+1} \frac{(p+1)!}{m!}\left(\frac{1,5}{a}\right)^m - \frac{1}{3} a^3 (p+3)! + \\
& + \frac{1}{3} a^3 e^{-\frac{1,5}{a}} \sum_{m=0}^{p+3} \frac{(p+3)!}{m!}\left(\frac{1,5}{a}\right)^m + \frac{2}{15} a^5 (p+5)! - \\
& - \frac{2}{15} a^5 e^{-\frac{1,5}{a}} \sum_{m=0}^{p+5} \frac{(p+5)!}{m!}\left(\frac{1,5}{a}\right)^m + \omega_b \Big[-\frac{1}{3} a^3 (p+2)! + \\
& + \frac{1}{3} a^3 e^{-\frac{1,5}{a}} \sum_{m=0}^{p+2} \frac{(p+2)!}{m!}\left(\frac{1,5}{a}\right)^m + \frac{2}{25} a^4 (n+4)! - \\
& - \frac{2}{25} a^4 e^{-\frac{1,5}{a}} \sum_{m=0}^{p+4} \frac{(p+4)!}{m!}\left(\frac{1,5}{a}\right)^m - \frac{1}{3} a^3 (p+3)! - \\
& - \frac{1}{3} a^3 e^{-\frac{1,5}{a}} \sum_{m=0}^{p+3} \frac{(p+3)!}{m!}\left(\frac{1,5}{a}\right)^m + \frac{2}{25} a^5 (p+5)! - \\
& - c^{-\frac{1,5}{a}} \sum_{m=0}^{p+5} \frac{(p+5)!}{m!}\left(\frac{1,5}{a}\right)^m \Big] + 2\Big\{ e^{-\frac{1,5}{a}} \Big[\sum_{m=0}^{p} \frac{p!}{m!}\left(\frac{1,5}{a}\right)^m + \\
& + \frac{\omega_b}{2\tau_x} \sum_{m=0}^{p-1} \frac{(p-1)!}{m!}\left(\frac{1,5}{a}\right)^m + \left(\frac{\omega_b}{2\tau_x}\right)^2 \sum_{m=0}^{p-2} \frac{(p-2)!}{m!}\left(\frac{1,5}{a}\right)^m + \qquad \text{[102} \\
& + \left(\frac{\omega}{2\tau_x}\right)^p \Big] - E_i\left(-\frac{1,5}{a}\right)\left(\frac{\omega_b}{2\tau_x}\right)^{p+1} + e^{-\frac{1,5}{a}} \frac{a}{1,5}\left(\frac{\omega}{2\tau_x}\right)^{p+2} + \\
& + E_i\left(-\frac{1,5}{a}\right)\left(\frac{\omega}{2\tau_x}\right)^{p+2} + \dots \Big\}. \qquad \text{(III.75)}
\end{aligned}
$$

The series in expression (III.75) converge rapidly and, for practical purposes, it is sufficient to retain

two terms in each series:

$$
\begin{aligned}
F = & -2[A_{01}(\bar{h}_1) - A_{01}(\bar{h} + 2\bar{h}_1) - A_{01}(\bar{h}) + A_{01}(2\bar{h} + \bar{h}_1) + \\
& + 2A_{01}[\bar{h}_1 + \bar{h}] - 2A_{01}[2(\bar{h}_1 + \bar{h})] + A_{01}(3\bar{h}_1 + 2\bar{h}) - A_{01}(3\bar{h} + \\
& + 4\bar{h}_1) - A_{01}(3\bar{h} + 2\bar{h}_1) + A_{01}(4\bar{h} + 3\bar{h}_1)] + 2A_{01}[3(\bar{h}_1 + \bar{h})] - 2A_{01}[4(\bar{h} + \bar{h}_1)] - \\
& - \frac{\omega}{2}\{B_{01}(\bar{h} + 2\bar{h}_1) - B_{01}(2\bar{h} + 3\bar{h}_1) + B_{01}(\bar{h}) - B_{01}(2\bar{h} + \bar{h}_1) - \\
& - 2B_{01}(\bar{h}_1 + \bar{h}) + 2B_{01}[2(\bar{h}_1 + \bar{h})] + B_{01}(3\bar{h} + 4\bar{h}_1) - \\
& - B_{01}(4\bar{h} + 5\bar{h}_1) + B_{01}(3\bar{h} + 2\bar{h}_1) - B_{01}(4\bar{h} + 3\bar{h}_1) - \\
& - 2B_{01}[3(\bar{h}_1 + \bar{h})] + 2B_0[4(\bar{h} + \bar{h}_1)]\}.
\end{aligned} \tag{III.76}
$$

For carrying out the approximate computations we may, for functions $A_{01}(\bar{x})$ and $B_{01}(\bar{x})$, assume the following:

$$
A_{01}(\bar{x}) = \frac{1}{4}\tau_x^2 + \dots \quad B_{01}(\bar{x}) = \frac{1}{2}\tau_x F_0 + \dots
$$

$$
\begin{aligned}
F_0 = & \frac{1}{1-\omega_h}\Big[1 - e^{-\frac{1,5}{a}} + a\Big(1 - e^{-\frac{1,5}{a}} - e^{-\frac{1,5}{a}}\frac{1,5}{a}\Big) - \\
& - \frac{a^3}{3}\Big(6 - 6e^{-\frac{1,5}{a}} - \frac{9}{a}e^{-\frac{1,5}{a}}\frac{6,75}{a^2}e^{-\frac{1,5}{a}} - \frac{3,375}{a^3}e^{-\frac{1,5}{a}}\Big) + \dots\Big] + \\
& + \omega_n\Big[-\frac{1}{3}a^2\Big(2 - 2e^{-\frac{1,5}{a}} - \frac{3}{a}e^{-\frac{1,5}{a}}\frac{4,5}{a^2}e^{-\frac{1,5}{a}}\Big) + \dots + \\
& + 2\Big[e^{-\frac{1,5}{a}} - E_i\Big(-\frac{1,5}{a}\Big)\frac{1}{2}\omega_\tau + \frac{1}{4}\omega_\tau^2\frac{1,5}{a} + \\
& + E_1\Big(-\frac{1,5}{a}\Big)\frac{1}{4}\omega_\tau^2\Big] + \dots
\end{aligned} \tag{III.77}
$$

By taking into account only two terms of the expression for a hydrofoil section, we have, in the first approximation, with Fr $\rightarrow \infty$, the following: [103

$$
F = \frac{1}{2}(\tau^2 - \tau_1^2), \tag{III.78}
$$

$$
\tau = \sqrt{4\bar{h}^2 + 1} - 2\bar{h}, \quad \tau_1 = \sqrt{4\bar{h}_1^2 + 1} - 2\bar{h}_1.
$$

To illustrate the effect of shallow water on the hydrofoil lifting force, Figure 11 shows curves for the relationships $\gamma = f(h_0, h)$, which were determined from formulas (III.68) and (III.76).

With $Fr_h \rightarrow \infty$, the problem of the hydrofoil motion in

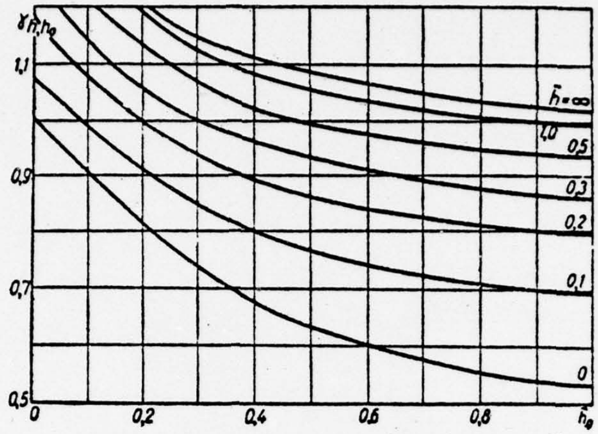

Fig. 11

a fluid of finite depth may be solved by the approximate method discussed in Chapter II. With $Fr_h \to \infty$, the boundary condition on the free surface is simplified and is in the form

$$\varphi_x = 0 \qquad \text{(III.79)}$$

or φ = const when $y = 0$.

A function, which has special points in a band $0 > > y > -h$ and satisfies boundary conditions (III.79), may be easily obtained with the aid of specular reflections. Let us illustrate this by considering vortex moving with a circulation Γ and located at a point $\zeta_0 = -ih$.

The complex potential of the vortex will be sought in the form

$$W(z) = \frac{\Gamma}{2\pi i} \ln(z + ih) + F(z),$$

where $F(z)$ is a function, analytical within the band $0 > > y > -h$.

In order to satisfy boundary condition (III.1) it is necessary to place a vortex with an intensity $-\Gamma$ at a point $\zeta = -i(h_0 + h_1)$ and, to satisfy the condition (III. [104
79), a vortex with an intensity Γ should be placed at a point $y = ih$.

In order to satisfy these boundary conditions caused by the imaginary vortexes it is necessary to place, at the

appropriate points, the imaginary vortexes which are the specular reflections of the former imaginary vortexes. In that way function F(z) may be determined as a characteristic function of an infinite chain of imaginary vortexes. By determining function F(z) in this way we obtain

$$F(z)=\sum_{k=1}^{\infty}\frac{\Gamma}{2\pi i}\ln(z+ih+ikt)+\frac{\Gamma}{2\pi i}\sum_{k=0}^{\infty}\{\ln(z-ih-ikt)+$$
$$+\ln[z+i(h+2h_1)+ikt]-\ln[z-i(n-2h_1)-ikt]-$$
$$-\ln[z+i(3h+2h_1)+ikt]-\ln[z-i(3h+2h_0)-ikt]-$$
$$-\ln[z+i(3h+4h_1)-ikt]+\ln[z-i(3h+rh_1)-ikt]\}.$$

Then, the complex velocity of the vortex in a fluid of finite depth may be written in the form

$$v(z)=\frac{\Gamma}{2\pi i(z+in)}+\frac{\Gamma}{2\pi i}\sum_{k=0}^{\infty}\left\{\frac{1}{(z+ih+i(k+1)t)}+\right.$$
$$+\frac{1}{(z-ih-ikt)}-\frac{1}{[z+i(h+2h_1)+ikt]}-\frac{1}{[z-i(h+2h_1)-ikt]}-$$
$$-\frac{1}{[z+i(3h+2h_1)+ikt]}-\frac{1}{[z-i(3h+2h_1)-ikt]}+$$
$$\left.+\frac{1}{[z+i(3h+4h_1)+ikt]}+\frac{1}{[z-i(3h+4h_1)-ikt]}\right\}. \quad \text{(III.80)}$$

This expression may be written in a simplified form with the aid of the hyperbolic functions, using the following representation:

$$\operatorname{cth} x=\frac{1}{x}+2x\sum_{k=1}^{\infty}\frac{1}{x^2+k^2\pi^2};$$

$$v(z)=\frac{\Gamma}{2it}\left\{\operatorname{cth}\frac{\pi(z+ih)}{t}+\operatorname{cth}\frac{\pi(z-ih)}{t}-\right.$$
$$\left.-\operatorname{cth}\frac{\pi[z+i(h+2h_1)]}{t}-\operatorname{cth}\frac{\pi[z-i(h+2h_1)]}{t}\right\}. \quad \text{(III.81)}$$

The force of the flow may be determined from the formula [105

$$P=\varrho\Gamma\left\{v_0-\frac{\Gamma}{2t}\left[\operatorname{ctg}\frac{2\pi h}{t}+\operatorname{ctg}\frac{2\pi h_0}{t}\right]\right\},$$

or

$$P = \varrho\Gamma\left(v_0 - \frac{\Gamma}{4h_0 \sin\frac{\pi h}{h_0}}\right). \qquad \text{(III.82)}$$

For $Fr_h \to \infty$ the expression (III.82) may be transformed into the form of (III.21). In the same way, we obtain for the source

$$\begin{aligned} v(z) = {} & \frac{Q}{2\pi(z+ih)} + \frac{Q}{2\pi}\sum_{k=0}^{\infty}\left\{\frac{1}{[z+ih+i(k+1)t]} - \right. \\ & - \frac{1}{[z-ih-ikt]} + \frac{1}{[z+i(h+2h_1)+ikt]} - \\ & - \frac{1}{[z-i(h+2h_1)-ikt]} - \frac{1}{[z+i(3h+2h_1)+ikt]} - \\ & - \frac{1}{[z-i(3h+2h_1)-ikt]} - \frac{1}{[z+i(3h+4h_1)+ikt]} - \\ & \left. - \frac{1}{[z-i(4h+4h_1)-ikt]}\right\}. \end{aligned} \qquad \text{(III.83)}$$

Determining the complex velocity by the first approximation we obtain

$$\begin{aligned} v_2(z) = {} & \frac{1}{2\pi i} \sum_{k=0}^{\infty}\left[\int_{C_1}\frac{v_\infty(\zeta)\,d\zeta}{z-\zeta+ikt} + \int_{C_1}\frac{\overline{v_\infty(\zeta)}\,d\bar\zeta}{z-\bar\zeta-ikt} - \right. \\ & - \int_{C_1}\frac{\overline{v_\infty(\zeta)}\,d\bar\zeta}{z-\bar\zeta+i(2h+2h_1+kt)} - \int_{C}\frac{v_\infty(\zeta)\,d\zeta}{z-\zeta-i(2h+2h_1+kt)} - \\ & - \int_{C}\frac{v_\infty(\zeta)\,d\zeta}{z-\zeta+i(2h+2h_1+kt)} - \int_{C_1}\frac{\overline{v_\infty(\zeta)}d\bar\zeta}{z-\bar\zeta-i(2h+2h_1+kt)} + \\ & \left. + \int_{C_1}\frac{\overline{v_\infty(\zeta)}\,d\bar\zeta}{z-\bar\zeta+i(4h+4h_1+kt)} + \int_{C_1}\frac{v_\infty(\zeta)\,d\zeta}{(z-\zeta-i(k+1)t}\right], \end{aligned} \qquad \text{(III.84)}$$

from which, using formula (II.69), we obtain for the lifting force [103] [106

$$\begin{aligned} P = {} & \varrho v_0\Gamma_\infty - \frac{\varrho}{2\pi}\left[\int_0 \left[\,|H_1(\lambda)|^2 \sum_{k=0}^{\infty}\left(e^{-kt\lambda} - e^{-\left(k+\frac{1}{2}\right)t\lambda}\right) + \right.\right. \\ & \left.\left. + |H_1(-\lambda)|^2\left(\sum_{k=0}^{\infty}\left(e^{-\left(k+\frac{1}{2}\right)t\lambda} - e^{-(k+1)t\lambda}\right)\right]d\lambda.\right. \end{aligned} \qquad \text{(III.85)}$$

Let us examine the motion of a cylinder with radius R and circulation Γ along the contour. Function $H(\lambda)$ will be determined from formula (II.66).

From formula (III.85) we obtain after transformations

$$P = \varrho v_0 \Gamma - \frac{\varrho \Gamma^2}{4h_0 \sin \frac{\pi h}{h_0}} - \frac{\varrho v_0 R^2 \Gamma}{t^2} \left\{ \zeta\left(2, \frac{2h}{t}\right) - \right.$$

$$- \zeta\left(2, \frac{2h_1}{t}\right) - \zeta\left[2, \frac{(h+2h_1)}{t} + \zeta\left(2, \frac{(h+4h_1)}{t}\right)\right]\Bigg\} - \frac{4\pi v_0^2 R^4}{t^3} \times$$

$$\times \left[\zeta\left(3, \frac{2h}{t}\right) + \zeta\left(3, \frac{3h_1}{t}\right) - \zeta\left(3, \frac{4h+h_1}{t}\right) - \zeta\left(3, \frac{2h+4h_1}{t}\right)\right], \quad \text{(III.86)}$$

where $\zeta(z, g) = \sum_{n=0}^{\infty} \frac{1}{(g+n)^2}$ - zeta-Rieman function [23]. Below are the final results for several hydrofoil shapes.

<u>Thin plate</u>:

$$\frac{\gamma}{h_0} = \frac{P_h}{P_\infty} = 1 - 4\pi \sin \alpha_n A_1 - \frac{\cos \alpha}{2} A_2; \quad \text{(III.87)}$$

$$A_1 = \frac{1}{4\sqrt{2}\pi}\left[\frac{1}{\sqrt{8\bar{h}^2+1}} F_1(h) + \frac{1}{\sqrt{8\bar{h}_1^2+1}} F_1(\bar{h}_1) + \right.$$

$$+ \sum_{k=1}^{\infty}\left[\frac{1}{\sqrt{2(2\bar{h}+kt)^2+1}} F_1\left(\bar{h} + \frac{kt}{2}\right) + \right.$$

$$\left. + \frac{1}{\sqrt{2(2\bar{h}_0+k\bar{t})^2+1}} F_1\left(\bar{h}_0 + \frac{k\bar{t}}{2}\right) - \frac{F_1\left(-h_0 + \frac{k\bar{t}}{2}\right)}{\sqrt{2(-2\bar{h}_0+k\bar{t})^2+1}}\right]; \quad \text{(III.88)}$$

$$A_2 = \frac{4h_0}{\sqrt{2}\sqrt{1+8\bar{h}_0^2}} F_1(h_0) + \frac{4\bar{h}}{\sqrt{2}\sqrt{8\bar{h}^2+1}} F_1(\bar{h}) +$$ [107

$$+ \sum_{k=1}^{\infty}\left[-\frac{4\bar{h}+2k\bar{t}}{\sqrt{2}\sqrt{1+(2\bar{h}+k\bar{t})^2}} F_1\left(\bar{h} + \frac{k\bar{t}}{2}\right) + \right.$$

$$+ \frac{4\bar{h}_0 + 2k\bar{t}}{\sqrt{2}\sqrt{1+2(2\bar{h}_0+kt)^2}} F_1\left(h_0 + \frac{kt}{2}\right) + \frac{-4\bar{h}_0 + 2kt}{\sqrt{2}\sqrt{1+2(-2\bar{h}_0+kt)^2}} \times$$

$$\times F_1\left(-h_0 + \frac{kt}{2}\right) + \frac{-4\bar{h}+2kt}{\sqrt{2}} \times$$

$$\left. \times \frac{1}{\sqrt{1+2(-2\bar{h}+kt)^2}} F_1\left(-\bar{h} + \frac{kt}{2}\right)\right]. \quad \text{(III.89)}$$

In the first approximation, A_1 and A_2 are determined from the following formulas:

$$A_1 = \frac{1}{4\sqrt{2}\pi}\left[\frac{1}{\sqrt{8\bar{h}^2+1}}F_1(\bar{h}) + \frac{1}{\sqrt{8\bar{h}_0^2+1}}F_1(\bar{h}_1)\right];$$

$$A_2 = \left[\frac{4\bar{\pi}_1}{\sqrt{2}}\cdot\frac{1}{\sqrt{8\bar{h}_1^2+1}}F_1(\bar{h}_1) - \frac{4\bar{h}}{\sqrt{2}}\cdot\frac{F_1(\bar{h})}{\sqrt{8\bar{h}^2+1}}\right]. \qquad \text{(III.90)}$$

It is not difficult to note that when determining the relative circulation using formula (III.68), formulas (III.68) and (III.87) yield similar results.

<u>Thin hydrofoil section</u>:

$$\gamma_{\bar{h}} = 1 - \frac{4\pi\sin(\alpha_0+\alpha_\kappa)}{\cos\alpha_0}A_1 - \frac{\cos\alpha_\kappa}{2\cos 2\alpha_0}A_2. \qquad \text{(III.91)}$$

<u>N. Ye. Zhukovskiy's hydrofoil</u>:

$$\gamma_{\bar{h}} = 1 - \frac{4\pi\sin(\alpha+\alpha_\kappa)}{\cos\alpha_0}A_1 - \frac{(1+\mu)^2}{\cos 2\alpha_0}A_2 - \frac{\delta}{(\alpha_0+\alpha_\kappa)}A_3; \qquad \text{(III.92)}$$

$$A_3 = \frac{1}{4\sqrt{2}}\Bigg\{\frac{K\delta}{(8h^2+1)^{3/2}}F_2(\bar{h}) + \frac{(1-K)\delta}{(8\bar{h}_0^2+1)^{3/2}}F_2(\bar{h}_0) +$$

$$+\sum_{k=1}^{\infty}\Bigg[\frac{K\delta}{[2(2\bar{h}+k\bar{t})^2+1]^{3/2}}F_2\left(\bar{h}+\frac{k\bar{t}}{2}\right) +$$

$$+\frac{(1-K)\delta}{[2(2\bar{h}_0+k\bar{t})^2+1]^{3/2}}F_2\left(\bar{h}_0+\frac{k\bar{t}}{2}\right) - \qquad [108$$

$$-\frac{K\delta}{[2(-2\bar{h}_0+k\bar{t})^2+1]^{3/2}}F_2\left(-h_0+\frac{k\bar{t}}{2}\right) +$$

$$+\frac{(1-K)\delta}{[2(-2\bar{h}+k\bar{t})^2+1]^{3/2}}F_2\left(-h+\frac{k\bar{t}}{2}\right)\Bigg]\Bigg\}. \qquad \text{(III.93)}$$

Approximately

$$A_3 = \frac{1}{4\sqrt{2}}\left[\frac{k}{(8\bar{h}^2+1)^{3/2}}F_2(\bar{h}) + \frac{(1-k)}{(8\bar{h}_1^2+1)^{3/2}}F_2(\bar{h}_1)\right]. \qquad \text{(III.94)}$$

Using the expansions in powers of parameter τ, we obtain in the first approximation

$$\gamma_h = 1 - \sin(\alpha_0+\alpha_k)(\tau+\tau_1) - \frac{\cos\alpha_k(1+\mu)^2}{2\cos 2\alpha_0}(\tau^2-\tau_1^2) -$$

$$-\frac{\delta}{a_0+a_k}[K\tau^3+(1.-K)\tau_1^3]. \qquad \text{(III.95)}$$

where $\tau_1=\sqrt{4\bar{h}_1^2+1}-2\bar{h}_1$.

In formulas $F_1(x)=F\left(\frac{1}{4}:\frac{3}{4}:1:\frac{1}{(8x^2+1)^2}\right)$ and $F_2(x)=F\left(\frac{3}{4}:\right.$ $\left.:\frac{5}{4}:2:\frac{1}{(8x^2+1)^2}\right)$ are the hypergeometric functions.

CHAPTER IV. INTERACTION OF HYDROFOILS IN A PLANE-PARALLEL FLOW [109

The problem concerning the interaction of hydrofoils is of interest, first of all, from the point of view of wave drag, since with the improper arrangement of hydrofoils in the system, the system wave drag may become greater than the total wave drag of individual hydrofoils in the system, while with a proper arrangement it may be considerably lower. At certain distances between hydrofoils pulling forces acting on individual hydrofoils may be produced, and, as a result, the total wave drag will be minimum. In addition, simultaneously with the change in wave drag, the lifting force of the system also changes considerably. An approximate solution of the problem concerning the interaction of two thin hydrofoils of infinite span is given by W. H. Isay [194].

In this chapter the general problem of the steady motion of a system of hydrofoils with an infinite span and arbitrary profile is examined.

4.1. Lifting Force and Wave Drag of a Hydrofoil System in a Fluid of Infinite Depth

Let us examine the steady-velocity motion (v_0) of a system consisting of n hydrofoils submerged under a free surface. Let us denote the distance between the conformal center of gravity of a hydrofoil and the non-disturbed free surface by h_j and the distance between the conformal centers of gravity of the i-th and j-th hydrofoils by L_{ij}.

For determining the characteristic function and complex velocity of the flow we will use boundary conditions (I.1) and (2) and the condition describing absence of disturbances in front of the system (33).

Let us take a point z in the lower half-plane and draw two contours C_{1_i} and C_{2_i} in such a way that point z would be located outside of contour C_{1_i} and inside contour [110
C_{2_i}. Contour C_{2_i} may be changed into a contour which would fully envelop the lower half-plane that does not contain any other contours for other hydrofoils in the system. Then, for the complex velocity of the flow we assume the following expression:

$$v(z) = v_1(z) + v_2(z),$$

where $v_1(z)$ - the analytical function outside the C_{1_i} contour;
$v_2(z)$ - the analytical function within the C_{2_i} contour.

Introducing functions $v_{ij}(z)$, which are analytical outside contours C_{ij}, C_{ij}, and $v_{2j}(z)$, which are analytical in the lower half-plane, one may write the expression for the complex velocity in the form

$$v(z) = v_{11}(z) + v_{12}(z) + \ldots + v_{1n}(z) + v_{21}(z) + v_{22}(z) + \ldots + v_{2n}(z). \qquad \text{(IV.1)}$$

From the expression for the complex velocity of the vortex source under the free surface, which satisfies boundary condition (I.1), we will obtain the following

$$v_{1j}(z) = \frac{1}{2\pi i} \int\limits_{C_j} \frac{v(\zeta)}{z-\zeta} d\zeta; \qquad \text{(IV.2)}$$

$$v_{2j}(z) = \frac{1}{2\pi i} \int\limits_{C_{ij}} \overline{v(\zeta)} \left[i \int\limits_0^\infty e^{-i\lambda(z-\bar{\zeta})} \frac{\lambda+\nu}{\lambda-\nu} d\lambda - \pi\nu e^{-i\nu(z-\bar{\zeta})} \right] d\bar{\zeta}. \qquad \text{(IV.3)}$$

Let us write function $H_{ji}(\lambda)$ as follows

$$H_{ji}(\lambda) = e^{i\lambda\zeta_{ji}} \int\limits_{C_j} e^{-i\lambda z} v(z)\, dz, \qquad \text{(IV.4)}$$

where $\zeta_{ij} = L_{ij} - ih_i$.

Then, after computations with the aid of (IV.2), (IV.3) and (IV.4), we obtain

$$(P_{-i}Q)_j = \varrho v_0 \Gamma_j - \frac{\varrho}{2\pi} \left[\sum_{l=0}^{n} \int\limits_0^\infty H_{jj}(\lambda) \overline{H_{jl}(\lambda)} \frac{\lambda+\nu}{\lambda-\nu} d\lambda + \right.$$

$$\left. + \sum_{l=1}^{n}{}' \int\limits_0^\infty H_{jj}^{\mathrm{Sign}(h_j-h_l)}(\lambda) H_{jl}(-\lambda)\, d\lambda \right] - i\varrho\nu \sum_{l=1}^{n} H_{jj}(\nu) \overline{H_{jl}}(\nu),$$

from which, by separating the real and imaginary parts, it is easy to obtain the expression for the lifting force magnitude and wave drag of the hydrofoil:

[111

$$F = \varrho v_0 \Gamma_j - \frac{\varrho}{2\pi} \left\{ \sum_{l=1}^{n} \int\limits_0^\infty [H'_{jj}(\lambda) H'_{jl}(\lambda) + H''_{jj}(\lambda) H''_{jl}(\lambda)] \frac{\lambda+\nu}{\lambda-\nu} d\lambda + \right.$$

$$+\sum_{l=1}^{n}{}' \int_0^{\infty\,\mathrm{Sign}(h_j-h_i)} [H'_{jl}(\lambda)\,H'_{jl}(-\lambda) - H''_{jl}(\lambda)\,H'_{jl}(-\lambda)]\,d\lambda \Big\} -$$

$$-\nu\varrho \sum_{i=1}^{n} [H'_{jl}(\nu)\,H''_{jl}(\nu) - H''_{jl}(\nu)\,H'_{jl}(\nu)], \qquad \text{(IV.5)}$$

$$Q_j = \sum_{i=1}^{n} \nu\varrho\,[H'_{jl}(\nu)\,H'_{jl}(\nu) + H''_{jl}(\nu)\,H''_{jl}(\nu)] -$$

$$-\frac{\varrho}{2\pi}\Big\{\sum \int_0^{\infty} [H'_{jl}(\lambda)\,H''_{jl}(\lambda) - H''_{jl}(\lambda)\,H'_{jl}(\lambda)]\,\frac{\lambda+\nu}{\lambda-\nu}\,d\lambda +$$

$$+\sum_{i=1}^{n}{}' \int_0^{\infty\,\mathrm{Sign}(h_j-h_i)} [H'_{jl}(\lambda)\,H''_{jl}(-\lambda) + H''_{jl}(\lambda)\,H'_{jl}(-\lambda)]\,d\lambda], \qquad \text{(IV.6)}$$

where H'_{ji} and H''_{ji} are real and imaginary parts of the $H_{ij}(\lambda)$ function and the sign ' at the $\sum$ symbol indicates that the summation contains no i = j terms.

From formulas (IV.5) and (IV.6) it is easy to obtain the formulas for P and Q for an isolated hydrofoil moving under a free surface.

The profile of the waves formed during the motion of a hydrofoil system we determine from formula (I.20):

$$\eta = \frac{v_0}{g}\,\mathrm{Re}\,v(x).$$

Let us examine the shape of waves far ahead and far behind the moving system.

From formula (IV.2) it follows:

$$\lim_{|x|\to\infty} \sum_{i=1}^{n} v_{1i}(x) = 0, \quad \lim_{x\to\infty} \sum_{i=1}^{n} v_{2i}(x) = 0,$$

$$\lim_{|x|\to\infty} \sum_{i=1}^{n} v_{3i}(x) = -2i\nu \sum_{i=1}^{n} \overline{H_{ii}(\nu)}\, e^{-i\nu(x-L_{i0})}.$$

Then, for the shape of the free surface [112

$$\eta = \frac{2v_0\nu}{g}\,\mathrm{Im}\left(\sum_{i=1}^{n} \overline{H_{ii}(\nu)}\, e^{-i\nu(x-L_{i0})}\right),$$

or

$$\eta = -\frac{2'}{v_0} \sum_{i=1}^{n} [H'_{ii}(\nu) \sin \nu(x - L_{i0}) + H''_{ii}(\nu) \cos \nu (x - L_{i0})]. \qquad \text{(IV.7)}$$

Condition $\eta = 0$ determines a system during the motion of which the free surface remains horizontal at infinity ahead and behind the system. The total wave drag of this system is equal to zero and such a system is the most advantageous system.

The particular optimum of a system with the change of a number of parameters will be determined by the condition

$$\frac{\partial A}{\partial x_i} = 0, \qquad \text{(IV.8)}$$

where A - amplitude of the waves formed;
x_i - i-th variable parameter.

As an example of the application of formulas (IV.5) and (IV.6) let us examine the problem of motion of two cylinders with radii R_1 and R_2, the depths of submergence h_1 and h_2, and circulations Γ_1 and Γ_2.

Using the complex velocity of the cylinder motion in an infinite flow determined by formula (IV.4) we obtain

$$H_{11}(\lambda) = e^{-\lambda h_1} (\Gamma_1 + 2\pi v_0 \lambda R_1^2);$$
$$H_{22}(\lambda) = e^{-\lambda h_2} (\Gamma_2 + 2\pi v_0 \lambda R_2^2);$$
$$H_{12}(\lambda) = e^{-\lambda (h_1 + iL)} (\Gamma_1 + 2\pi v_0 \lambda R_1^2);$$
$$H_{21}(\lambda) = e^{-\lambda (h_2 - iL)} (\Gamma_2 + 2\pi v_0 \lambda R_2^2). \qquad \text{(IV.9)}$$

Let us examine a case when $h_1 < h_2$. Then, formulas (8) and (9) will acquire the form

$$P_1 = \varrho v_0 \Gamma_1 - \frac{\varrho}{2\pi} \left[\int_0^{\infty} \left(H_{11}^2(\lambda) + H_{11}(\lambda) H'_{21}(\lambda) \right) \frac{\lambda + \nu}{\lambda - \nu} d\lambda - \right.$$
$$\left. - \int_0^{-\infty} H'_{21}(-\lambda) H_{11}(\lambda) d\lambda \right] - \nu \varrho H_{11}(\nu) H''_{21}(\nu), \qquad \text{(IV.10)}$$

$$Q_1 = \nu\varrho [H_{11}^2(\nu) + H_{11}(\nu) H'_{21}(\nu)] - \frac{\varrho}{2\pi}\int_0^{\infty} H_{11}(\lambda) H''_{21}(\lambda) \frac{\lambda+\nu}{\lambda-\nu}\, d\lambda +$$ [113

$$+ \frac{\varrho}{2\pi}\int_0^{-\infty} H_{11}(\lambda) H''_{21}(-\lambda)\, d\lambda. \tag{IV.11}$$

$$P_2 = \varrho v_0 \Gamma_2 - \frac{\varrho}{2\pi}\left\{\int_0^{\infty} [H_{22}^2(\lambda) + H_{22}(\lambda) H'_{12}(\lambda)] \frac{\lambda+\nu}{\lambda-\nu}\, d\lambda -\right.$$

$$\left. - \int_0^{\infty} H_{22}(\lambda) H'_{12}(-\lambda)\, d\lambda \right\} - \nu\varrho H_{22}(\nu) H''_{12}(\nu). \tag{IV.12}$$

$$Q_2 = \nu\varrho [H_{22}^2(\nu) + H_{22}(\nu) H'_{21}(\nu)] - \frac{\varrho}{2\pi}\int_0^{\infty} H_{22}(\lambda) H''_{12}(\lambda) \frac{\lambda+\nu}{\lambda-\nu}\, d\lambda +$$

$$+ \frac{\varrho}{2\pi}\int_0^{\infty} H_{22}(\lambda) H''_{12}(-\lambda)\, d\lambda. \tag{IV.13}$$

After performing calculations we obtain:

$$P_1 = P_{1\infty} - \frac{\varrho\Gamma_1\Gamma_2 h_{cp}}{\pi(4h_{cp}^2 + L^2)} + \frac{\varrho v_0(\Gamma_1 R_2^2 + \Gamma_2 R_1^2)(L^2 - 4h_{cp}^2)}{(L^2 + 4h_{cp}^2)^2} +$$

$$+ \frac{4\pi\varrho v_0^2 R_1^2 R_2^2 (6h_{cp} L^2 - 8h_{cp}^3)}{(L^2 + 4h_{cp}^2)^3} - \frac{\varrho\Gamma_1\Gamma_2 a}{2\pi(a^2 + L^2)} +$$

$$+ \frac{\varrho v_0(\Gamma_1 R_2^2 - \Gamma_2 R_1^2)(L^2 - a^2)}{(L^2 + a^2)^2} + \frac{4\pi\varrho v_0^2 R_1^2 R_2^2 (3aL^2 - a^3)}{(L^2 + a^2)^3} +$$

$$+ \nu\varrho\left\{\frac{1}{\pi} e^{-2\nu h_{cp}} [\Gamma_1\Gamma_2 + \nu(2\pi\Gamma_1 v_0 R_2^2 + 2\pi\Gamma_2 v_0^2 R_1^2) +\right.$$

$$+ 4\pi^2\nu^2 v_0^2 R_1^2 R_2^2]\, \mathrm{Re}\,(\mathrm{Ci}\bar{b} - i\,\mathrm{Si}\bar{b})\, e^{-i\nu L} -$$

$$\left. - \frac{4v_0 h_{cp}[\Gamma_1 R_2^2 + \Gamma_2 R_1^2 + 2\pi v_0 \nu R_1^2 R_2^2]}{4(h_{cp}^2 + L^2)} + \frac{4\pi v_0^2 R_1^2 R_2^2 (L - 4h_{cp}^2)}{(4h_{cp}^2 + L^2)^2}\right\}, \tag{IV.14}$$

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$$Q_1 = Q_{1\infty} + \frac{\varrho\Gamma_1\Gamma_2 L}{2\pi(4h_{cp}^2 + L^2)} + \frac{4\varrho v_0 h_{cp} L(\Gamma_1 R_2^2 + \Gamma_2 R_1^2)}{(L^2 + 4h_{cp}^2)^2} -$$

$$- \frac{4\pi\varrho v_0^2 R_1^2 R_2^2 (L^3 - 12h_{cp}^2 L)}{(L^2 + 4h_{cp}^2)^3} - \frac{\varrho\Gamma_1\Gamma_2 L}{2\pi(a^2 + L^2)} -$$

$$- \frac{2a\varrho v_0(\Gamma_2 R_1^2 - \Gamma_1 R_2^2)}{(L^2 + a^2)^2} - \frac{4\pi\varrho v_0^2 R_1^2 R_2^2 (L^3 - 3a^2 L)}{(L^2 + a^2)^3} +$$

$$
\begin{aligned}
&+\nu\varrho\Bigg\{-\frac{1}{\pi}e^{-2\nu h_{cp}}[\Gamma_1\Gamma_2+2\pi\nu v_0^2(\Gamma_1R_2^2+\Gamma_2R_1^2)+4\pi\nu v_0^2R_1^2R_2^2]\times\\
&\times\operatorname{Im}(\mathrm{Ci}\,\bar{b}+\mathrm{Si}\,\bar{b})\,e^{-i\nu L}+\frac{2v_0L\,[\Gamma_1R_2^2+\Gamma_2R_1^2+2\pi\nu v_0R_1^2R_2^2]}{4h_{cp}^2+L^2}+\\
&\qquad+\frac{16\pi v_0^2R_1^2R_2^2h_{cp}L}{(4h_{cp}^2+L^2)^2}\Bigg\},
\end{aligned}
\tag{IV.15}
$$

$$
\begin{aligned}
P_2={}&P_{2\infty}-\frac{\varrho\Gamma_1\Gamma_2h_{cp}}{\pi(4h_{cp}^2+L^2)}+\frac{\varrho v_0(\Gamma_1R_2^2+\Gamma_2R_1^2)(L^2-4h_{cp}^2)}{(L^2+4h_{cp}^2)^2}+\\
&+\frac{4\pi\varrho v_0^2R_1^2R_2^2(6h_{cp}^2L-8h_{cp}^2)}{(L^2+4h_{cp}^2)^3}+\frac{\varrho\Gamma_1\Gamma_2a}{\pi(a^2+L^2)}-\\
&-\frac{\varrho v_0(\Gamma_2R_1^2-\Gamma_1R_2^2)(L^2-a^2)}{(L^2+a^2)^2}-\frac{4\pi\varrho v_0^2R_1^2R_2^2(3aL-a^3)}{(L^2+a^2)^3}+\\
&+\nu\varrho\Bigg\{e^{-2\nu h_{cp}}[\Gamma_1\Gamma_2+2\pi\nu v_0(\Gamma_1R_2^2+\Gamma_2R_1^2)+4\pi\nu^2v_0^2R_1^2R_2^2]\times\\
&\times\left[2\sin\nu L+\frac{1}{\pi}\operatorname{Re}(\mathrm{Ci}\,\bar{b}-i\,\mathrm{Si}\,\bar{b})\,e^{i\nu L}\right]-\\
&-\frac{[(\Gamma_1R_2^2+\Gamma_2R_1^2)+2\pi\nu v_0R_1^2R_2]\,4v_0h_{cp}}{(4h_{cp}^2+L^2)}+\\
&+\frac{4\pi v_0^2R_1^2R_2^2(L^2-4h_{cp}^2)}{(4h_{cp}^2+L^2)^2}\Bigg\},
\end{aligned}
\tag{IV.16}
$$

$$
\begin{aligned}
Q_2={}&Q_{2\infty}-\frac{\varrho\Gamma_1\Gamma_2L}{2\pi(4h_{cp}^2+L^2)}-\frac{4\varrho v_0h_{cp}(\Gamma_1R_2^2+\Gamma_2R_1^2)}{(L^2+4h_{cp}^2)^2}+\\
&+\frac{4\pi\varrho v_0^2R_2^2R_1^2(L^3-12h_{cp}^2L)}{(L^2+4h_{cp}^2)^3}+\frac{\varrho\Gamma_1\Gamma_2L}{2\pi(a^2+L^2)}+\frac{2a\varrho v_0(\Gamma_2R_1^2-\Gamma_1R_2^2)}{(a^2+L^2)^2}+
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4\pi\varrho v_0^2R_1^2R_2^2(L^3-3a^2L)}{(L^2+a^2)^3}+\nu\varrho\Bigg\{e^{-2\nu h_{cp}}[\Gamma_1\Gamma_2+2\pi\nu v_0(R_1^2\Gamma_2+R_2^2\Gamma_1)+\\
&+4\pi\nu^2v_0^2R_1^2R_2^2]\left[2\cos\nu L-\frac{1}{\pi}\operatorname{Im}(\mathrm{Ci}\,\bar{b}-i\,\mathrm{Si}\,\bar{b})\,e^{i\nu L}\right]-\\
&-\frac{2v_0L\,[(\Gamma_1R_2^2+\Gamma_2R_1^2)]+2\pi\nu v_0R_1^2R_2^2]}{(4h_{cp}^2+L^2)}-\frac{16\pi v_0^2R_1^2R_2^2Lh_{cp}}{(4h_{cp}^2+L^2)^2}\Bigg\},
\end{aligned}
\tag{IV.17}
$$
[115

where $\mathrm{Ci}\,\bar{b}$ and $\mathrm{Si}\,\bar{b}$ are the integral cosine and sine; $b = L + i2h_{cp}$; $a = h_1 - h_2$.

As seen from the formulas obtained, an additional cylinder in the flow causes changes in the lifting force and drag. It is easy to note that at long distances between the cylinders, forces acting on the front cylinder will differ little from those acting on the isolated cylinder, while those acting on the rear cylinder will depend considerably on the value of νL.

For the total wave drag of the system

$$\sum Q = Q_{1\infty} + Q_{2\infty} + 2\nu\varrho e^{-2\nu h_{cp}}[\Gamma_1\Gamma_2 + 2\pi\nu v_0(\Gamma_1 R_2^2 + \Gamma_2 R_1^2) + + 4\pi\nu^2 v_0^2 R_1^2 R_2^2]\cos\nu L. \quad (IV.18)$$

From formula (IV.18) it is easy to obtain the values of practical and impractical distances between the cylinders.

With $\nu L = 2\pi$ and 4π the wave drag will be maximum.

With $\nu L = \frac{\pi}{2}$ and $\frac{3}{2}\pi$, $\sum Q = Q_{1\infty} + Q_{2\infty}$

and with $\nu L = \pi$ and 3π, $\sum Q < Q_{1\infty} + Q_{2\infty}$.

For cylinders with equal radii and circulations formula (IV.18) will acquire the form

$$\sum Q = 2Q_{\infty}(1 + \cos\nu L).$$

From this formula it follows that with $\nu L = 2\pi$ and 4π, $\sum Q = 4Q_{\infty}$, and with $\nu L = \pi$ and 3π, $\sum Q = 0$.

The Γ_j values in formulas (IV.5) and (IV.6) depend on the submergence and position of the hydrofoil in the system. These values are determined from the condition of finiteness of the flow velocity at the trailing edge.

Let us present an expression for the complex velocity $v(z)$ which satisfies the condition of the hydrofoil's contour.

Let us map conformally the shape of the contour Γ_j on the circle with radius R in such a way that the infinitely distant point in plane z would change into an infinitely distant point in plane u and that $\left(\frac{du}{dz}\right)_{t=\infty} = 1$.

Boundary condition (2) may be written in the form [116

$$\operatorname{Im}\left[\sum_{l=1}^{n}\left(v_{1l}(z)+v_{2l}(z)\right)-v_0\right]dz=0 \tag{IV.19}$$

or

$$\operatorname{Re} v_{1j}\frac{dz}{d\sigma}\sigma=-\operatorname{Re}\left[\sum_{l=1}^{n}{}' v_{1l}(z)+\sum_{l=1}^{n} v_{2l}(z)-v_0\right]\frac{dz}{d\sigma}\sigma.$$

From formula of Schwarz, which determines the analytical function outside the circle by means of its real part on the circle, we obtain the following expression:

$$v_{1j}(z)\frac{dz}{du}u=-\frac{1}{4\pi i}\int_k\left[\left(\sum_{l=1}^{n}{}' v_{1l}(z)+\sum_{l=1}^{n} v_{2l}(z)-v_0\right)\times\right.$$

$$\left.\times\frac{dz}{d\sigma}\sigma+\overline{\left(\sum_{l=1}^{n}{}' v_{1l}(z)+\sum_{l=1}^{n} v_{2l}(z)-v_0\right)}\frac{d\bar z}{d\bar\sigma}\bar\sigma\right]\frac{u+\sigma}{u-\sigma}\frac{d\sigma}{\sigma}+Ci,$$

which after transformations can be written in the form

$$v_{1j}(z)\frac{dz}{du}u=\frac{u}{2\pi i}\int_k\frac{\sum_{l=1}^{n}{}' v_{1l}(z)+\sum_{l=1}^{n} v_{2l}(z)-v_0}{\sigma-u}dz+$$

$$+\frac{R^2}{2\pi i u}\int_k\frac{\overline{\sum_{l=1}^{n}{}' v_{1l}(z)+\sum_{l=1}^{n} v_{2l}(z)-v_0}}{\bar\sigma-\frac{R^2}{\bar u}}d\bar z+C_l. \tag{IV.20}$$

Functions $v_{2i}(z)$ may be expressed through functions $H_{ij}(\lambda)$:

$$v_{2l}(z)=\frac{1}{2\pi}\int_0^\infty e^{-i\lambda z}H_{lj}(\lambda)\frac{\lambda+\nu}{\lambda-\nu}d\lambda+i\nu e^{-i\nu z}\overline{H_{lj}(\nu)}. \tag{IV.21}$$

Introducing functions $G_j(\lambda, u)$ we obtain:

$$\begin{aligned}\int_{kj}\frac{v_{2l}(z)}{\sigma-u}dz&=\frac{1}{2\pi}\int_0^\infty \overline{H_{lj}(\lambda)}G_j(\lambda,u)\frac{\lambda+\nu}{\lambda-\nu}d\lambda+\\&+i\nu G_j(\nu,u)\overline{H_{lj}(\nu)};\end{aligned} \tag{IV.22}$$

$$\int_{kj}\frac{\overline{v_{2l}(z)}}{\bar\sigma-\frac{R^2}{\bar u}}d\bar z=\frac{1}{2\pi}\int_0^\infty H_{lj}(\lambda)\overline{G_j\left(\lambda,\frac{R^2}{\bar u}\right)}\frac{\lambda+\nu}{\lambda-\nu}d\lambda-$$

[117

$$-i\nu\,\overline{G_l\left(\nu, \frac{R^2}{u}\right)}\,H_{lj}(\nu); \qquad \text{(IV.23)}$$

$$\int_{kl} \frac{\overline{v_{1l}(z)}}{\sigma - u}\,dz = \frac{1}{2\pi} \int_0^{\infty\,\mathrm{Sign}\,(h_j - h_l)} H_{lj}(-\lambda)\,G_l(\lambda, u)\,d\lambda; \qquad \text{(IV.24)}$$

$$\int_{kl} \frac{\overline{v_{1l}(z)}}{\sigma - \frac{R^2}{\bar{u}}}\,dz = \frac{1}{2\pi} \int_0^{\infty\,\mathrm{Sign}\,(h_j - h_l)} \overline{H_{lj}(-\lambda)}\,\overline{G_l\left(\lambda, \frac{R^2}{\bar{u}}\right)}\,d\lambda. \qquad \text{(IV.25)}$$

Then, taking into account expressions (IV.22)-(IV.25), we obtain from (IV.20)

$$v_{lj}(z) = \frac{du}{dz}\Bigg\{ -v_0 + \bar{v}_0 \frac{R_l^2}{u^2} + \frac{Ci}{u} +$$

$$+ \frac{1}{4\pi^2 i} \sum_{l=1}^{n} \int_0^{\infty} \left[\overline{H_{lj}(\lambda)}\,G_l(\lambda, u) + H(\lambda)\,\overline{G_l\left(\lambda, \frac{R^2}{\bar{u}}\right)} \right] \frac{\lambda + \nu}{\lambda - \nu}\,d\lambda +$$

$$+ \frac{1}{4\pi^2 i} \sum_{l=1}^{n} \int_0^{\infty\,\mathrm{Sign}\,(h_j - h_l)} \left[H_{lj}(-\lambda)\,G_l(\lambda, u) + \overline{H_{lj}(-\lambda)\,G_l\left(\lambda, \frac{R^2}{\bar{u}}\right)} \right] d\lambda +$$

$$+ \frac{\nu}{2\pi} \sum_{l=1}^{n} \left[G_l(\nu, u)\,\overline{H_{lj}(\nu)} - \frac{R^2}{u^2}\,\overline{G_l\left(\nu, \frac{R^2}{\bar{u}}\right)}\,H_{lj}(\nu) \right] \Bigg\}. \qquad \text{(IV.26)}$$

This is the principal expression used in determining the velocity circulation along the contour of the j-th hydrofoil and will be used for these purposes below.

4.2. <u>Motion of Two Hydrofoils Under the Free Surface of a Fluid</u>

From the general formulas for forces acting on hydrofoils we obtain:

$$P_1 = \varrho v_0 \Gamma_1 - \frac{\varrho}{2\pi} \int_0^{\infty} |H_{11}(\lambda)|^2 \frac{\lambda + \nu}{\lambda - \nu}\,d\lambda -$$

$$- \frac{\varrho}{2\pi} \int_0^{\infty} [H'_{11}(\lambda)\,H'_{21}(\lambda) + H''_{11}(\lambda)\,H''_{21}(\lambda)] \frac{\lambda + \nu}{\lambda - \nu}\,d\lambda - \qquad \text{[118}$$

$$- \frac{\varrho}{2\pi} \int_0^{\infty\,\mathrm{Sign}\,(h_1 - h_2)} [H'_{21}(-\lambda)\,H'_{11}(\lambda) - H''_{21}(-\lambda)\,H''_{11}(\lambda)]\,d\lambda +$$

$$+ \nu\varrho [H''_{11}(\nu) H'_{21}(\nu) - H'_{11}(\nu) H''_{21}(\nu)], \quad \text{(IV.27)}$$

$$Q_1 = \nu\varrho [H'_{21}(\nu) H'_{11}(\nu) + H''_{21}(\nu) H''_{11}(\nu) + |H_{11}(\nu)|^2] +$$

$$+ \frac{\varrho}{2\pi} \int_0^\infty [H'_{21}(\lambda) H''_{11}(\lambda) - H''_{21}(\lambda) H'_{11}(\lambda)] \frac{\lambda+\nu}{\lambda-\nu} d\lambda -$$

$$- \frac{\varrho}{2\pi} \int_0^{\infty \operatorname{Sign}(h_1 - h_2)} [H''_{11}(\lambda) H''_{21}(-\lambda) + H''_{11}(\lambda) H'_{21}(-\lambda)] d\lambda, \quad \text{(IV.28)}$$

$$P_2 = \varrho v_0 \Gamma_2 - \frac{\varrho}{2\pi} \int_0^\infty |H_{22}(\lambda)|^2 \frac{\lambda+\nu}{\lambda-\nu} d\lambda -$$

$$- \frac{\varrho}{2\pi} \int_0^\infty [H'_{22}(\lambda) H'_{12}(\lambda) + H''_{22}(\lambda) H''_{12}(-\lambda)] \frac{\lambda+\nu}{\lambda-\nu} d\lambda -$$

$$- \frac{\varrho}{2\pi} \int_0^{\infty \operatorname{Sign}(h_2 - h_1)} [H'_{12}(-\lambda) H'_{22}(\lambda) - H''_{12}(-\lambda) H''_{22}(\lambda)] d\lambda +$$

$$+ \nu\varrho [H''_{22}(\nu) H'_{12}(\nu) - H'_{22}(\nu) H''_{12}(\nu)], \quad \text{(IV.29)}$$

$$Q_2 = \nu\varrho [H'_{12}(\nu) H'_{22}(\nu) + H''_{12}(\nu) H''_{22}(\nu) + |H_{22}(\nu)|^2 +$$

$$+ \frac{\varrho}{2\pi} \int_0^\infty [H'_{12}(\lambda) H''_{22}(\lambda) - H''_{12}(\lambda) H'_{22}(\lambda)] \frac{\lambda+\nu}{\lambda-\nu} d\lambda -$$

$$- \frac{\varrho}{2\pi} \int_0^{\infty \operatorname{Sign}(h_2 - h_1)} [H''_{12}(-\lambda) H'_{22}(\lambda) + H'_{12}(-\lambda) H''_{12}(\lambda)] d\lambda, \quad \text{(IV.30)}$$

where P_i - lifting force on the i-th hydrofoil;

Q_i - wave drag on the i-th hydrofoil; [119

$\nu = \frac{g}{v_0^2}$;

v_0 - velocity of the hydrofoil.

Function H_{ij} should be determined from the formula

$$H_{ij}(\lambda) = e^{i\lambda\zeta_{ij}} \int_C e^{-i\lambda f(z)} v(z)\, dz, \quad \text{(IV.31)}$$

where f(z) is the transforming function.

Let us examine hydrofoils obtained with the aid of

the transforming function by N. Ye. Zhukovskiy:

$$f(z) = z + \frac{R^2}{z}.$$

Let us determine function $H_{ij}(\lambda)$ in the first approximation using the complex velocity of motion of the i-th hydrofoil in an infinite fluid.

$$\left.\begin{aligned}
H_{11}(\lambda) &= e^{-\lambda h_1}\Gamma_{1\infty}[J_0(2\lambda R_1) - iJ_1(2\lambda R_1)]\\
H_{22}(\lambda) &= e^{-\lambda h_2}\Gamma_{2\infty}[J_0(2\lambda R_2) - iJ_1(2\lambda R_2)]\\
H_{21}(\lambda) &= e^{-\lambda(h_2 - iL)}\Gamma_{2h}[J_0(2\lambda R_2) - iJ_1(2\lambda R_2)]\\
H_{12}(\lambda) &= e^{-\lambda(h_1 + iL)}\Gamma_{1h}[J_0(2\lambda R_1) - iJ_1(2\lambda R_1)]
\end{aligned}\right\}, \qquad \text{(IV.32)}$$

where $J_0(x)$, $J_1(x)$ - Bessel functions;

Γ_{ih} - circulation at the i-th hydrofoil moving under the free surface.

From formulas (IV.27)-(IV.32) we obtain:

$$\begin{aligned}
P_1 = {}& \varrho v_0\Gamma_1 - \frac{\varrho\Gamma_{1\infty}^2}{2\pi R_1}\{A_{00}(R_1) + A_{11}(R_1) - 2\nu R_1[B_{00}(R_1) + B_{11}(R_1)]\} -\\
& - \frac{\varrho\Gamma_{1\infty}\Gamma_{2h}}{2\pi R_1}[D_{00} + G_{01} - G_{10} + D_{11} - 2\nu R_1(F_{00} + P_{01} - P_{10} + F_{11}) +\\
& + 2\nu\pi R_1(L_{10} + N_{11} - N_{00} - L_{01}) + D_{00}(a) + G_{01}(a) -\\
& - G_{10}(a) + D_{11}(a)],
\end{aligned} \qquad \text{(IV.33)}$$

$$\begin{aligned}
P_2 = {}& \varrho v_0\Gamma_2 - \frac{\varrho\Gamma_{2\infty}^2}{2\pi R_2}\{A_{00}(R_2) + A_{11}(R_2) - 2\nu R_2[B_{00}(R_2) + B_{11}(R_2)]\} -\\
& - \frac{\varrho\Gamma_{1h}\Gamma_{2\infty}}{2\pi R_1}[D_{00} + G_{01} - G_{10} + D_{11} - 2\nu R_1(F_{00} + P_{01} - P_{10} + F_{11}) +\\
& + 2\nu\pi R_1(-L_{10} - N_{11} - N_{10} + L_{11}) + D_{00}(-a) - G_{10} -\\
& - (-a) + G_{01}(-a) + D_{11}(a)],
\end{aligned} \qquad \text{(IV.34)}$$

$$\begin{aligned}
Q_1 = {}& Q_{1\infty} + \frac{\varrho\Gamma_{1\infty}\Gamma_{2h}}{2\pi R_1}[-D_{10} - G_{11} - G_{00} + D_{01} + 2\nu\pi R_2(L_{00} + N_{01} -\\
& - N_{10} + L_{01}) + G_{00}(a) - D_{01}(a) + D_{10}(a) + G_{11}(a)] +\\
& + 2\nu R_1(F_{10} + P_{11} + P_{10} - F_{01})],
\end{aligned} \qquad \text{(IV.35)}$$ [120

$$\begin{aligned}
Q_2 = {}& Q_{2\infty} + \frac{\varrho\Gamma_{1h}\Gamma_{2\infty}}{2\pi R_1}[D_{10} + G_{11} + G_{00} - D_{01} - 2\nu R_1(F_{10} + P_{11} +\\
& + P_{00} - F_{01}) + 2\nu\pi R_1(L_{00} + N_{01} - N_{10} + L_{11}) -\\
& - G_{00}(-a) + D_{01} - D_{10}(-a) - G_{11}(-a),
\end{aligned} \qquad \text{(IV.36)}$$

where

$$A_{nn}(R_i) = R_i \int_0^\infty e^{-2h_i\lambda} J_n^2(2\lambda R_i)\, d\lambda; \qquad (IV.37)$$

$$B_{nn}(R_i) = \int_{-\infty}^{1} e^{-2\nu(1-\lambda)h_i} J_n^2[2\nu(1-\lambda)R_i] \frac{d\lambda}{\lambda}; \qquad (IV.38)$$

$$D_{ij} = R_1 \int_0^\infty e^{-2h_{cp}\lambda} J_i(2\lambda R_1) J_j(2\lambda R_2)\, d\lambda; \qquad (IV.39)$$

$$G_{ij} = R_1 \int_0^\infty e^{-2h_{cp}\lambda} J_i(2\lambda R_1) J_j(2\lambda R_2) \sin\lambda L d\lambda; \qquad (IV.40)$$

$$F_{ij} = \int_{-\infty}^{1} e^{-2\nu(1-\lambda)h_{cp}} J_i[2\nu(1-\lambda)R_1] J_j[2\nu(1-\lambda)R_2] \cos\nu(1-\lambda)L \frac{d\lambda}{\lambda}; \qquad (IV.41)$$

$$P_{ij} = \int_{-\infty}^{1} e^{-2\nu(1-\lambda)h_{cp}} J_i[2\nu(1-\lambda)R_1] J_j(2\nu(1-\lambda)R_2] \times$$

$$\times \sin\nu(1-\lambda) L \frac{d\lambda}{c\lambda}; \qquad (IV.42)$$

$$N_{nm} = J_n(2\nu R_1) J_m(2\nu R_2) \sin\nu L; \qquad (IV.43)$$

$$L_{nm} = J_n(2\nu R_1) J_m(2\nu R_2) \cos\nu L;$$

$$h_{cp} = \frac{h_1 + h_2}{2}.$$

Functions $D_{ij}(a)$ $D_{ij}(-a)$ and $G_{ij}(a)$ $G_{ij}(-a)$ will be also determined from formulas (IV.39) and (IV.40), in which $2h_{cp}$ should be replaced by $\pm a = h_1 - h_2$ and the infinite limit multiplied by $\mathrm{Sign}(h_j - h_i)$. [121

Let us show that functions F_{ij} and P_{ij} may be represented in the form

$$F_{ij} = N_{ij}\pi + \mathrm{Re} f_{ij}(\nu, 2h_{cp} - iL),$$
$$P_{ij} = -L_{ij}\pi + J_m f_{ij}(\nu, 2h_{cp} - iL) \qquad (IV.44)$$

and that function $f_{ij}(\nu, 2h_{cp} - iL) \to 0$ when $\pm L \to \infty$.

Let us examine the integral

$$J = \int_{-\infty}^{1} e^{-\nu(1-\lambda)[2h_{cp} - iL]} J_0[2\nu(1-\lambda)R_1] J_0[2\nu(1-\lambda)R_2] \frac{d\lambda}{\lambda}.$$

We obtain

$$\operatorname{Re} J = F_{00}. \qquad \operatorname{Im} J = P_{00}.$$

Let us represent the product of the Bessel functions in the form [25]

$$J_0(b_1) J_0(b_2) = \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{s+p}}{\pi(s)\pi(s)\pi(p)\pi(p)} \left(\frac{b_1}{2}\right)^{2s} \left(\frac{b_2}{2}\right)^{2p}.$$

Then

$$J = \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{s+p} R_2^{2s} R_1^{2p}}{\pi(s)\pi(s)\pi(p)\pi(p)} \int_{-\infty}^{1} e^{-\nu(1-\lambda)(2h_{cp}-iL)} \frac{[\nu(1-\lambda)]^{2(s+p)}}{\lambda} d\lambda;$$

$$\int_{-\infty}^{1} e^{-\nu(1-\lambda)(2h_{cp}-iL)} [\nu(1-\lambda)]^{2(s+p)} \frac{d\lambda}{\lambda} =$$

$$= \frac{e^{-\nu(2h_{cp}-iL)}}{(2h_{cp}-iL)^{2(s+p)}} \int_{-\infty}^{\nu(2h_{cp}-iL)} e^{u} \frac{[\nu(2h_{cp}-iL)-u]^{-2(s+p)}}{u} du =$$

$$= f_{2(s+p)} [\nu(2h_{cp}-iL)] \times (2h_{cp}-iL)^{-2(s+p)}$$

We have

$$\left.\begin{aligned} & f_m(x) = x f_{m-1}(x) - (m-1)! \\ & f_m(x) = x^m e^{-x} \int_{-\infty}^{\nu(2h_{cp}-iL)} \frac{e^u}{u} du - x^{m-1} - 1!\, x^{m-2} - \\ & \quad - 2!\, x^{m-3} - \ldots - (m-1)! \\ & \int_{-\infty}^{\nu(2h_{cp}-iL)} \frac{e^u}{u} du = \operatorname{Ci}[-\nu(L+i2h_{cp})] + i \operatorname{Si}[-\nu(L+i2h_{cp})] = \\ & \quad = \operatorname{Ci}[\nu(L+i2h_{cp})] - i \operatorname{Si}[\nu(L+i2h_{cp})] - i\pi \end{aligned}\right\} \quad \text{(IV.45)}$$

[122

Then

$$J = \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{s+p} R_2^{2s} R_1^{2p} \nu^{2(s+p)} e^{-2\nu h_{cp}} e^{iL\nu}}{\pi(s)\pi(s)\pi(p)\pi(p)} \Big[\operatorname{Ci}[\nu(L+i2h_{cp})] -$$

$$- i \operatorname{Si}[\nu(L+i2h_{cp})] - i\pi - \frac{1}{\nu(2h_{cp}-iL)} - \frac{1}{[\nu(2h_{cp}-iL]^2} -$$

$$- \frac{[2(s+\varrho)-1]!}{[\nu(2h_{cp}-iL)]^{2(s+p)}} \Big], \quad \text{(IV.46)}$$

from which we obtain

$$\operatorname{Im} I = -\pi L_{00} + \operatorname{Im} f_{00}(\nu,\, 2h_{cp} - iL);$$

$$\operatorname{Re} I = \pi N_{\infty} + \operatorname{Re} f_{00}(\nu,\, 2h_{cp} - iL);$$

$$f(\nu_1 2h_{cp} - iL) = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-1)^{s+p} R_2^{2s} R_1^{2p} \nu^{2(s+p)} e^{-2h_{cp}\nu} e^{iL}}{\Pi(s)\,\Pi(s)\,\Pi(p)\,\Pi(p)} \times$$

$$\times \Big[\operatorname{Ci}(\nu(L + 2ih_{cp})] - i\operatorname{Si}[\nu(L + 2ih_{cp})] - \frac{1}{\nu(2h_{cp} - iL)} -$$

$$- \frac{1!}{[\nu(2h_{cp} - iL)]^2} - \ldots - \frac{[2(s+p)-1]!}{[\nu(2h_{cp} - iL)]^{2(s+p)}}\Big]. \qquad \text{(IV.47)}$$

The other relationships (IV.44) are obtained similarly. Functions $f_{n,m}(\nu_1 2h_{cp} - iL)$ will be determined by the formula

$$f_{n,m}(\nu,\, 2h_{cp} - iL) = \sum_{s=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-1)^{s+p}(R_2,\nu)^{n+m+2(s+p)} K^{m+2p} e^{\omega i L} e^{-2\omega h_{cp}}}{s!\,p!\,(n+s)!\,(m+p)!\,[\nu(2h_{cp} - iL)]^{2(s+p)}} \times$$

$$\times \{[\operatorname{Ci}\nu(L + 2ih_{cp}) - i\operatorname{Si}\nu(2ih_{cp} + L)][\nu(2h_{cp} - iL)]^{2(s+p)} -$$

$$- [\nu(2h_{cp} - iL)]^{2(s+p)-1} - [2(s+p)-1]!\}, \qquad \text{(IV.48)}$$

$$K = \frac{R_1}{R_2}.$$

Taking into account relations (IV.44), formulas (IV.33)-(IV.36) will be in the form [123

$$P_1 = \rho v_0 \Gamma_1 - \frac{\rho\Gamma_{1\infty}^2}{2\pi R_1}\{A_{00}(R_1) + A_{11}(R_1) - 2\nu R_1[B_{00}(R_1) +$$

$$+ B_{11}(R_1)]\} - \frac{\rho\Gamma_{1\infty}\Gamma_{2h}}{2\pi R_1}\{D_{00} + G_{01} - G_{10} + D_{11} -$$

$$- 2\nu R_1[\operatorname{Re} f_{00}(\nu,\, 2h_{cp} - iL) + \operatorname{Im} f_{01}(\nu,\, 2h_{cp} - iL) -$$

$$- \operatorname{Im} f_{10}(\nu,\, 2h_{cp} - iL) + \operatorname{Re} f_{11}(\nu_1\, 2h_{cp} - iL) + D_{00}(a) +$$

$$+ G_{01}(a) - G_{10}(a) - D_{11}(a)]\}; \qquad \text{(IV.49)}$$

$$P_2 = \rho v_0 \Gamma_2 - \frac{\rho\Gamma_{2\infty}}{2\pi R_2}\{A_{00}(R_2) + A_{11}(R_2) - 2\nu R_2[B_{00}(R_2) +$$

$$+ B_{11}(R_2)]\} - \frac{\rho\Gamma_{1h}\Gamma_{2\infty}}{2\pi R_2}[D_{00} + G_{01} - G_{10} + D_{11} -$$

$$- 2\nu R_1[\operatorname{Re} f_{00}(\nu,\, 2h_{cp} - iL) + \operatorname{Im} f_{01}(\nu,\, 2h_{cp} - iL) -$$

$$- \operatorname{Im} f_{10}(\nu,\, 2h_{cp} - iL) + \operatorname{Re} f_{11}(\nu,\, 2h_{cp} - iL) - 4\nu\pi(L_{10} + N_{11} -$$

$$- N_{00} - L_{01})] + D_{00}(-a) - G_{10}(-a) + G_{01}(-a) + D_{11}(-a); \qquad \text{(IV.50)}$$

$$Q_1 = Q_{1\infty} + \frac{\rho\Gamma_{1\infty}\Gamma_{2h}}{2\pi R}\{-D_{10} + G_{11} - G_{00} + D_{01} +$$

$$+ 2\nu R_1 [\operatorname{Re} f_{10}(\nu, 2h_{cp} - iL) + \operatorname{Im} f_{11}(\nu, 2h_{cp} - iL) +$$
$$+ \operatorname{Im} f_{00}(\nu, 2h_{cp} - iL) - \operatorname{Re} f_{01}(\nu, 2h_{cp} - iL) + G_{00}(a) -$$
$$- D_{01}(a) + D_{10}(a) + G_{11}(a)]]; \qquad (IV.51)$$

$$Q_2 = Q_{2\infty} + \frac{\varrho \Gamma_{1h} \Gamma_{2\infty}}{2\pi R_1} \{D_{10} + G_{11} + G_{00} - D_{01} -$$
$$- 2\nu R_1 [\operatorname{Re} f_{10}(\nu, 2h_{cp} - iL) + \operatorname{Im} f_{11}(\nu, 2h_{cp} - iL) +$$
$$+ \operatorname{Im} f_{00}(\nu, 2h_{cp} - iL) - \operatorname{Re} f_{01}(\nu, 2h_{cp} - iL) + 4\pi\nu R_1 ((L_{00} + N_{01} +$$
$$+ L_{11} - N_{10}) - G(-a) + D_{01}(-a) - D_{10}(-a) - G_{11}(-a)]\}. \qquad (IV.52)$$

Let us determine the values of circulation Γ_1 and Γ_2 at the front and rear hydrofoils.

From expression (IV.26) we obtain

$$v_{11}(z) = \frac{du}{dz} \Bigg\{ - v_0 + \frac{v_0 R_1^2}{u^2} + \frac{1}{4\pi^2 i} \int_0^\infty \Bigg[\overline{H}_{11}(\lambda) G_1(\lambda, u) +$$

$$+ \frac{R_1^2}{u^2} H_{11}(\lambda) \overline{G_1\left(\lambda, \frac{R_1^2}{\overline{u}}\right)} \Bigg] \frac{\lambda + \nu}{\lambda - \nu} d\lambda + \frac{\nu}{2\pi} \Bigg[G_1(\nu_1, u) \overline{H}_{11}(\nu) - \quad [124$$

$$- \frac{R_1^2}{u^2} \overline{G_1\left(\nu, \frac{R_1^2}{\overline{u}}\right)} H_{11}(\nu) \Bigg] + \frac{1}{4\pi^2 i} \int_0^\infty \Bigg[\overline{H}_{21}(\lambda) G_1(\lambda, u) +$$

$$+ \frac{R_1^2}{u^2} H_{21}(\lambda) \overline{G_1\left(\lambda, \frac{R_1^2}{\overline{u}}\right)} \Bigg] \frac{\lambda + \nu}{\lambda - \nu} d\lambda +$$

$$+ \frac{1}{4\pi^2 i} \int_0^{\infty \operatorname{Sign}(h_1 - h_2)} \Bigg[H_{21}(-\lambda) G_1(\lambda, u) + \frac{R_1^2}{u^2} \overline{H_{21}(-\lambda)} \overline{G_1\left(\lambda, \frac{R_1^2}{\overline{u}}\right)} \Bigg] d\lambda +$$

$$+ \frac{\nu}{2\pi} \Bigg[G_1(\nu, u) \overline{H_{21}(\nu)} - \frac{R_1^2}{u^2} \overline{G_1\left(\nu, \frac{R_1^2}{\overline{u}}\right)} H_{21}(\nu) \Bigg] \Bigg\}; \qquad (IV.53)$$

$$v_{12}(z) = \frac{du}{dz} \Bigg\{ - v_0 + \frac{\overline{v}_0 R_2^2}{u^2} +$$

$$+ \frac{1}{4\pi^2 i} \int_0^\infty \Bigg[\overline{H}_{22}(\lambda) G_2(\lambda, u) + \frac{R_2^2}{u^2} H_{22}(\lambda) \overline{G_2\left(\lambda, \frac{R_2^2}{\overline{u}}\right)} \Bigg] \frac{\lambda + \nu}{\lambda - \nu} d\lambda +$$

$$+ \frac{\nu}{2\pi} \Bigg[G_2(\nu, u) \overline{H}_{22}(\nu) - \frac{R_2^2}{u^2} \overline{G_2\left(\nu, \frac{R^2}{u}\right)} H_{22}(\nu) \Bigg] +$$

$$+\frac{1}{4\pi^2 i}\int_0^{\infty}\left[H_{12}(\lambda)G_2(\lambda,u)+\frac{R_2^2}{u^2}H_{12}(\lambda)G_2\left(\overline{\lambda,\frac{R_2^2}{u}}\right)\right]d\lambda+$$

$$+\frac{1}{4\pi^2 i}\int_0^{\infty\,\mathrm{Sign}\,(h_2-h_1)}\left[H_{12}(-\lambda)G_1(\lambda,u)+\frac{R_2^2}{u^2}\overline{H}_{12}(-\lambda)G_1\left(\overline{\lambda,\frac{R_2^2}{u}}\right)\right]d\lambda+$$

$$+\frac{\nu}{2\pi}\left[G_2(\nu,u)\overline{H}_{12}(\nu)-\frac{R_2^2}{u^2}G_2\left(\overline{\nu\frac{R_2^2}{u}}\right)H_{12}(\nu)\right]\Bigg\}. \qquad \text{(IV.54)}$$

Functions $G_1(\lambda,u)$, $\overline{G\left(\lambda,\frac{R^2}{u}\right)}$ are determined from formulas (II.35) and (II.36). Then, from (IV.54) we obtain

$$v_{11}=\frac{du}{dz}\Bigg\{-\Bigg[v_0+\frac{1}{2\pi}\Bigg(\int_0^{\infty}H_{11}(\lambda)J_0(2\lambda R_1)e^{-\lambda h_1}\frac{\lambda+\nu}{\lambda-\nu}d\lambda+$$ [125

$$+\int_0^{\infty}H_{21}(\lambda)J_0(2\lambda R_1)\frac{\lambda+\nu}{\lambda-\nu}d\lambda+\int_0^{\infty\,\mathrm{Sign}\,(h_1-h_2)}H_{21}(-\lambda)J_0\times$$

$$\times(2\lambda R_1)e^{-\lambda h_1}d\lambda-2\pi i\nu e^{-\lambda h_1}J_0(2\nu R_1)\;(H_{11}(\nu)+H_{21}(\nu))\Bigg]+$$

$$+\frac{R_1^2}{u^2}\Bigg[\overline{v}_0+\frac{1}{2\pi}\Bigg(\int_0^{\infty}\overline{H}_{11}(\lambda)J_0(2\lambda R_1)e^{-\lambda h_1}\frac{\lambda+\nu}{\lambda-\nu}d\lambda+$$

$$+\int_0^{\infty}H_{21}(\lambda)J_0(2\lambda R_1)e^{-h\lambda_1}\frac{\lambda+\nu}{\lambda-\nu}d\lambda+$$

$$+\int_0^{\infty\,\mathrm{Sign}\,(h_1-h_2)}H_{21}(-\lambda)J_0(2\lambda R_1)e^{-\lambda h_1}d\lambda)+2\nu\pi i e^{-\nu h_1}J_0(2\nu R_1)\times$$

$$\times(H_{11}(\nu)+H_{21}(\nu)\Bigg)\Bigg]+\frac{\Gamma}{2\pi i u}\Bigg\}+$$

$$+\frac{1}{4\pi^2 i}\int_0^{\infty\,\mathrm{Sign}\,(h_1-h_2)}\Bigg[B(\lambda,u)H_{21}(-\lambda)+\frac{R^2}{u^2}B\left(\overline{\lambda,\frac{R^2}{u}}\right)\times$$

$$\times\frac{du}{d\frac{R^2}{u}}H_{21}(-\lambda)\Bigg]d\lambda+\int_0^{\infty}\Bigg[B(\lambda,u)\Big(\overline{H}_{21}(\lambda)+\overline{H}_{11}(\lambda))+$$

$$+\frac{R^2}{u^2}\cdot\frac{du}{d\frac{R^2}{u}}\bar{B}_1\left(\lambda,\frac{R^2}{\bar{u}}\right)(H_{21}(\lambda)+H_{11}(\lambda))\Bigg]\frac{\lambda+\nu}{\lambda-\nu}d\lambda+$$

$$+\frac{\nu}{2\pi}\left\{[\bar{H}_{21}(\nu)+\bar{H}_{11}(\nu)]B_1(\nu,u)-\frac{R^2}{u^2}\left[H_{21}(\nu)+H_{11}(\nu)B_1\left(\overline{\nu,\frac{R^2}{\bar{u}}}\right)\right]\frac{du}{d\frac{R^2}{u}}\right\}; \qquad \text{(IV.55)}$$

[126

$$v_{12}=\frac{du}{dz}\Bigg\{-\Bigg[v_0+\frac{1}{2\pi}\Bigg(\int_0^\infty H_{22}(\lambda)J_0(2\lambda R_2)e^{-\lambda h_2}\frac{\lambda+\nu}{\lambda-\nu}d\lambda+$$

$$+\int_0^\infty H_{12}(\lambda)J_0(2\lambda R_2)e^{-\lambda h_2}\frac{\lambda+\nu}{\lambda-\nu}d\lambda+\int_0^{\infty\,\mathrm{Sign}\,(h_2-h_1)}\bar{H}_{21}(-\lambda)J_0\times$$

$$\times(2\lambda R_1)e^{-\lambda h_2}d\lambda\Bigg)-2\pi i\nu e^{-\lambda h_2}J_0(2\lambda R_2)(H_{22}(\nu)+H_{12}(\nu))\Bigg]+$$

$$+\frac{R_2^2}{u^2}\Bigg[\bar{v}_0+\frac{1}{2\pi}\Bigg(\int_0^\infty\overline{H_{22}(\lambda)}J_0(2\lambda R_2)e^{-\lambda h_2}\frac{\lambda+\nu}{\lambda-\nu}d\lambda+$$

$$+\int_0^\infty\overline{H_{12}(\lambda)}J_0(2\lambda R_2)e^{-\lambda h_2}\frac{\lambda+\nu}{\lambda-\nu}d\lambda+\int_0^{\infty\,\mathrm{Sign}\,(h_2-h_1)}H_{12}(-\lambda)J_0\times$$

$$\times(2\lambda R_2)e^{-\lambda h_2}d\lambda\Bigg)+2\pi i\nu e^{-\nu h_2}J_0(2\lambda R_2)(\bar{H}_{22}(\nu)+\bar{H}_{12}(\nu))+\frac{\Gamma}{2\pi i u}\Bigg\}+$$

$$+\frac{1}{4\pi^2 i}\int_0^{\infty\,\mathrm{Sign}\,(h_2-h_1)}\Bigg[B_2(\lambda,u)H_{12}(-\lambda)+$$

$$+\frac{R^2}{u^2}\overline{B_2\left(\lambda,\frac{R^2}{\bar{u}^2}\right)}\frac{du}{d\frac{R^2}{u}}\bar{H}_{12}(-\lambda)\Bigg]d\lambda+\int_0^\infty\Bigg\{B_2(\lambda,u)[\bar{H}_{12}(\lambda)+$$

$$+\overline{H_{22}(\lambda)}]+\frac{R^2}{u^2}\cdot\frac{du}{d\frac{R^2}{u}}\overline{B_2\left(\lambda,\frac{R^2}{\bar{u}}\right)}[H_{12}(\lambda)+H_{11}(\lambda)]\Bigg\}\frac{\lambda+\nu}{\lambda-\nu}d\lambda+$$

$$+\frac{\nu}{2\pi}\Bigg\{[\overline{H_{21}(\nu)}+H_{11}(\nu)]B_2(\nu,u)-\frac{R^2}{u^2}[H_{21}(\nu)+$$

$$+H_{11}(\nu)]B_2\left(\nu,\frac{R^2}{u}\right)\frac{du}{d\frac{R^2}{u}}\Bigg\}. \qquad \text{(IV.56)}$$

Assuming that conditions of the Zhukovskiy-Chaplygin

postulate are satisfied at points $u_1 = -R_1$ and $u_2 = -R_2$, from expressions (IV.55) and (IV.56) we obtain a system of equations for determining $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$:

$$\Gamma_1 = 4\pi R_1 \operatorname{Im}\left\{ v_0 + \frac{1}{2\pi}\left[\left[\int_0^\infty H_{11}(\lambda) J_0(2\lambda R_1) e^{-\lambda h_1} \frac{\lambda+\nu}{\lambda-\nu} d\lambda + \right.\right.\right.$$ [127

$$+ \int_0^\infty H_{21}(\lambda) J_0(2\lambda R_1) e^{-\lambda h_1} \frac{\lambda+\nu}{\lambda-\nu} d\lambda + \int_0^{\infty \operatorname{Sign}(h_1-h_2)} \bar{H}_{21}(-\lambda) J_0 \times$$

$$\left.\left.\left. \times (2\lambda R_1) e^{-\lambda h_1} d\lambda - 2\pi i \nu e^{-\nu h_1} J_0(2\nu R_1)(H_{11}(\nu) + H_{21}(\nu)) \right]\right]\right\}, \qquad \text{(IV.57)}$$

$$\Gamma_2 = 4\pi R_2 \operatorname{Im}\left\{ v_0 + \frac{1}{2\pi}\left[\left[\int_0^\infty H_{22}(\lambda) J_0(2\lambda R_2) e^{-\lambda h_2} \frac{\lambda+\nu}{\lambda-\nu} d\lambda + \right.\right.\right.$$

$$+ \int_0^\infty H_{12}(\lambda) J_0(2\lambda R_2) e^{-\lambda h_2} \frac{\lambda+\nu}{\lambda-\nu} d\lambda + \int_0^{\infty \operatorname{Sign}(h_2-h_1)} \bar{H}_{12}(-\lambda) J_0 \times$$

$$\left.\left.\left. \times (2\lambda R_2) e^{-\lambda h_2} d\lambda - 2\pi i \nu e^{-\nu h_2} J_0(2\lambda R_2)(H_{22}(\nu) + H_{12}(\nu)) \right]\right]\right\}.$$

Let us write a system of functional equations for determining functions $H_{ij}(\lambda)$. According to (IV.31)

$$H_{ij}(\lambda) = e^{i\lambda L_{ij}} H_{ii}(\lambda).$$

Then, in the usual way, from expressions (IV.55) and (IV.56) we obtain:

$$H_{11}(\mu) = e^{-\mu h_1} \left\{ \Gamma_1 [J_0(2\mu R_1) - iJ_1(2\mu R_2)] + \right.$$

$$+ \frac{1}{4\pi^2 i} \left\{ \int_0^{\infty \operatorname{Sign}(h_1-h_2)} [e^{i\lambda L_{12}} H_{22}(-\lambda) N_{01}(\lambda, \mu) + \bar{H}_{22}(-\lambda) e^{-i\lambda L_{12}} N_{11} \times \right.$$

$$\times (\lambda, \mu)] \, d\lambda + \int_0^\infty \{N_{01}(\lambda, \mu) [e^{i\lambda L_{12}} \bar{H}_{22}(\lambda) + \bar{H}_{11}(\lambda)] + N_{11}(\lambda, \mu) \times$$

$$\times \{e^{-i\lambda L_{12}} H_{22}(\lambda) + H_{11}(\lambda)]\} \frac{\lambda+\nu}{\lambda-\nu} d\lambda + \frac{\nu}{2\pi} [N_{01}(\nu, \mu) \times$$

$$\times (e^{i\lambda L_{12}} H_{22}(\nu) + H_{11}(\nu)) - N_{11}(\nu, \mu) (e^{-i\lambda L_{12}} H_{22}(\nu) + H_{11}(\nu))] \Bigg] \Bigg\}. \quad (IV.58)$$ [128

$$H_{22}(\mu) = e^{-\mu h_2} \Bigg\{ \Gamma_2 [J_0(2\mu R_2) - iJ_1(2\mu R_2)] +$$

$$+ \frac{1}{4\pi^2 i} \Bigg[\int_0^{\infty \operatorname{Sign}(h_2 - h_1)} (e^{i\lambda L_{12}} H_{11}(-\lambda) N_{02}(\lambda, \mu) + \overline{H}_{11}(-\lambda) e^{-i\lambda L_{12}} N_{12} \times$$

$$\times (\lambda, \mu)) d\lambda + \int_0^{\infty} \{ N_{02}(\lambda, \mu) [e^{-i\lambda L_{12}} \overline{H}_{11}(\lambda) + \overline{H}_{22}(\lambda)] +$$

$$+ N_{12}(\lambda, \mu) [e^{i\lambda L_{12}} H_{11}(\lambda) + H_{22}(\lambda)] \} \frac{\lambda + \nu}{\lambda - \nu} d\lambda \Bigg] + \frac{\nu}{2\pi} \Bigg[N_{02}(\nu, \mu) \times$$

$$\times (e^{-i\lambda L_{12}} \overline{H}_{11}(\nu) + H_{22}(\nu)) - N_{12}(\nu, \mu) (e^{i\lambda L_{12}} H_{22}(\nu) + H_{11}(\nu)) \Bigg] \Bigg\},$$

where

$$N_{0l}(\lambda, \mu) = e^{\mu h_l} \int_{C_l} e^{-i\mu z} B(\lambda, u) \, dz;$$

$$N_{1l}(\lambda, \mu) = e^{\mu h_l} \int_{C_l} e^{-i\mu z} B\left(\lambda, \frac{\overline{R^2}}{u}\right) dz.$$

Let us determine approximately $H_{ij}(\lambda)$ by two methods, as was done earlier:

$$H_{ll}(\mu) = e^{-\mu h_l} \Gamma_{l\infty} [J_0(2\mu R_l) - iJ_1(2\mu R_l)];$$

$$H_{ll}(\mu) = e^{-\mu h_l} \Gamma_l [J_0(2\mu R_l) - iJ_1(2\mu R_l)].$$

In the first case the system of equations degenerates into two formulas for determining Γ_j:

$$\Gamma_1 = 4\pi R_1 \Bigg\{ \sin\alpha_1 - \frac{\Gamma_{1\infty}}{2\pi R_1} A_{01}(R_1) + \nu \frac{\Gamma_{1\infty}}{\pi} B_{01}(R_1) +$$

$$+ \frac{\Gamma_{2\infty}}{2\pi} [G_{00} - D_{01} - 2\nu (P_{00} - F_{01}) + G_{00}\ (a) - D_{01}\ (a)] -$$

$$- e^{-2\nu h_1} \Gamma_{1\infty} \nu J_0^2 (2\nu R_2) - \Gamma_{2\infty} \nu (L_{00} + N_{01}) \Bigg\}, \quad (IV.59)$$

$$\Gamma_2 = 4\pi R_2 \Bigg\{ \sin\alpha_2 - \frac{\Gamma_{2\infty}}{2\pi R_2} A_{01}(R_2) + \nu \frac{\Gamma_{2\infty}}{\pi} B_{01}(R_2) +$$ [129

$$+\frac{\Gamma_{1\infty}}{2\pi}[-G_{00}-D_{10}-2\nu(P_{00}+F)-G_{00}(-a)-D_{10}(a)]-$$

$$\left.-e^{-2\nu h_2}\Gamma_{2\infty}\nu J_0^2(2\nu R_2)-\Gamma_{1\infty}\nu(L_{00}-N_{10})\right\} \quad \text{(IV.60)}$$

or, taking into account relations (IV.44),

$$\Gamma_1=\Gamma_{1\infty}\Big\{1-2A_{01}(R_1)+4\nu R_1B_{01}(R_1)-4\pi\nu R_1J_0^2(2\nu R_1)e^{-2\nu h_1}+$$

$$+2R_1\frac{\Gamma_{2\infty}}{\Gamma_{1\infty}}[G_{00}-D_{01}-2\nu(\operatorname{Im}f_{00}(\nu,2h_{cp}-iL)-$$

$$\left.-\operatorname{Re}f_{00}(\nu,2h_{cp}-iL)+G_{00}(a)-D_{01}(a)]\right\}, \quad \text{(IV.61)}$$

$$\Gamma_2=\Gamma_{2\infty}\Big\{1-2A_{01}(R_2)+4\nu R_2B_{01}(R_2)-4\pi\nu[R_2J_0^2(2\nu R_2)e^{-2\nu h_1}+$$

$$+2R_2\frac{\Gamma_{1\infty}}{\Gamma_{2\infty}}(L_{00}-N_{10})]+2R_2\frac{\Gamma_{1\infty}}{\Gamma_{2\infty}}[-G_{00}-D_{10}+2\nu\Big(\operatorname{Im}f_{00}(\nu,2h_{cp}-iL)+$$

$$\left.+\operatorname{Re}f_{10}(\nu,2h_{cp}-iL)\Big)-G_{00}(-a)-D_{10}(-a)]\right\}. \quad \text{(IV.62)}$$

In the second case we arrive at the system of two algebraic equations, the solution of which is in the form

$$\Gamma_1=\frac{(1+F_2)\Gamma_{1\infty}-\Gamma_{2\infty}F_3}{(1+F_1)(1+F_2)-F_3F_4}, \quad \text{(IV.63)}$$

$$\Gamma_2=\frac{(1+F_1)\Gamma_{2\infty}-\Gamma_{1\infty}F_4}{(1+F_1)(1+F_2)-F_3F_4}, \quad \text{(IV.64)}$$

where

$$F_j=2A_{01}(R_j)-4\nu R_jB_{01}(R_j)+4\pi\nu R_jJ_0^2(2\nu R_j)e^{-2\nu h_j},\ j=1,2$$

$$F_3=-2R_1[G_{00}-D_{01}-2\nu(\operatorname{Im}f_{00}(\nu,2h_{cp}-iL)-$$

$$-\operatorname{Re}f_{01}(\nu,2h_{cp}-iL)+G_{00}(a)-D_{01}(a)],$$

$$F_4=-2R_1[-G_{00}-D_{10}+2\nu\Big(\operatorname{Im}f_{00}(\nu,2h_{cp}-iL)+$$

$$+\operatorname{Re}f_{10}(\nu,2h_{cp}-iL)\Big)-G_{00}(-a)-D_{10}(-a)-4\pi\nu(L_{00}-N_{10})].$$

The combination $A_{nm}(R_l)-2\nu R_lB_{nm}(R_l)$ is determined in the form of a series with respect to powers of parameter $\tau_l=\sqrt{4h_l+1}-2h_l$ from formulas (II.56) and (II.58). Functions [130
D_{nm} and G_{nm} may also be determined in the form of the expansions with respect to parameter τ. With $L\to\infty$ the formulas will be

$$P_1 = \varrho v_0 \Gamma_2 - \frac{\varrho \Gamma_{1\infty}^2}{2\pi R_1}\left\{A_{00}(R_1) + A_{11}(R_1) - \frac{\omega_1}{2}[B_{00}(R_1) + B_{11}(R_1)]\right\}; \qquad \text{(IV.65)}$$

$$P_2 = \varrho v_0 \Gamma_2 - \frac{\varrho \Gamma_{2\infty}^2}{2\pi R_2}\left\{A_{00}(R_2) + A_{11}(R_2) - \frac{\omega_2}{2}[B_{00}(R_2) + B_{11}(R_2)] - \right.$$

$$\left. - 2\nu\varrho\Gamma_{1\infty}\Gamma_{2\infty}(L_{10} + N_{11} - N_{00} - L_{01})\right\}; \qquad \text{(IV.66)}$$

$$Q_1 = Q_{1\infty}; \qquad \text{(IV.67)}$$

$$Q_2 = Q_{2\infty} + 2\nu\varrho\Gamma_{1\infty}\Gamma_{2\infty}(L_{00} + N_{01} - N_{10} + L_{11}); \qquad \text{(IV.68)}$$

$$\Gamma_1 = \Gamma_{2\infty}\left\{1 - 2\left[A_{01}(R_1) - \frac{\omega_1}{2}B_{01}(R_1)\right] - \pi\omega_1 e^{-2\omega_1\bar{h}_1} J_0^2\left(\frac{\omega_1}{2}\right)\right\}; \qquad \text{(IV.69)}$$

$$\Gamma_2 = \Gamma_{2\infty}\left\{1 - 2\left[A_{01}(R_2) - \frac{\omega_2}{2}B_{01}(R_2)\right] - \pi\omega_2\left[J_0^2\left(\frac{\omega_2}{2}\right)e^{-2\omega_2\bar{h}_2} + \right.\right.$$

$$\left.\left. + 2\frac{\Gamma_{1\infty}}{\Gamma_{2\infty}}(L_{00} - N_{10})\right]\right\}; \qquad \text{(IV.70)}$$

$$F_3 = 0;$$

$$F_4 = 2\pi\omega_1(L_{00} - N_{10}). \qquad \text{(IV.71)}$$

When determining function $H_{ij}(\lambda)$ according to the second method, $\Gamma_1\Gamma_2$ should be used instead of the product $\Gamma_{1\infty}\Gamma_{2\infty}$ in formulas (IV.65) and (IV.66). Formulas (IV.65) and (IV.67) give us the hydrodynamic characteristics of the hydrofoil in an undisturbed flow. They agree with the formulas discussed in Chapter II. Formulas (IV.66) and (IV.68) contain terms which take into consideration the effect of the bow hydrofoil. The interaction in this case is apparently determined by the effect of stationary waves on the hydromechanical characteristics of the hydrofoil.

Let us also examine the motion of a biplane system without stagger. In this case formulas for forces may be written in the following way:

$$P_1 = \varrho v_0 \Gamma_1 - \frac{\varrho\Gamma_{1\infty}^2}{2\pi}\left\{A_{00}(R_1) + A_{11}(R_1) - 2\nu[B_{00}(R_1) + B_{11}(R_1)]\right\} -$$

$$- \frac{\varrho\Gamma_{1\infty}\Gamma_{2h}}{2\pi}[D_{00} + D_{11} - 2\nu(F_{00} + F_{11}) + 2\pi\nu[L_{10} - L_{01}] +$$

$$+ D_{00}(a) + D_{11}(a)]; \qquad \text{(IV.72)}$$

$$P_2 = \varrho v_0 \Gamma_2 - \frac{\varrho\Gamma_{2\infty}^2}{2\pi}\{A_{00}(R_2) + A_{11}(R_2) - 2\nu[B_{00}(R_2) + B_{11}(R_2)]\} -$$ [131

$$-\frac{\varrho\Gamma_{1h}\Gamma_{\infty}}{2\pi}[D_{00}+D_{11}-2\nu(F_{00}+F_{11})+2\nu\pi(-L_{10}+L_{01})+$$

$$+D_{00}(-a)+D_{11}(-a)]; \qquad (IV.73)$$

$$Q_1=Q_{1\infty}+\frac{\varrho\Gamma_{1\infty}\Gamma_{2h}}{2\pi}[+D_{01}-D_{10}+2\nu(F_{10}-F_{01})+2\nu\pi(L_{00}+L_{11})-$$

$$-D_{01}(a)+D_{10}(a)]; \qquad (IV.74)$$

$$Q_2=Q_{2\infty}+\frac{\varrho\Gamma_{1h}\Gamma_{2\infty}}{2\pi}[D_{10}-D_{01}-2\nu(F_{10}-F_{01})+2\nu\pi(L_{00}+L_{11})+$$

$$+D_{01}(-a)-D_{10}(-a)]. \qquad (IV.75)$$

In determining $H(\lambda)$ functions when using the complex velocity of a hydrofoil in an infinite flow, Γ_1 and Γ_2 are calculated from formulas

$$\Gamma_1=4\pi R_1\Big\{\sin\alpha_1-\frac{\Gamma_{1\infty}}{2\pi}[A_{01}(R_1)-2\nu B_{01}(R_1)]+$$

$$+\frac{\Gamma_{2\infty}}{2\pi}[-D_{01}+2\nu F_{01}-D_{01}(a)]+\nu[\Gamma_{1\infty}J_0^2(2\nu R_1)e^{-2h_1}+\Gamma_{2\infty h}L_{00}]\Big\}; \qquad (IV.76)$$

$$\Gamma_2=4\pi R_2\Big\{\sin\alpha_2-\frac{\Gamma_{2\infty}}{2\pi}[A_{01}(R_2)-2\nu B_{01}(R_2)]+\frac{\Gamma_{1\infty}}{2\pi}[-D_{10}+$$

$$+2\nu F_{10}-D_{10}(-a)]+\nu[\Gamma_{2\infty}J_0^2(2\nu R_2)e^{-2\nu h_2}+\Gamma_{1\infty}L_{00}]\Big\}. \qquad (IV.77)$$

In the second case of determining the $H(\lambda)$ functions, F_3 and F_4 in the formulas are in the form

$$F_3=-2R_1[-D_{01}+2\nu F_{01}-D_{01}(a)+2\pi\nu L_{00}],$$
$$F_4=-2R_1[D_{10}+2\nu F_{10}-D_{10}(-a)+2\pi\nu L_{00}]. \qquad (IV.78)$$

The biplane system can be calculated from formulas (IV.72)-(IV.78) without any difficulties. The functions in these formulas can be easily determined in the form of power series in powers of parameter τ. These formulas are particularly simple for a biplane with indentical foils. Then

$$D_{nm}=A_{nm}(R,h_{cp}), \quad F_{nm}=B_{nm}(R,h_{cp})$$

and formulas will acquire the following form: [132

$$P_1=\varrho v_0\Gamma_1-\frac{\varrho\Gamma_{1\infty}^2}{2\pi}\{A_{00}(R_1,h_{cp})+A_{11}(R_1,h_1)-2\nu[B_{00}(R_1h_1)+$$

$$+B_{11}(R,h_1)]\}-\frac{\varrho\Gamma_{1\infty}\Gamma_{2\infty}}{2\pi}\{A_{00}(R_1,h_{cp})+A_{11}(R_1,h_{cp})-$$

$$-2\nu[B_{00}(R_1,h_{cp})+B_{11}(R_1,h_{cp})]+A_{00}(R_1,a)+A_{11}(R_1,a)\}. \quad \text{(IV.79)}$$

$$P_2=\varrho v_0\Gamma_2-\frac{\varrho\Gamma_{2\infty}^2}{2\pi}\{A_{00}(R_1,h_2)+A_{11}(R_1,h_2)-2\nu[B_{00}(R_1,h_2)+$$

$$+B_{11}(R_1,h_2)]\}-\frac{\varrho\Gamma_{1\infty}\Gamma_{2\infty}}{2\pi}\{A_{00}(R_1\,a)+A_{11}(R_1,a)+A_{00}(R_1,h_{cp})+$$

$$+A_{11}(R_1,h_{cp})-2\nu[B_{00}(R_1,h_{cp})+B_{11}(R_1,h_{cp})]\}; \quad \text{(IV.80)}$$

$$Q_1=Q_{1\infty}+\nu\varrho\Gamma_1\Gamma_2(L_{00}+L_{11}); \quad \text{(IV.81)}$$

$$Q_2=Q_{2\infty}+\nu\varrho\Gamma_1\Gamma_2(L_{00}+L_{11}); \quad \text{(IV.82)}$$

$$\Gamma_1=4\pi R\Big\{\sin\alpha-\frac{\Gamma_{1\infty}}{2\pi}[A_{01}(R_1h_1)-2\nu B_{01}(R_1h_1)]-$$

$$-\frac{\Gamma_{2\infty}}{2\pi R_1}[A_{01}(R_1,h_{cp})-2\nu B_{01}(R_1,h_{cp})+A_{01}(a)]\Big\}+$$

$$+\nu[\Gamma_{1\infty}J_0^2(2\nu R_1)e^{-2\nu h_1}+\Gamma_{2\infty}J_0^2(2\nu R_2)e^{-2\nu h_{cp}}]; \quad \text{(IV.83)}$$

$$\Gamma_2=4\pi R\Big\{\sin\alpha-\frac{\Gamma_{1\infty}}{2\pi}[A_{01}(R_1,h_2)-2\nu B_{01}(R_1,h_2)]-$$

$$-\frac{\Gamma_{1\infty}}{2\pi}[A_{00}(R_1,h_{cp})-2\nu B_{01}(R_1,h_{cp})+A_{01}(a)]\Big\}+$$

$$+\nu[\Gamma_{2\infty}J_0^2(2\nu R_2)e^{-2\nu h_2}+\Gamma_{1\infty}J_0^2(2\nu R_1)^{-2\nu h_{cp}}]; \quad \text{(IV.84)}$$

$$F_3=2[A_{01}(R_1,h_{cp})-2\nu R_1B_{01}(R_1,h_{cp})+A_{01}(a)-2\pi\nu R_1J_0^2(2\nu R_2)e^{-2\nu h_{cp}}]; \quad \text{(IV.85)}$$

$$F_4=2[A_{01}(R_1,h_{cp})-2\nu R_1B_{01}(R_1,h_{cp})+A_{01}(a)-2\pi\nu R_1J_0^2(2\nu R_1)e^{-2\nu h_{cp}}.$$

Functions A_{nm} and B_{nm} are determined by formulas (II.56). Evidently, the principal effect of the free surface and of the biplane foil on the lift is determined by formulas (IV.83)-(IV.85), since the terms in the expressions (IV.79) and (IV.80), containing $\Gamma_{i\infty}^2$ and $\Gamma_{1\infty}\Gamma_{j\infty}$, result in terms of the second order of smallness with respect to α.

It follows from formulas (IV.83)-(IV.85) that when [133

$Fr\to\infty$ functions $\gamma_i=\dfrac{\Gamma_i}{\Gamma_{i\infty}}$ are:

$$\gamma_1=1-2A_{01}(R_1,h_1)-\frac{2}{\overline{\Gamma}}[A_{01}(R_1h_{cp})+A_{01}(R_1,a)]; \quad \text{(IV.86)}$$

$$\gamma_2 = 1 - 2A_{01}(R_1, h_2) - 2\bar{\Gamma}[A_{01}(R_1, h_{cp}) + A_{01}(R_1, a)]; \qquad (IV.87)$$

$$\gamma_1 = \frac{1 + 2A_{01}(R_1, h_2) - \dfrac{2}{\bar{\Gamma}}[A_{01}(R_1, h_{cp}) + A_{01}(R_1 a)]}{[1+2A_{01}(R_1, h_2)][1+2A_{01}(R_1, h_1)] - 4[A_{01}(R_1, h_{cp}) + A_{01}(R_1, a)]^2}. \qquad (IV.88)$$

$$\gamma_2 = \frac{1 + 2A_{01}(R_1 h_1) - 2\bar{\Gamma}[A_{01}(R_1 h_{cp}) + A_{01}(R_1 a)]}{[1+2A_{01}(R_1, h_2)][1+2A_{01}(R_1, h_1)] - 4[A_{01}(R_1, h_{cp}) + A_{01}(R_1, a)]^2}. \qquad (IV.89)$$ [134

$$\bar{\Gamma} = \frac{\Gamma_{1\infty}}{\Gamma_{2\infty}}.$$

[133

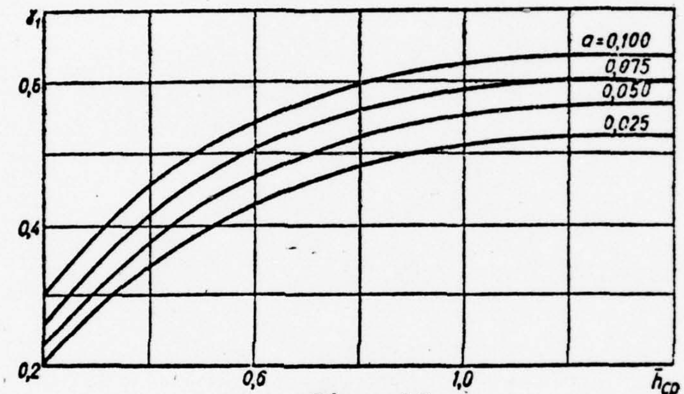

Fig. 12

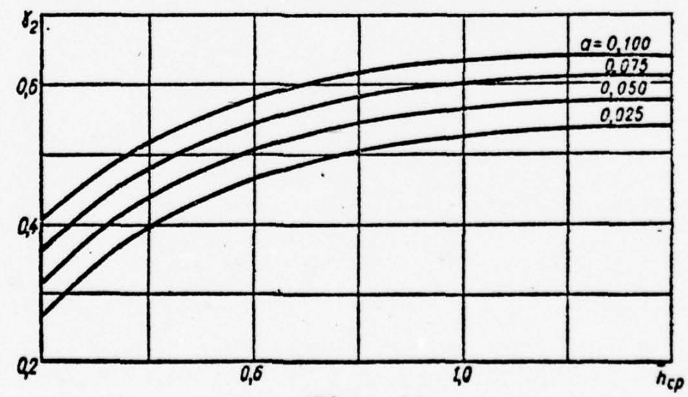

Fig. 13

With small submersions the results obtained with the aid of formulas (IV.88) and (IV.89) are more accurate than those obtained from formulas (IV.86) and (IV.87). [134

Let us determine functions A_{01} by the first terms in

the expansion with respect to τ.

Formulas (IV.88) and (IV.89) will be in the form:

$$\gamma_1 = \frac{\left(1+\frac{\tau_2^2}{2}\right) - \frac{1}{2\bar{\Gamma}}(\tau_3^2+\tau_4^2)}{\left(1+\frac{\tau_1^2}{2}\right)\left(1+\frac{\tau_2^2}{2}\right) - \frac{1}{4}(\tau_3^2+\tau_4^2)^2}, \qquad \text{(IV.90)}$$

$$\gamma_2 = \frac{\left(1+\frac{\tau_1^2}{2}\right) - \frac{\bar{\Gamma}}{2}(\tau_3^2+\tau_4^2)}{\left(1+\frac{\tau_1^2}{2}\right)\left(1+\frac{\tau_2^2}{2}\right) - \frac{1}{4}(\tau_3^2+\tau_4^2)^2}, \qquad \text{(IV.91)}$$

where

$$\tau_1 = \sqrt{4\bar{h}_1^2+1} - 2\bar{h}_1; \quad \tau_2 = \sqrt{4\bar{h}_2^2+1} - 2\bar{h}_2;$$

$$\tau_3 = \sqrt{4\bar{h}_{cp}^2+1} - 2\bar{h}_{cp}; \quad \tau_4 = \sqrt{4(\bar{h}_2-\bar{h}_1)^2+1} - 2(\bar{h}_2-\bar{h}_1).$$

With $h_1 \to \infty$ and $h_2 \to \infty$ the formulas will correspond to those describing the motion of a biplane in an infinite fluid.

Graphs of the relationships $\gamma_1 = f(h_{cp}, a)$ and $\gamma_2 = f(h_{cp}, a)$ determined from formulas (IV.83)-(IV.91) are shown in Figures 12 and 13.

4.3. The Lift and Wave Drag of a Hydrofoil System in a Fluid of Finite Depth

In order to solve the problem of motion of a hydrofoil system in a fluid of finite depth let us use the complex velocity representation in the form of (IV.1), where $v_{1j}(z)$ is now an analytical function in the area between the straight lines $0 > y > -h$ and contour C_{1j}; and $v_{2j}(z)$ is an analytical function in the band $0 > y > -h$.

The expression for the $v_{2j}(z)$ function satisfying the conditions on the free surface and in front of the system at infinity will easily follow from the expression (III.49). [135
The forces acting on the hydrofoils of the system we will determine from formula (II.9).

Let us first write the formula for $J_j = \int_{C_j} v_1(z)\, v_2(z)\, dz$.

Introducing functions $H_{ij}(\lambda)$ into expression (III.49) one can easily obtain

$$J_l = \sum_{i=1}^{n} \frac{1}{2\pi} \left\{ \int_0^\infty \left[H_{jl}(-\lambda) H_{il}(-\lambda) e^{-2\lambda h_0} + \frac{(\nu+\lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} \times \right.\right.$$

$$\left. \times \left(H_{jl}(-\lambda) \overline{H_{il}(-\lambda)} e^{-2\lambda h_0} - H_{jl}(\lambda) \overline{H_{il}}(\lambda) e^{2\lambda h_0} \right) \right] d\lambda -$$

$$- \frac{\pi i \nu}{2(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} [H_{jl}(-\lambda_0) \overline{H}_{il}(-\lambda_0) e^{-2\lambda_0 h_0} +$$

$$\left. + H_{jl}(\lambda_0) \overline{H}_{il}(\lambda_0) e^{2\lambda_0 h_0} - H_{jl}(-\lambda_0) \overline{H}_{il}(\lambda_0) - H_{jl}(\lambda_0) H_{il}(-\lambda_0)] \right\} +$$

$$+ \frac{1}{2\pi} \sum_{i=1}^{n}{}' \left[\int_0^{\infty \operatorname{Sign}(h_j - h_l)} H^{*}_{jl}(\lambda) H^{*}_{il}(-\lambda) \, d\lambda + \right.$$

$$\left. + \frac{1}{2} \int_0^\infty \frac{(\nu+\lambda) e^{-\lambda h_0}}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} (H_{jl}(\lambda) H_{il}(-\lambda) - H_{jl}(-\lambda) H_{il}(\lambda)) \, d\lambda \right]. \quad \text{(IV.92)}$$

Now, using expressions (IV.3) and (IV.92) one may derive the general formulas for forces acting on the hydrofoils of the system.

Separating the real part from the imaginary part in the formula, we obtain:

$$P_l = \varrho v_0 \Gamma_l - \frac{\varrho}{2\pi} \sum_{i=1}^{n} \left\{ \int_0^\infty \left\{ [H'_{jl}(-\lambda) H'_{il}(-\lambda) + H''_{jl}(-\lambda) H''_{il}(-\lambda)] \times \right.\right.$$

$$\times e^{-2\lambda h_0} + \frac{(\nu+\lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} (H'_{jl}(-\lambda) H'_{il}(-\lambda) + H''_{jl}(-\lambda) H''_{il}(-\lambda)) \times$$

$$\left. \times e^{-2\lambda h_0} + (H'_{jl}(\lambda) H'_{il}(\lambda) + H''_{jl}(\lambda) H''_{il}(\lambda)) e^{2\lambda h_0}] \right\} d\lambda - \frac{\pi \nu}{2(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} \times$$

$$\times [(H'_{jl}(-\lambda_0) H''_{il}(-\lambda_0) - H''_{jl}(-\lambda_0) H'_{il}(-\lambda_0)) e^{-2\lambda_0 h_0} + (H'_{jl}(\lambda_0) H''_{il}(\lambda_0) -$$

$$- H''_{jl}(\lambda_0) H'_{il}(\lambda_0)) e^{2\lambda_0 h_0} + H'_{jl}(-\lambda_0) H''_{il}(\lambda_0) + H''_{il}(-\lambda_0) H'_{il}(\lambda_0) +$$

$$\left. + H'_{jl}(\lambda_0) H''_{il}(-\lambda_0) + H''_{jl}(\lambda_0) H''_{il}(-\lambda_0)] \right\} + \sum_{i=1}^{n}{}' \int_0^{\infty \operatorname{sign}(h_j - h_l)} [H'_{jl}(\lambda) H'_{il}(-\lambda) -$$

$$- H''_{il}(\lambda) H''_{il}(-\lambda)] \, d\lambda + \frac{1}{2} \int_0^\infty \frac{(\nu+\lambda) e^{-\lambda h_0}}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} [-H'_{jl}(-\lambda) H'_{il}(\lambda) +$$ [136

$$+ H''_{jl}(-\lambda) H''_{il}(\lambda) + H'_{il}(\lambda) H'_{il}(-\lambda) - H''_{jl}(\lambda) H''_{il}(-\lambda)] \, d\lambda; \quad \text{(IV.93)}$$

$$Q_l = - \frac{\varrho \nu}{4(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} \sum_{i=1}^{n} \{[H'_{jl}(-\lambda_0) H'_{il}(-\lambda_0) +$$

$$+H''_{jj}(-\lambda_0)H''_{ij}(-\lambda_0)]e^{-2\lambda_0 h_0}+[H'_{jj}(\lambda_0)H'_{ij}(\lambda_0)+H''_{jj}(\lambda_0)H''_{ij}(\lambda_0)]e^{2\lambda_0 h_0}-$$

$$-H'_{jj}(-\lambda_0)H'_{ij}(\lambda_0)+H''_{jj}(-\lambda_0)H''_{ij}(\lambda_0)-H'_{jj}(\lambda_0)H'_{ij}(-\lambda_0)+$$

$$+H''_{jj}(\lambda_0)H''_{ij}(-\lambda_0)\}-\frac{\varrho}{2\pi}\sum_{i=1}^{n}\left[\int_0^\infty\{[H'_{ij}(-\lambda)H''_{ij}(-\lambda)-H''_{jj}(-\lambda)H'_{ij}(-\lambda)]\times\right.$$

$$\times e^{-2\lambda h_0}+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}[(H'_{jj}(-\lambda)H''_{ij}(-\lambda)-H''_{jj}(-\lambda)H''_{ij}(-\lambda))\times$$

$$\times e^{-2\lambda h_0}-(H'_{jj}(\lambda)H''_{ij}(\lambda)-H''_{jj}(\lambda)H'_{ij}(\lambda))e^{2\lambda h_0}]\,d\lambda-$$

$$-\frac{\varrho}{2\pi}\sum_{i=1}^{n\,\prime}\left\{\int_0^{\infty\operatorname{Sign}(h_j-h_i)}[(H'_{jj}(\lambda)H''_{ij}(-\lambda)+H''_{jj}(\lambda)H'_{ij}(-\lambda)]\,d\lambda+\right.$$

$$+\frac{1}{2}\int_0^\infty\frac{(\nu+\lambda)e^{-\lambda h_0}}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}[H'_{jj}(\lambda)H''_{ij}(-\lambda)+H''_{jj}(\lambda)H'_{ij}(-\lambda)-$$

$$\left.-H'_{jj}(-\lambda)H''_{ij}(\lambda)-H''_{jj}(\lambda)H'_{ij}(-\lambda)]\,d\lambda\right\}. \qquad (IV.94)$$

Using formula (IV.79), let us write an expression for $v_{2j}(z)$ which will satisfy the conditions along the profile of the j-th hydrofoil:

$$\int_{K_j}\frac{v_{2j}(z)}{\sigma-u}\,d\sigma=\frac{1}{2\pi}\sum_{i=1}^{n}\left\{\int_0^\infty\{\overline{H}_{ij}(-\lambda)G_j(-\lambda,u)e^{-2\lambda h_0}+\right.$$

$$+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}[\overline{H}_{ij}(-\lambda)G_j(-\lambda,u)e^{-2\lambda h_0}-\overline{H}_{ij}(\lambda)G_j(\lambda,u)e^{2\lambda h_0}-$$

$$-H_{ij}(\lambda)G_j(-\lambda,u)+H_{ij}(-\lambda)G_j(\lambda,u)]\}\,d\lambda-$$

$$-\frac{\nu\pi i}{2(\nu h-\operatorname{ch}^2\lambda_0 h_0)}[\overline{H}_{ij}(-\lambda_0)G_j(-\lambda_0,u)e^{-2\lambda_0 h_0}+ \qquad [137$$

$$\left.+\overline{H}_{ij}(\lambda_0)G_j(\lambda_0,u)e^{2\lambda_0 h_0}-H_{ij}(\lambda_0)G_j(-\lambda_0,u)-H_{ij}(-\lambda_0)G_j(\lambda_0,u)]\right\}. \qquad (IV.95)$$

$$\int_{K_j}\frac{\overline{v_{2j}(z)}}{\overline{\sigma}-\dfrac{R^2}{\overline{u}}}\,\overline{dz}=\frac{1}{2\pi}\sum_{i=1}^{n}\left\{\int_0^\infty\{H_{ij}(-\lambda)G_j\left(\overline{-\lambda,\frac{R^2}{\overline{u}}}\right)e^{-2\lambda h_0}+\right.$$

$$+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\Big[H_{il}(-\lambda)\,G_l\left(\overline{-\lambda,\frac{R^2}{\overline{u}}}\right)e^{-2\lambda h_0}-$$

$$-H_{il}(\lambda)\,\overline{G_l\left(-\lambda,\frac{R^2}{\overline{u}}\right)}\,e^{2\lambda h_0}-\overline{H}_{il}(\lambda)\,G_l\left(\overline{-\lambda,\frac{R^2}{\overline{u}}}\right)+$$

$$+\overline{H_{il}(-\lambda)}\,G_l\left(\overline{-\lambda,\frac{R^2}{\overline{u}}}\right)\Big]\Big|\,d\lambda+$$

$$+\frac{\nu\pi i}{2\,(\nu h-\mathrm{ch}^2\,\lambda_0 h_0)}\Big[H_{il}(-\lambda_0)\,G_l\left(\overline{-\lambda_0,\frac{R^2}{\overline{u}}}\right)e^{-2\lambda_0 h_0}+$$

$$+H_{il}(\lambda_0)\,\overline{G_l\left(\lambda_0,\frac{R^2}{\overline{u}}\right)}\,e^{2\lambda_0 h_0}-\overline{H}_{il}(\lambda_0)\,G_l\left(\overline{-\lambda_0,\frac{R^2}{\overline{u}}}\right)-$$

$$-\overline{H}_{il}(-\lambda_0)\,G_l\left(\overline{\lambda_0,\frac{R^2}{\overline{u}}}\right)\Big]\Big\} \qquad \text{(IV.96)}$$

and from formula (IV.69) we obtain

$$v_{1l}(z)=\frac{du}{dz}\Big(-v_0+\frac{\overline{v}_0R_l^2}{u^2}+\frac{C_l}{u}+$$

$$+\frac{1}{4\pi^2 i}\sum_{i=1}^{n}\Big\{\Big|\int_0^\infty\Big\{\overline{H}_{il}(-\lambda)\,G_i(-\lambda,u)\,e^{-2\lambda h_0}+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\times$$

$$\times\,[\overline{H}_{il}(-\lambda)\,G_l(-\lambda,u)\,e^{-2\lambda h_0}-\overline{H}_{il}(\lambda)\,G_i(\lambda,u)\,e^{2\lambda h_0}-$$

$$-H_{il}(\lambda)\,G_l(-\lambda,u)+H_{il}(-\lambda)\,G_l(\lambda,u)]\Big|\,d\lambda-$$

$$-\frac{\nu\pi i}{2\,(\nu h-\mathrm{ch}^2\,\lambda_0 h_0)}\,[\overline{H}_{il}(-\lambda_0)\,G_i(-\lambda_0,u)\,e^{-2\lambda_0 h_0}+\overline{H}_{il}(\lambda_0)\,G_i(\lambda_0,u)\,\times$$

$$\times\,e^{2\lambda_0 h_0}-H_{il}(\lambda_0)\,G_i(-\lambda_0,u)-H_{il}(\lambda_0)\,G_l(\lambda_0,u)]\Big\}+$$ [138

$$+\frac{1}{4\pi^2 i}\frac{R^2}{\overline{u}}\sum_{i=1}^{n}\Big\{\Big|\int_0^\infty\Big\{H_{il}(-\lambda)\,G_l\left(\overline{-\lambda,\frac{R^2}{\overline{u}}}\right)e^{-2\lambda h_0}+$$

$$+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\Big[H_{il}(-\lambda)\,G_i\left(\overline{-\lambda,\frac{R^2}{\overline{u}}}\right)e^{-2\lambda h_0}-H_{il}(\lambda)\,\times$$

$$\times\,\overline{G_i\left(\lambda,\frac{R^2}{\overline{u}}\right)}\,e^{2\lambda h_0}-\overline{H}_{il}(\lambda)G_l\left(\overline{-\lambda,\frac{R^2}{\overline{u}}}\right)+\overline{H}_{il}(-\lambda)G_l\left(\overline{-\lambda,\frac{R^2}{\overline{u}}}\right)\Big]\Big|\,d\lambda+$$

$$+\frac{\nu\pi i}{2\,(\nu h-\mathrm{ch}^2\,\lambda_0 h_0)}\Big[H_{il}(-\lambda_0)\,G_i\left(\overline{-\lambda_0,\frac{R^2}{\overline{u}}}\right)e^{-2\lambda_0 h_0}+$$

$$+ H_{ij}(\lambda_0)\overline{G_j\left(\lambda_0, \frac{R^2}{u}\right)} e^{2\lambda_0 h_0} - \overline{H}_{ij}(\lambda_0) G_j\left(-\lambda_0, \frac{R^2}{u}\right) -$$

$$- H_{ij}(-\lambda_0) G_j\left(\lambda_0, \frac{R^2}{u}\right]\Bigg)\Bigg\} + \frac{1}{4\pi^2 i}\sum_{i=1}^{n}{}' \int_0^{\infty \operatorname{Sign}(h_j - h_i)} \Bigg[H_{ij}(-\lambda) G_j(\lambda, u) +$$

$$+ \frac{R^2}{u^2}\overline{H_{ij}(-\overline{\lambda})}\, G_j\left(\lambda, \overline{\frac{R^2}{u}}\right)\Bigg] d\lambda. \qquad \text{(IV.97)}$$

4.4. Motion of Two Hydrofoils Under a Free Surface of a Fluid of Finite Depth

Forces acting on hydrofoils of a system consisting of two hydrofoils are determined by the formulas

$$P_j = \varrho v_0 \Gamma_j - \frac{\varrho}{2\pi}\int_0^{\infty}\Bigg[|H_{jj}(-\lambda)|^2 e^{-2\lambda h_0} + (\nu + \lambda) e^{-\lambda h_0} \times$$

$$\times \frac{|H_{jj}(-\lambda)|^2 e^{-2\lambda h_0} - |H_{jj}(+\lambda)|^2 e^{2\lambda h_0}}{2(\nu \operatorname{sh}\lambda h_0 - \lambda \operatorname{ch}\lambda h_0)}\Bigg] d\lambda +$$

$$+ \nu\varrho \frac{\operatorname{Im}[H_{jj}(\lambda_0) H_{jj}(-\lambda_0)]}{2(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} -$$

$$- \frac{\varrho}{2\pi}\int_0^{\infty}\Big\{[H'_{jj}(-\lambda) H'_{ij}(-\lambda) + H''_{jj}(-\lambda) H''_{ij}(-\lambda)] e^{-2\lambda h_0} +$$

$$+ \frac{(\nu+\lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh}\lambda h_0 - \lambda \operatorname{ch}\lambda h_0)}\Big[\Big(H'_{jj}(-\lambda) H'_{ij}(-\lambda) +$$ [139

$$+ H''_{jj}(-\lambda) H''_{ij}(-\lambda)\Big)\Big] e^{-2\lambda h_0} - [H'_{jj}(\lambda) H'_{ij}(\lambda) + H''_{jj}(\lambda) H''_{ij}(\lambda)] e^{2\lambda h_0} -$$

$$- H'_{jj}(-\lambda) H'_{ij}(\lambda) + H''_{jj}(-\lambda) H''_{ij}(\lambda) + H'_{jj}(\lambda) H'_{ij}(-\lambda) -$$

$$- H''_{jj}(\lambda) H''_{ij}(-\lambda)]\Big\} d\lambda + \frac{\pi\nu}{2(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} \{[H'_{jj}(-\lambda_0) H''_{ij}(-\lambda_0) -$$

$$- H''_{jj}(-\lambda_0) H_{ij}(-\lambda_0)] e^{-2\lambda h_0} + [H'_{jj}(\lambda_0) H''_{ij}(\lambda_0) -$$

$$- H''_{ij}(\lambda_0) H'_{ij}(\lambda_0)] e^{2\lambda_0 h_0} + H'_{jj}(-\lambda_0) H''_{ij}(\lambda_0) + H''_{jj}(-\lambda_0) H'_{ij}(\lambda_0) +$$

$$+ H'_{jj}(\lambda_0) H''_{ij}(-\lambda_0) + H''_{jj}(\lambda_0) H''_{ij}(-\lambda_0)\} + \int_0^{\infty \operatorname{Sign}(h_j - h_i)} [H'_{ij}(\lambda) H'_{ij}(-\lambda) -$$

$$- H''_{jj}(\lambda) H''_{ij}(-\lambda)] d\lambda; \qquad \text{(IV.98)}$$

$$Q_i = -\frac{\varrho\nu\, |\overline{H_{jj}(\lambda)}\, e^{\lambda_0 h_0} - H_{jj}(-\lambda_0) e^{-\lambda_0 h_0}|^2}{4(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} -$$

$$- \frac{\varrho\nu}{4(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} \{[H'_{jj}(-\lambda_0) H'_{ij}(-\lambda_0) + H''_{jj}(-\lambda_0) H''_{ij}(-\lambda_0)] e^{-2\lambda_0 h_0} +$$

$$+[H'_{ij}(\lambda_0)H'_{ij}(\lambda_0)+H''_{ij}(\lambda_0)H''_{ij}(\lambda_0)]e^{2\lambda_0 h_0}-[H'_{ij}(-\lambda_0)H'_{ij}(\lambda_0)+$$

$$+H''_{ij}(-\lambda_0)H''_{ij}(\lambda_0)-H'_{ij}(\lambda_0)H'_{ij}(-\lambda_0)+H''_{ij}(\lambda_0)H'_{ij}(-\lambda_0)]-$$

$$-\frac{\varrho}{2\pi}\int_0^\infty\{[H_{ij}-(\lambda)H_{ij}(-\lambda)-H''_{ij}(-\lambda)H_{ij}(-\lambda)]e^{-2\lambda_0 h_0}\times$$

$$\times\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}[(H'_{ij}(-\lambda)H''_{ij}(-\lambda)-H''_{ij}(-\lambda)H''_{ij}(-\lambda))e^{-2\lambda h_0}-$$

$$-(H'_{ij}(\lambda)H''_{ij}(\lambda)-H'_{ij}(\lambda)H'_{ij}(\lambda))e^{2\lambda h_0}+$$

$$+H'_{ij}(\lambda)H''_{ij}(-\lambda)+H''_{ij}(\lambda)H'_{ij}(-\lambda)-H'_{ij}(-\lambda)H''_{ij}(\lambda)-$$

$$\left.-H''_{ij}(-\lambda)H'_{ij}(\lambda)]\right\}d\lambda-\frac{\varrho}{2\pi}\int_0^{\infty\operatorname{Sign}(h_j-h_i)}(H'_{ij}(\lambda)H''_{ij}(-\lambda)+$$

$$+H''_{ij}(\lambda)H'_{ij}(-\lambda))d\lambda.\quad j=1,2\ i=1,2\ i\neq j. \tag{IV.99}$$

For a tandem system let us transform formulas (IV.98) and (IV.99) into a different form. Let us examine the integral [140

$$J=\int_0^\infty\frac{\varphi(\lambda)e^{ix\lambda}}{\psi(\lambda)}d\lambda.$$

Let us assume that function $\Psi(\lambda)$ has a simple zero at point λ_0.

For $x > 0$ and $x < 0$ the integral J may be written in the form

$$\int_0^\infty\frac{\varphi(\lambda)}{\psi(\lambda)}e^{ix\lambda}d\lambda=\int_{L_1}\frac{\varphi(\lambda)}{\psi(\lambda)}e^{ix\lambda}d\lambda+\frac{\pi i\varphi(\lambda_0)}{\varphi'(\lambda_0)}e^{ix\lambda},\quad x>0,$$

$$\int_0^\infty\frac{\varphi(\lambda)}{\psi(\lambda)}e^{ix\lambda}d\lambda=\int_{L_2}\frac{\varphi(\lambda)}{\psi(\lambda)}e^{ix\lambda}d\lambda-\frac{\pi i\varphi}{\psi'(\lambda_0)}e^{ix\lambda},\quad x<0. \tag{IV.100}$$

where L_1 is a contour consisting of a section of the real axis $0-\lambda_0-\xi$ of a small semicircle with radius ξ, which passes over the special point, and a section of the real axis $\lambda_0+\xi-\infty$.

Let L_2 consist of the identical sections, i.e., the real axis and a small semicircle which passes under the

the special point.

Integration by parts gives us

$$\int_{L1,2} \frac{\varphi(\lambda)}{\psi(\lambda)} e^{ix\lambda} d\lambda = i \frac{\varphi'(0)}{x\psi'(0)} + \frac{i}{x} \int_{L1,2} e^{i\lambda x} \frac{d}{d\lambda}\left(\frac{\varphi(\lambda)}{\psi(\lambda)}\right) d\lambda,$$

from which it is evident that

$$\lim_{|x|\to\infty} \int_{L1,2} \frac{\varphi(\lambda)}{\psi(\lambda)} e^{ix\lambda} d\lambda = 0. \qquad \text{(IV.101)}$$

Applying formula (IV.100) to formulas (IV.98) and (IV.99) we obtain after transformations

$$P_1 = \varrho v_0 \Gamma_1 - \frac{\varrho}{2\pi} \int_0^\infty \Big[|H_{11}(-\lambda)|^2 e^{-2\lambda h_0} + (\nu + \lambda) e^{-\lambda h_0} \times$$

$$\times \frac{|H_{11}(-\lambda)|^2 e^{-2\lambda h_0} - |H_{11}(\lambda)|^2 e^{2\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} \Big] d\lambda + \nu\varrho \frac{\operatorname{Im}[H_{11}(\lambda_0) H_{11}(-\lambda_0)]}{2(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} -$$

$$- \frac{\varrho}{2\pi} \int_0^\infty [H'_{11}(-\lambda) H'_{21}(-\lambda) + H''_{11}(-\lambda) H''_{21}(-\lambda)] e^{-2\lambda h_0} d\lambda -$$

$$- \frac{\varrho}{4\pi} \Big[\int_{L_1} \frac{(\nu + \lambda) e^{-\lambda h_0}}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} \{ [H'_{11}(-\lambda) H'_{21}(-\lambda) + H''_{11}(-\lambda) H''_{21}(-\lambda)] \times$$ [141

$$\times e^{-2\lambda h_0} - H'_{11}(-\lambda) H'_{21}(\lambda) + H''_{11}(-\lambda) H''_{21}(\lambda) \} d\lambda -$$

$$- \int_{L_1} \frac{(\nu + \lambda) e^{-\lambda h_0}}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} \{ [H'_{11}(\lambda) H'_{21}(\lambda) + H''_{11}(\lambda) H''_{21}(\lambda)] e^{2\lambda h_0} -$$

$$- H'_{11}(\lambda) H'_{21}(-\lambda) + H''_{11}(\lambda) H''_{21}(-\lambda) \} d\lambda \Big] -$$

$$- \frac{\varrho}{2\pi} \int_0^{\infty \operatorname{Sign}(h_j - h_i)} [H'_{11}(\lambda) H'_{21}(-\lambda) - H''_{11}(\lambda) H''_{21}(-\lambda)] d\lambda, \qquad \text{(IV.102)}$$

$$Q_1 = - \frac{\varrho\nu |\overline{H}_{11}(\lambda_0) e^{\lambda_0 h_0} - H_{11}(-\lambda_0) e^{-\lambda_0 h_0}|^2}{4(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} -$$

$$- \frac{\varrho}{2\pi} \Big[\int_0^\infty [H'_{11}(-\lambda) H''_{21}(-\lambda) - H''_{11}(-\lambda) H'_{21}(-\lambda)] e^{-2\lambda h_0} d\lambda +$$

$$+ \frac{1}{2} \int_{L_1} \frac{(\nu + \lambda) e^{-\lambda h_0}}{\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0} \{ [H'_{11}(-\lambda) H''_{21}(-\lambda) - H''_{11}(-\lambda) H'_{21}(-\lambda)] \times$$

$$\times e^{-2\lambda h_0} - H'_{11}(-\lambda) H''_{21}(\lambda) - H''_{11}(-\lambda) H'_{21}(\lambda) \} d\lambda +$$

$$+\frac{1}{2}\int\limits_{L_2}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\,[(H'_{11}(\lambda)\,H''_{21}(\lambda)-H''_{11}(\lambda)\,H'_{21}(\lambda))\,e^{2\lambda h_0}+$$

$$+H'_{11}(\lambda)\,H''_{21}(-\lambda)+H''_{11}(\lambda)\,H'_{11}(-\lambda)]\,d\lambda+$$

$$+\int\limits_{0}^{\infty\,\mathrm{Sign}(h_j-h_l)}[H'_{11}(\lambda)\,H''_{21}(-\lambda)+H''_{11}(\lambda)\,H'_{21}(-\lambda)]\,d\lambda; \qquad \text{(IV.103)}$$

$$P_2=\varrho v_0\Gamma_2-\frac{\varrho}{2\pi}\int\limits_0^{\infty}\Big[\,|H_{22}(-\lambda)|^2\,e^{-2\lambda h_0}+(\nu+\lambda)\,e^{-\lambda h_0}\times$$

$$\times\frac{|H_{22}(-\lambda)|^2\,e^{-2\lambda h_0}-|H_{22}(\lambda)|^2\,e^{2\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\Big]\,d\lambda+\nu\varrho\,\frac{\mathrm{Im}\,\{H_{22}(\lambda)\,H_{22}(-\lambda)\}}{2(\nu h-\mathrm{ch}^2\,\lambda_0 h_0)}-$$

$$-\frac{\varrho}{2\pi}\Bigg[\int\limits_0^{\infty}[H'_{22}(-\lambda)\,H'_{12}(-\lambda)+H''_{22}(-\lambda)\,H''_{12}(-\lambda)]\,e^{-2\lambda h_0}d\lambda+$$

$$+\frac{1}{2}\int\limits_{L_2}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\,[H'_{22}(-\lambda)\,H'_{12}(-\lambda)+H''_{22}(-\lambda)\,H''_{12}(-\lambda)]\times$$ [142

$$\times e^{-2\lambda h_0}-H'_{22}(-\lambda)\,H'_{12}(\lambda)+H''_{12}(\lambda)\,H''_{12}(-\lambda)\}\,d\lambda-$$

$$-\frac{1}{2}\int\limits_{L_1}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\{[H'_{22}(\lambda)\,H'_{21}(\lambda)+H''_{11}(\lambda)\,H''_{12}(\lambda)]\,e^{2\lambda h_0}-$$

$$-H'_{22}(\lambda)\,H'_{12}(-\lambda)+H''_{22}(\lambda)\,H''_{12}(-\lambda)\}\,d\lambda+$$

$$+\int\limits_{0}^{\infty\,\mathrm{Sign}(h_j-h_l)}[H'_{22}(\lambda)\,H'_{12}(-\lambda)-H''_{22}(\lambda)\,H''_{12}(-\lambda)]\,d\lambda-$$

$$-\frac{\pi\nu}{\nu h-\mathrm{ch}^2\,\lambda_0 h_0}\,\{[H'_{22}(-\lambda_0)\,H''_{12}(-\lambda_0)-H''_{22}(-\lambda_0)\,H'_{12}(-\lambda_0)]\times$$

$$\times e^{-2\lambda_0 h_0}+[H'_{22}(\lambda_0)\,H''_{12}(\lambda_0)-H''_{22}(\lambda_0)\,H'_{12}(\lambda_0)]\,e^{2\lambda_0 h_0}+H'_{22}(-\lambda_0)\,H''_{12}(\lambda_0)+$$

$$+H''_{22}(-\lambda_0)\,H'_{12}(\lambda_0)+H'_{22}(\lambda_0)\,H''_{12}(-\lambda_0)+H''_{22}(\lambda_0)\,H'_{12}(-\lambda_0)\}; \qquad \text{(IV.104)}$$

$$Q_2=-\frac{\varrho\nu\,|\overline{H}_{22}(\lambda_0)\,{}^{\lambda_0 h_0}-H_{22}(-\lambda_0)\,e^{-\lambda_0 h_0}|^2}{4(\nu h_0-\mathrm{ch}^2\,\lambda_0 h_0)}-$$

$$-\frac{\varrho}{2\pi}\Bigg[\int\limits_0^{\infty}\Big\{[H'_{22}(-\lambda)\,H''_{12}(-\lambda)-H''_{22}(-\lambda)\,H'_{12}(-\lambda)]\,e^{-2\lambda h_0}d\lambda+$$

$$+\frac{1}{2}\int\limits_{L_2}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\{[H'_{22}(-\lambda)\,H''_{12}(-\lambda)-H''_{22}(-\lambda)\,H'_{12}(-\lambda)]\times$$

$$\times e^{-2\lambda h_0}-H'_{22}(-\lambda)\,H''_{12}(\lambda)-H''_{22}(-\lambda)\,H'_{12}(\lambda)\}\,d\lambda+$$

$$+\frac{1}{2}\int_{L_1}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\{[H'_{22}(\lambda)H''_{12}(\lambda)-H''_{22}(\lambda)\,H'_{12}(\lambda)]\,e^{2\lambda h_0}+$$

$$+H'_{22}(\lambda)\,H''_{12}(-\lambda)+H''_{22}(\lambda)\,H'_{12}(-\lambda)\}\,d\lambda+$$

$$\left.+\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_l)}[H'_{22}(\lambda)\,H''_{12}(-\lambda)+H''_{22}(\lambda)\,H'_{12}(-\lambda)]\,d\lambda\right]-$$

$$-\frac{\varrho\nu}{2\,(\nu h_0-\mathrm{ch}^2\lambda_0 h_0)}\{[H'_{22}(-\lambda_0)\,H'_{12}(-\lambda_0)+H''_{22}(-\lambda_0)\,H''_{12}(-\lambda_0)]\times$$

$$\times e^{-2\lambda_0 h_0}+[H'_{22}(\lambda_0)\,H'_{12}(\lambda_0)+H''_{22}(\lambda_0)\,H''_{12}(\lambda_0)]\,e^{2\lambda_0 h_0}-H''_{22}(-\lambda_0)\,H'_{12}(\lambda_0)+$$ [143

$$+H''_{22}(-\lambda_0)\,H''_{12}(\lambda_0)-H'_{22}(\lambda_0)\,H'_{12}(-\lambda_0)+H''_{22}(\lambda_0)\,H'_{12}(-\lambda_0)\}. \qquad \text{(IV.105)}$$

Let us examine motion of a system consisting of two foils by N. Ye. Zhukovskiy.

Function $G(\lambda, u)G\left(\lambda, \frac{R^2}{u}\right)$ in this case is determined from formulas (II.35) and (II.36).

From formula (IV.97) we obtain

$$v_{1j}(z)=\frac{du}{dz}\Bigg\{-v_0+\frac{1}{2\pi}\int\limits_0^{\infty}J_0(2\lambda R_j)\Big\{[H_{jl}(-\lambda)+H_{lj}(-\lambda)]\,e^{-\lambda(2h_0-h_j)}+$$

$$+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[H_{lj}(-\lambda)+H_{jl}(-\lambda)]\,e^{-\lambda(2h_0-h_j)}-$$

$$-[H_{jl}(\lambda)+H_{lj}(\lambda)]\,e^{\lambda(2h_0-h_j)}-[\overline{H}_{lj}(\lambda)+\overline{H}_{jl}(\lambda)]\,e^{\lambda h_j}+$$

$$+[\overline{H}_{jl}(-\lambda)+\overline{H}_{lj}(-\lambda)]e^{-\lambda h_j}\Big\}\,d\lambda+\frac{i\nu}{4}\,\frac{J_0(2\lambda_0 R_j)}{(\nu h_0-\mathrm{ch}^2\,\lambda_0 h_0)}\times$$

$$\times[(H_{jl}(-\lambda_0)+H_{lj}(-\lambda_0))\,e^{-\lambda_0(2h_0-h_j)}+(H_{jl}(\lambda_0)+H_{lj}(\lambda_0)e^{\lambda_0(2h_0-h_j)}-$$

$$-(\overline{H}_{jl}(\lambda_0)+\overline{H}_{lj}(\lambda_0))\,e^{\lambda h_j}+(\overline{H}_{jl}(-\lambda_0)+\overline{H}_{jl}(-\lambda_0))\,e^{-\lambda h_i}+$$

$$+\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_l)}\overline{H}_{lj}(-\lambda)\,J_0(2\lambda R_j)\,e^{-\lambda h_j}\,d\lambda+\frac{\Gamma}{2\pi iu}\Bigg\}+$$

$$+\frac{R_j^2}{u^2}\Bigg\{\bar{v}_0+\frac{1}{2\pi}\int\limits_0^{\infty}J_0(2\lambda R_j)\Big\{(\overline{H}_{lj}(-\lambda)+\overline{H}_{lj}(-\lambda))\,e^{-\lambda(2h_0-h_j)}+$$

$$+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2\,(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[(\overline{H}_{lj}(-\lambda)+\overline{H}_{jl}(-\lambda))e^{-\lambda(2h_0-h_j)}-$$

$$-(\overline{H}_{lj}(\lambda)+\overline{H}_{jl}(\lambda))\,e^{\lambda(2h_0-h_j)}-$$

$$-\left(H_{ji}(\lambda)+H_{il}(\lambda)\right)e^{\lambda h_l}+\left(H_{ji}(-\lambda)+H_{il}(-\lambda)\right)e^{-\lambda h_l}\Big]\Bigg\}\,d\lambda-$$

$$-\frac{i\nu}{4}\frac{J_0(2\lambda_0R_j)}{\nu h_0-\mathrm{ch}^2\lambda_0h_0}\Bigg\{[\overline{H}_{ji}(-\lambda_0)+\overline{H}_{il}(-\lambda_0)]\,e^{-\lambda_0(2h_0-h_j)}+$$

$$+[\overline{H}_{jl}(\lambda_0)+\overline{H}_{il}(\lambda_0)]\,e^{\lambda_0(2h_0-h_j)}-[H_{jl}(\lambda_0)+H_{ii}(\lambda_0)]\,e^{\lambda h_l}+$$ [144

$$+[H_{jl}(-\lambda_0)+H_{il}(-\lambda_0)]\,e^{-\lambda h_l}+$$

$$+\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_l)}H_{il}(-\lambda)\,J_0(2\lambda R_i)\,e^{-\lambda h_l}d\lambda\Bigg\}+$$

$$+\frac{1}{4\pi^2 i}\int\limits_0^{\infty}\{[\overline{H}_{jl}(-\lambda)+\overline{H}_{il}(-\lambda)]\,B_j(-\lambda,u)\,e^{-\lambda(2h_0-h_j)}+$$

$$+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[\overline{H}_{il}(-\lambda)+\overline{H}_{il}(-\lambda)]\,B_l(-\lambda,u)\,e^{-\lambda(2h_0-h_j)}-$$

$$-[\overline{H}_{jl}(\lambda)+\overline{H}_{il}(\lambda)]\,B_j(\lambda,u)\,e^{\lambda(2h_0-h_j)}-[H_{jl}(\lambda)+H_{il}(\lambda)]B_i(-\lambda,u)e^{\lambda h_l}+$$

$$+[H_{ji}(-\lambda)+H_{ii}(-\lambda)]\,B_i(\lambda,u)\,e^{-\lambda h_l}\Big\}d\lambda-$$

$$-\frac{\nu}{8\pi(\nu h_0-\mathrm{ch}^2\lambda_0h_0)}\Big\{[\overline{H}_{ji}(-\lambda_0)+\overline{H}_{ii}(-\lambda_0)]\,B_l(-\lambda_0,u)\,e^{-\lambda_0(2h_0-h_j)}+$$

$$+[\overline{H}_{il}(\lambda_0)+\overline{H}_{il}(\lambda_0)]\,B_l(\lambda_0,u)\,e^{\lambda(2h_0-h_j)}-[H_{ji}(\lambda_0)+H_{ti}(\lambda_0)]\times$$

$$\times B_l(\lambda_0,u)\,e^{\lambda_0h_l}+[H_{jl}(-\lambda_0)+H_{il}(-\lambda_0)]\,B_l(\lambda_0,u)\,e^{\lambda_0h_l}\}+$$

$$+\frac{1}{4\pi^2 i}\cdot\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_l)}\overline{H}_{il}(-\lambda)\,B_l(\lambda,u)\,d\lambda+$$

$$+\frac{du}{d\frac{R_l^2}{u}}\frac{R_l^2}{u^2}\left(\frac{1}{4\pi^2 i}\int\limits_0^{\infty}\Big\{[H_{ji}(-\lambda)+H_{il}(-\lambda)]\,B_l\overline{\left(-\lambda,\frac{R^2}{u}\right)}\,e^{-\lambda(2h_0-h_j)}+\right.$$

$$+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2(\nu\mathrm{sh}\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[H_{il}(-\lambda)+H_{il}(-\lambda)]\,\overline{B_l\left(-\lambda,\frac{R^2}{u}\right)}e^{-\lambda(2h_0-h_j)}-$$

$$-[H_{ji}(\lambda)+H_{ii}(\lambda)]\overline{B_l\left(\lambda,\frac{R^2}{u}\right)}\,e^{\lambda(2h_0-h_j)}-[H_{li}(\lambda)+\overline{H}_{il}(\lambda)]\times$$

$$\times\overline{B_l\left(-\lambda,\frac{R^2}{u}\right)}\,e^{\lambda h_l}+\left(\overline{H}_{il}(-\lambda)+\overline{H}_{ii}(-\lambda)\right)\overline{B_l\left(\lambda,\frac{R^2}{u}\right)}\,e^{-\lambda h_l}\Big]\Big\}\times$$

$$\times d\lambda+\frac{\nu}{8\pi(\nu h_0-\mathrm{ch}^2\lambda_0h_0)}\Big[\left(H_{ji}(-\lambda_0)+H_{il}(-\lambda_0)\right)\overline{B_l\left(-\lambda_0,\frac{R^2}{u}\right)}\times$$

$$\times e^{-\lambda_0(2h_0-h_j)}+\left(H_{il}(\lambda_0)+H_{il}(\lambda_0)\right)\overline{B_l\left(-\lambda_0,\frac{R^2}{u}\right)}\,e^{\lambda(2h_0-h_j)}-$$

$$-(\overline{H}_{ll}(\lambda_0)+\overline{H}_{ll}(\lambda_0))\overline{B\left(-\lambda_0\frac{\overline{R^2}}{u}\right)}e^{\lambda h_l}+(\overline{H}_{ll}(-\lambda_0)+\overline{H}_{ll}(-\lambda_0))\times$$ [145

$$\times B\left(\overline{\lambda_0,\frac{\overline{R^2}}{u}}\right)e^{-\lambda h_l}\Bigg]+\int_0^{\infty\,\mathrm{Sign}(h_l-h_l)} H_{ll}(-\lambda)\overline{B_l\left(\lambda,\frac{R^2}{u}\right)}d\lambda. \qquad \text{(IV.106)}$$

In the same way as it was done with formulas (IV.98) and (IV.99) let us transform the expression (IV.106) with the aid of formulas (IV.100):

$$v_{11}(z)=\frac{du}{dz}\Bigg\{-\Bigg\{v_0+\frac{1}{2\pi}\int_0^\infty J_0(2\lambda R_l)\Big\{[H_{11}(-\lambda)+$$

$$+H_{21}(-\lambda)]e^{-\lambda(2h_0-h_1)}+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}[H_{11}(-\lambda)e^{-\lambda(2h_0-h_1)}-$$

$$-H_{11}(\lambda)e^{\lambda(2h_0-h_1)}-\overline{H}_{11}(\lambda)e^{\lambda h_1}+\overline{H}_{11}(-\lambda)e^{-\lambda h_1}]\Big\}d\lambda+$$

$$+\frac{1}{2\pi}\int_0^{\infty\,\mathrm{Sign}(h_1-h_2)}\overline{H}_{21}(-\lambda)J_0(2\lambda R_1)e^{-\lambda h_1}d\lambda+$$

$$+\frac{1}{4\pi}\int_{L_1}\frac{(\nu+\lambda)e^{-\lambda h_0}}{\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0}[-H_{21}(\lambda)e^{\lambda(2h_0-h_1)}+\overline{H}_{21}(-\lambda)e^{-\lambda h_1}]d\lambda+$$

$$+\frac{1}{4\pi}\int_{L_2}\frac{(\nu+\lambda)e^{-\lambda h_0}}{(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}[H_{21}(-\lambda)e^{-\lambda(2h_0-h_1)}-$$

$$-\overline{H}_{21}(\lambda)e^{\lambda h_1}]d\lambda+\frac{i\nu}{4}\frac{J_0(2\lambda_0R_1)}{(\nu h_0-\operatorname{ch}^2\lambda_0h_0)}[H_{11}(-\lambda_0)e^{-\lambda_0(2h_0-h_1)}+$$

$$+H_{11}(\lambda_0)e^{\lambda_0(2h_0-h_1)}-\overline{H}_{11}(\lambda_0)e^{\lambda h_1}+\overline{H}_{11}(-\lambda_0)e^{-\lambda h_1}]\Bigg\}+\frac{\Gamma}{2\pi iu}+$$

$$+\frac{R_1^2}{u^2}\Bigg\{\bar{v}_0+\frac{1}{2\pi}\int_0^\infty J_0(2\lambda R_1)\Big\{[\overline{H}_{11}(-\lambda)+\overline{H}_{21}(-\lambda)]e^{-\lambda(2h_0-h_1)}+$$

$$+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}[\overline{H}_{11}(-\lambda)e^{-\lambda(2h_0-h_1)}-\overline{H}_{11}(\lambda)e^{\lambda(2h_0-h_1)}-$$

$$-H_{11}(\lambda)e^{\lambda h_1}+H_{11}(-\lambda)e^{-\lambda h_1}]\Big\}d\lambda+$$

$$+\frac{1}{2\pi}\int_0^{\infty\,\mathrm{Sign}(h_1-h_2)}H_{21}(-\lambda)J_0(2\lambda R_1)e^{-\lambda h_1}d\lambda+$$

$$+\frac{1}{4\pi}\int_{L_2}\frac{(\nu+\lambda)e^{-\lambda h_0}}{(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}[-\overline{H}_{21}(\lambda)e^{\lambda(2h_0-h)}+H_{21}(-\lambda)e^{-\lambda h_1}]d\lambda+$$ [146

$$+\frac{1}{4\pi}\int_{L_1}\frac{(\nu+\lambda)e^{-\lambda h_0}}{(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}[\overline{H}_{21}(-\lambda)e^{-\lambda(2h_0-h_1)}-H_{21}(\lambda)e^{\lambda h_1}]d\lambda-$$

$$
-\frac{i\nu}{4}\frac{J_0(2\lambda_0R_1)}{(\nu h_0-\mathrm{ch}^2\lambda_0h_0)}[\overline{H}_{11}(-\lambda_0)e^{-\lambda_0(2h_0-h_1)}+
$$

$$
+\overline{H}_{11}(\lambda_0)e^{\lambda_0(2h_0-h_1)}-H_{11}(\lambda_0)e^{\lambda_0h_0}+H_{11}(-\lambda_0)e^{-\lambda_0h_1}]\Bigg\}\Bigg\}+
$$

$$
+\frac{1}{4\pi^2 i}\left\{\int_0^\infty\left\{[\overline{H}_{11}(-\lambda)+\overline{H}_{21}(-\lambda)]B_1(-\lambda,u)e^{-\lambda(2h_0-h_1)}+\right.\right.
$$

$$
+\frac{(\nu+\lambda)e^{-2\lambda h_0}}{2(\nu\mathrm{sh}\lambda h_0-\lambda\mathrm{ch}\lambda h_0)}[\overline{H}_{11}(-\lambda)e^{-\lambda(2h_0-h_1)}B_1(-\lambda,u)-
$$

$$
-\overline{H}_{11}(\lambda)e^{\lambda(2h_0-h_1)}B_1(\lambda,u)-H_{11}(\lambda)B_1(-\lambda,u)e^{\lambda h_1}+
$$

$$
+H_{11}(-\lambda)B_1(\lambda,u)e^{-\lambda h_1}]\Big\}d\lambda+\frac{1}{2}\int_{L_0}\frac{(\nu+\lambda)e^{-\lambda h_0}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\times
$$

$$
\times[-\overline{H}_{21}(\lambda)e^{\lambda(2h_0-h_1)}+H_{21}(-\lambda)e^{-\lambda h_1}]B_1(\lambda,u)\,d\lambda+
$$

$$
+\frac{1}{2}\int_{L_1}\frac{(\nu+\lambda)e^{-\lambda h_0}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[\overline{H}_{21}(-\lambda)e^{-\lambda(2h_0-h_1)}-H_{21}(\lambda)e^{\lambda h_1}]B_1(-\lambda,u)d\lambda+
$$

$$
+\int_0^{\infty\,\mathrm{Sign}(h_1-h_2)}\overline{H}_{21}(-\lambda)B_1(\lambda,u)\,d\lambda\Bigg\}-\frac{\nu}{8\pi(\nu h_0-\mathrm{ch}^2\lambda_0h_0)}\times
$$

$$
\times\left\{\left[\overline{H}_{11}(-\lambda_0)B_1(-\lambda_0,u)-H_{11}(-\lambda_0)\frac{R^2}{u^2}\frac{dz}{d\frac{R^2}{u}}\overline{B_1\left(-\lambda_0,\frac{R^2}{u}\right)}\right]\times\right.
$$

$$
\times e^{-\lambda_0(2h_0-h_1)}+\left[\overline{H}_{11}(\lambda_0)B_1(\lambda_0,u)-H_{11}(\lambda_0)\times\right.
$$

$$
\left.\times\frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u}}\overline{B_1\left(\lambda_0,\frac{R^2}{u}\right)}\right]e^{\lambda(2h_0-h_1)}-
$$

$$
-\left[H_{11}(\lambda_0)B_1(-\lambda_0,u)-\overline{H}_{11}(\lambda_0)\frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u}}\overline{B_1\left(-\lambda_0,\frac{R_2}{u}\right)}\right]e^{\lambda h_1}+ \qquad [147
$$

$$
+\left[H_{11}(-\lambda_0)B_1(\lambda_0u)-\overline{H}_{11}(\lambda_0)\frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u^2}}\overline{B_1\left(-\lambda_0,\frac{R^2}{u}\right)}\right]e^{-\lambda h_1}\Bigg\}\Bigg\}+
$$

$$
+\frac{du}{d\frac{R^2}{u}}\frac{R^2}{u^2}\left\{\frac{1}{4\pi^2 i}\left\{\int_0^\infty\left[(\overline{H}_{11}(-\lambda)+\overline{H}_{21}(-\lambda))\overline{B_1\left(-\lambda,\frac{R^2}{u}\right)}\times\right.\right.\right.
$$

$$\times e^{-\lambda(2h_0-h_1)}+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}\left(H_{11}(-\lambda)\,e^{-\lambda(2h_0-h_1)}\,\overline{B_1\left(-\lambda,\frac{R^2}{\bar u}\right)}-\right.$$

$$-H_{11}(\lambda)\,e^{\lambda(2h_0-h_1)}\,\overline{B_1\left(\lambda,\frac{R^2}{\bar u}\right)}-\overline{H}_{11}(\lambda)\,\overline{B_1\left(-\lambda,\frac{R^2}{\bar u}\right)}\,e^{\lambda h_1}+$$

$$\left.\left.+\overline{H}_{11}(-\lambda)\,\overline{B_1\left(\lambda,\frac{R^2}{\bar u}\right)}\,e^{-\lambda h_1}\right)\right]d\lambda+\frac{1}{2}\int\limits_{L_1}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}\times$$

$$\times\left[-H_{21}(\lambda)\,e^{\lambda(2h_0-h_1)}+\overline{H}_{21}(-\lambda)\,e^{-\lambda h_1}\right]\overline{B_1\left(\lambda,\frac{R^2}{\bar u}\right)}\,d\lambda+$$

$$+\frac{1}{2}\int\limits_{L_2}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}\left[H_{21}(-\lambda)\,e^{-\lambda(2h_0-h_1)}-\right.$$

$$\left.-\overline{H}_{21}(\lambda)\,e^{\lambda h_1}\right]\overline{B_1\left(-\lambda,\frac{R^2}{\bar u}\right)}\,d\lambda+\int\limits_0^{\infty\operatorname{Sign}(h_1-h_2)}H_{21}(-\lambda)\,\overline{B_1\left(\lambda,\frac{R^2}{\bar u}\right)}\,d\lambda\Bigg\};\qquad\text{(IV.107)}$$

$$v_{12}(z)=\frac{du}{dz}\left\{-\left\{v_0+\frac{1}{2\pi}\int\limits_0^{\infty}J_0(2\lambda R_2)\left[(H_{22}(-\lambda)+H_{12}(-\lambda))\,e^{-\lambda(2h_0-h_2)}+\right.\right.\right.$$

$$+\frac{(\nu+\lambda)\,e^{-2\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}\left(H_{22}(-\lambda)\,e^{-\lambda(2h_0-h_2)}-H_{22}(\lambda)\,e^{\lambda(2h_0-h_2)}-\right.$$

$$\left.\left.-\overline{H}_{22}(\lambda)\,e^{\lambda h_2}+\overline{H}_{22}(-\lambda)\,e^{-\lambda h_2}\right)\right]d\lambda+$$

$$+\frac{1}{2\pi}\int\limits_0^{\infty\operatorname{Sign}(h_2-h_1)}\overline{H}_{12}(-\lambda)\,J_0(2\lambda R_1)\,e^{-\lambda h_1}\,d\lambda+$$

$$+\frac{1}{4\pi}\int\limits_{L_2}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}\left[-H_{12}(\lambda)\,e^{\lambda(2h_0-h_2)}+\overline{H}_{12}(-\lambda)\,e^{-\lambda h_2}\right]d\lambda+$$ [148

$$+\frac{1}{4\pi}\int\limits_{L_1}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}\left[H_{12}(-\lambda)\,e^{-\lambda(2h_0-h_2)}-\overline{H}_{12}(\lambda)\,e^{\lambda h_2}\right]d\lambda+$$

$$+\frac{i\nu}{4}\,\frac{J_0(2\lambda_0R_2)}{(\nu h_0-\operatorname{ch}^2\lambda_0h_0)}\left[(H_{22}(-\lambda_0)+2H_{12}(-\lambda_0))\,e^{-\lambda_0(2h_0-h_2)}+\right.$$

$$+(H_{22}(\lambda_0)+2H_{12}(\lambda_0))\,e^{\lambda_0(2h_0-h_2)}-(\overline{H}_{22}(\lambda_0)+2H_{12}(\lambda_0))\,e^{\lambda h_2}+$$

$$\left.\left.+(\overline{H}_{22}(-\lambda_0)+2\overline{H}_{12}(-\lambda_0))\,e^{-\lambda h_2}\right]\right\}+\frac{\Gamma}{2\pi iu}+$$

$$+\frac{R^2}{u^2}\left\{v_0+\frac{1}{2\pi}\int\limits_0^{\infty}J_0(2\lambda R_2)\left[(\overline{H}_{22}(-\lambda)+\overline{H}_{21}(-\lambda))\,e^{-\lambda(2h_0-h_2)}+\right.\right.$$

$$+\frac{(\nu+\lambda)\,e^{-2\lambda h_0}}{2(\nu\operatorname{sh}\lambda h_0-\lambda\operatorname{ch}\lambda h_0)}\left(\overline{H}_{22}(-\lambda)\,e^{-\lambda(2h_0-h_2)}-\right.$$

$$-\overline{H}_{22}(-\lambda)\,e^{\lambda(2h_0-h_2)}-H_{22}(\lambda)\,e^{\lambda h_2}+H_{22}(-\lambda)\,e^{-\lambda h_2}\Big)\Big]\,d\lambda+$$

$$+\frac{1}{2\pi}\int_0^{\infty\,\mathrm{Sign}(h_2-h_1)} H_{12}(-\lambda)\,J_0(2\lambda R_2)\,e^{-\lambda h_2}d\lambda+$$

$$+\frac{1}{4\pi}\int_{L_1}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\,[-\overline{H}_{12}(\lambda)\,e^{+\lambda(2h_0-h_2)}+$$

$$+H_{12}(-\lambda)\,e^{-\lambda h_2}]\,d\lambda+\frac{1}{4\pi}\int_{L_2}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\,[\overline{H}_{12}(-\lambda)\,e^{-\lambda(2h_0-h_2)}-$$

$$-H_{12}(\lambda)\,e^{\lambda h_2}]\,d\lambda-\frac{i\nu}{4}\frac{J_0(2\lambda R_1)}{(\nu h_0-\mathrm{ch}^2\lambda_0 h_0)}\,[(\overline{H}_{22}(-\lambda_0)+$$

$$+2\overline{H}_{12}(-\lambda_0))\,e^{-\lambda_0(2h_0-h_2)}+(\overline{H}_{22}(\lambda_0)+2\overline{H}_{12}(\lambda_0))\,e^{\lambda_0(2h-h_2)}-$$

$$-(H_{22}(\lambda_0)+2H_{12}(\lambda_0))\,e^{\lambda h_2}+(H_{22}(-\lambda_0)+2H_{12}(-\lambda_0))\,e^{-\lambda h_2}]\Big\}\Big\}+$$

$$+\frac{1}{4\pi^2 i}\Big\{\int_0^{\infty}\Big[(\overline{H}_{22}(-\lambda)+\overline{H}_{12}(-\lambda))\,B_2(-\lambda,u)\,e^{-\lambda(2h_0-h_2)}+$$

$$+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}(\overline{H}_{22}(-\lambda)\,e^{-\lambda(2h_0-h_2)}\,B_2(-\lambda,u)-$$ [149

$$-\overline{H}_{22}(\lambda)\,e^{\lambda(2h_0-h_2)}B_2(\lambda,u)-H_{22}(\lambda)\,B_2(-\lambda,u)\,e^{\lambda h_2}+$$

$$+H_{22}(-\lambda)\,B_2(\lambda,u)\,e^{-\lambda h_2})\Big]\,d\lambda+\frac{1}{2}\int_{L_1}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\times$$

$$\times[-\overline{H}_{12}(\lambda)\,e^{\lambda(2h_0-h_2)}+H_{12}(-\lambda)\,e^{-\lambda h_2}]\,B_2(\lambda,u)\,d\lambda+$$

$$+\frac{1}{2}\int_{L_2}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[\overline{H}_{12}(-\lambda)e^{-\lambda(2h_0-h_2)}-H_{12}(\lambda)e^{\lambda h_2}]B_2(-\lambda,u)d\lambda+$$

$$+\int_0^{\infty\,\mathrm{Sign}(h_2-h_1)}\overline{H}_{12}(-\lambda)\,B_2(\lambda,u)\,d\lambda\Big\}-\frac{\nu}{8\pi(\nu h-ch^2\lambda_0 h_0)}\times$$

$$\times\Big\{\Big[(\overline{H}_{22}(-\lambda_0)+2\overline{H}_{12}(-\lambda_0))\,B_2(-\lambda_0,u)-(H_{22}(-\lambda_0)+2H_{12}(-\lambda_0))\times$$

$$\times\frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u}}\overline{B_2\left(-\lambda_0\frac{R^2}{u}\right)}\Big]e^{-\lambda_0(2h_0-h_2)}+\Big[(\overline{H}_{22}(\lambda_0)+2H_{12}(\lambda_0))B_2(\lambda_0,u)-$$

$$-(H_{22}(\lambda_0)+2H_{12}(\lambda_0))\frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u}}\overline{B_2\left(\lambda_0,\frac{R^2}{u}\right)}\Big]e^{\lambda_0(2h_0-h_2)}-\Big[(H_{22}(\lambda_0)+$$

$$+2H_{12}(\lambda_0))\,B_2(-\lambda_0,u)-(H_{22}(\lambda_0)+2\overline{H}_{12}(\lambda_0))\frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u}}\times$$

$$\times\,\overline{B_2\left(-\lambda_0,\frac{R^2}{u}\right)}\Bigg]e^{\lambda h_2}+\Bigg[(H_{22}(-\lambda_0)+2H_{12}(-\lambda_0))\,B_2(\lambda_0,u)-$$

$$-(\overline{H}_{22}(-\lambda_0)+2\overline{H}_{12}(-\lambda_0))\frac{R^2}{u^2}\frac{du}{d\frac{R^2}{u}}\overline{B_2\left(-\lambda_0,\frac{R^2}{u}\right)}\Bigg]e^{-\lambda h_2}\Bigg\}+$$

$$+\frac{du}{d\frac{R^2}{u}}\frac{R^2}{u^2}\cdot\frac{1}{4\pi^2 i}\Bigg\{\int_0^\infty\Bigg[(\overline{H}_{22}(-\lambda)+\overline{H}_{12}(-\lambda))\,\overline{B_2\left(-\lambda_0,\frac{R^2}{u}\right)}\,e^{-\lambda(2h_0-h_2)}+$$

$$+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\Bigg(H_{22}(-\lambda)e^{-\lambda(2h_0-h_2)}\overline{B_2\left(\lambda,\frac{R^2}{u}\right)}-$$ [150

$$-H_{22}(\lambda)\,e^{\lambda(2h_0-h_2)}\,\overline{B_2\left(\lambda,\frac{R^2}{u}\right)}-\overline{H}_{22}(\lambda)\,\overline{B_2\left(-\lambda,\frac{R^2}{u}\right)}\,e^{\lambda h_2}+$$

$$+\overline{H}_{22}(-\lambda)\,\overline{B_2\left(\lambda,\frac{R^2}{u}\right)}\,e^{-\lambda h_2}\Bigg)\Bigg]d\lambda+\frac{1}{2}\int_{L_1}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}\times$$

$$\times[-H_{12}(\lambda)\,e^{\lambda(2h_0-h_2)}+\overline{H}_{12}(-\lambda)\,e^{-\lambda h_2}]\,\overline{B_2\left(-\lambda,\frac{R^2}{u}\right)}\,d\lambda+$$

$$+\frac{1}{2}\int_{L_1}\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0}[H_{12}(-\lambda)\,e^{-\lambda(2h_0-h_2)}-\overline{H}_{21}(\lambda)\,e^{\lambda h_2}]\times$$

$$\times\,\overline{B_2\left(-\lambda,\frac{R^2}{u}\right)}\,d\lambda+\int_0^{\infty\,\mathrm{Sign}(h_2-h_1)}H_{12}(-\lambda)\,B_2\left(\overline{\lambda,\frac{R^2}{u}}\right)d\lambda\Bigg\}. \qquad \text{(IV.108)}$$

Let condition (II.33) be satisfied at points $u_1 = -R_1$ and $u_2 = -R_2$. Then from expressions (IV.107) and (IV.108) we obtain a system of equations for determining Γ_j:

$$\Gamma_j = 4\pi R_j \mathrm{Im}\Bigg\{v_0+\frac{1}{2\pi}\int_0^\infty J_0(2\lambda R_i)\Big([H_{jj}(-\lambda)+H_{ij}(-\lambda)]\,e^{-\lambda(2h_0-h_j)}+$$

$$+\frac{(\nu+\lambda)\,e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[(H_{ij}(-\lambda)+H_{ij}(-\lambda))\,e^{-\lambda(2h_0-h_j)}-$$

$$-(H_{jj}(\lambda)+H_{ij}(\lambda))\,e^{\lambda(2h_0-h_j)}-(\overline{H}_{jj}(\lambda)+\overline{H}_{ij}(\lambda))\,e^{\lambda h_j}+$$

$$+(\overline{H}_{jj}(-\lambda)+\overline{H}_{ij}(-\lambda))\,e^{-\lambda h_j}\Big)\,d\lambda+$$

$$+\frac{i\nu}{4}\frac{J_0(2\lambda_0R_j)}{(\nu h_0-\mathrm{ch}^2\lambda_0h_0)}[(H_{jj}(-\lambda_0)+H_{ij}(-\lambda_0))e^{-\lambda_0(2h_0-h_j)}+(H_{ij}(\lambda_0)+ H_{ij}(\lambda_0))e^{\lambda_0(2h_0-h_j)}-(\overline{H}_{jj}(\lambda_0)+H_{ij}(\lambda_0))e^{\lambda_0h_j}-(\overline{H}_{jj}(-\lambda_0)+ \overline{H}_{ij}(-\lambda_0))e^{-\lambda_0h_j}]+\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_i)}\overline{H}_{ij}(-\lambda)J_0(2\lambda R_j)e^{-\lambda h_j}d\lambda\Bigg\}. \qquad (j=1,2,\ i\neq j) \tag{IV.109}$$

For the hydrofoils in a tandem system, expression (IV.109) will become [151

$$\Gamma_j=4\pi R_j\,\mathrm{Im}\Bigg\{v_0+\frac{1}{2\pi}\int\limits_0^{\infty}J_0(2\lambda R_j)\Big[(H_{jj}(-\lambda)+H_{ij}(-\lambda))e^{-\lambda(2h_0-h_j)}+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\lambda h_0)}(H_{jj}(-\lambda)e^{-\lambda(2h_0-h_j)}-H_{jj}(\lambda)e^{\lambda(2h_0-h_j)}-\overline{H}_{jj}(\lambda)e^{\lambda h_j}+\overline{H}_{jj}(-\lambda)e^{-\lambda h_j})\Big]d\lambda+\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_i)}\overline{H}_{ij}(-\lambda)J_0(2\lambda R_j)e^{-\lambda h_j}d\lambda+$$
$$+\frac{1}{4\pi}\int\limits_{L_j}\frac{(\nu+\lambda)e^{-\lambda h_0}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[-H_{jj}(\lambda)e^{\lambda(2h_0-h_j)}+H_{jj}(-\lambda)e^{-\lambda h_j}]d\lambda+\frac{1}{4\pi}\int\limits_{L_i}\frac{(\nu+\lambda)e^{-\lambda h_0}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[H_{ij}(-\lambda)e^{-\lambda(2h_0-h_j)}-\overline{H}_{ij}(\lambda)e^{\lambda h_j}]d\lambda+$$
$$+\frac{i\nu}{4}\frac{J_0(2\lambda_0R_j)}{(\nu h_0-\mathrm{ch}^2\lambda_0h_0)}[H_{jj}(-\lambda_0)e^{-\lambda_0(2h_0-h_j)}+H_{jj}(\lambda_0)e^{\lambda_0(2h_0-h_j)}-\overline{H}_{jj}(\lambda_0)e^{\lambda_0h_j}-\overline{H}_{jj}(-\lambda_0)e^{-\lambda_0h_j}+F_j(\lambda_0)]\Bigg\}. \quad (j=1,2,\ i\neq j) \tag{IV.110}$$

$$F_j(\lambda_0)=\begin{cases}0 \text{ with } j=1,\\ 2[H_{12}(-\lambda_0)e^{-\lambda_0(2h_0-h_2)}+H_{12}(\lambda_0)e^{\lambda_0(2h_0-h_2)}-\\ -\overline{H}_{12}(\lambda_0)e^{\lambda_0h_2}-\overline{H}_{12}(-\lambda_0)e^{-\lambda_0h_2}] \text{ with } i=2.\end{cases}$$

The system of functional equations for determining functions $H_{ij}(\mu)$ may be derived in the usual way:

$$H_{ij}(\mu)=e^{-\mu h_j}\Bigg\{\Gamma_j[J_0(2\mu R_j)-iJ_1(2\mu R_j)]+\frac{1}{4\pi^2i}\Bigg(\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_i)}[e^{-i\lambda L_{ij}}H_{ij}(-\lambda)N_{0j}(\lambda,\mu)+\overline{H}_{ij}(-\lambda)e^{i\lambda L_{ij}}N_{1j}(\lambda,\mu)]d\lambda+$$

$$+\int_0^\infty [(\overline{H}_{ij}(-\lambda)+e^{-i\lambda L_{ij}}\overline{H}_{jj}(-\lambda))N_{0j}(-\lambda,\mu)+(H_{ij}(-\lambda)+$$

$$+e^{-i\lambda L_{ij}}H_{ii}(-\lambda))N_{1j}(-\lambda,\mu)]e^{-\lambda(2h_0-h_j)}+$$ [152

$$+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[(\overline{H}_{ij}(-\lambda)+e^{i\lambda L_{ij}}H_{ii}(-\lambda))\times$$

$$\times N_{0i}(-\lambda,\mu)e^{-\lambda(2h_0-h_j)}+(H_{ji}(-\lambda)+e^{-i\lambda L_{ij}}H_{ii}(-\lambda))N_{1j}(-\lambda,\mu)\times$$

$$\times e^{-\lambda(2h_0-h_j)}-(\overline{H}_{ij}(\lambda)+e^{-i\lambda L_{ij}}\overline{H}_{ii}(\lambda))N_{0j}(\lambda,\mu)e^{\lambda(2h_0-h_j)}-$$

$$-(H_{ij}(\lambda)+e^{i\lambda L_{ij}}H_{ii}(\lambda))N_{1j}(\lambda,u)e^{\lambda(2h_0-h_j)}-(H_{ij}(\lambda)+e^{i\lambda L_{ij}}H_{ii}(\lambda))\times$$

$$\times N_{0i}(-\lambda,\mu)e^{\lambda h_j}-(\overline{H}_{ji}(\lambda)+e^{-i\lambda L_{ij}}\overline{H}_{ii}(\lambda))N_{1j}(-\lambda,\mu)e^{\lambda h_j}+$$

$$+(H_{ij}(-\lambda)+e^{-i\lambda L_{ij}}H_{ii}(-\lambda))N_{0i}(\lambda,\mu)e^{-\lambda h_j}+$$

$$\left.+(\overline{H}_{ji}(-\lambda)+e^{i\lambda L_{ij}}\overline{H}_{ij}(-\lambda))N_{1j}(\lambda,\mu)e^{-\lambda h}\right\}d\lambda-$$

$$-\frac{\nu}{8\pi(\nu h_0-\mathrm{ch}^2\lambda_0 h_0)}[(\overline{H}_{ji}(-\lambda_0)+e^{i\lambda_0 L_{ij}}\overline{H}_{ij}(-\lambda_0))N_{0i}(-\lambda,\mu)\times$$

$$\times e^{-\lambda(2h_0-h_j)}-(H_{ij}(-\lambda_0)+e^{-i\lambda L_{ij}}H_{ii}(-\lambda_0))N_{ii}(-\lambda_0,\mu)e^{-\lambda_0(2h_0-h_j)}+$$

$$+(\overline{H}_{ij}(\lambda)+e^{-i\lambda L_{ij}}\overline{H}_{ii}(\lambda_0))N_{0j}(\lambda_0,\mu)e^{\lambda_0(2h_0-h_j)}-$$

$$-(H_{ji}(\lambda_0)+e^{i\lambda L_{ij}}H_{ii}(\lambda_0))N_{1i}(\lambda_0,\mu)e^{\lambda_0(2h_0-h_j)}-$$

$$-(H_{ji}(\lambda_0)+e^{i\lambda L_{ij}}H_{ii}(\lambda_0))N_{0j}(-\lambda_0,\mu)e^{\lambda h_j}+$$

$$+(\overline{H}_{jj}(\lambda_0)+e^{-i\lambda L_{ij}}\overline{H}_{ii}(\lambda_0))N_{1i}(-\lambda_0,\mu)e^{\lambda h_j}+$$

$$+(H_{ij}(-\lambda_0)+e^{-i\lambda L_{ij}}H_{ii}(-\lambda_0))N_{0j}(\lambda_0,\mu)e^{-\lambda h_j}-$$

$$\left.-(\overline{H}_{ij}(-\lambda_0)+e^{i\lambda L_{ij}}\overline{H}_{ii}(-\lambda_0))N_{1j}(\lambda_0,\mu)e^{-\lambda h_j}]\right\}. \qquad \text{(IV.111)}$$

For the tandem hydrofoils the system of functional equations is obtained from formulas (IV.106) and (IV.107):

$$H_{jj}(\mu)=e^{\mu h_j}\left\{\Gamma_j[J_0(2\mu R_j)-iJ_1(2\mu R_j)]+\right.$$

$$+\frac{1}{4\pi^2 i}\left(\int_0^{\infty\,\mathrm{Sign}(h_j-h_i)}[e^{-i\lambda L_{ij}}H_{ii}(-\lambda)N_{0j}(\lambda,\mu)+\overline{H}_{ii}(-\lambda)e^{i\lambda L_{ij}}N_{1i}(\lambda,\mu)]\,d\lambda+\right.$$

$$+\int_0^\infty\left[(\overline{H}_{jj}(-\lambda)+e^{i\lambda L_{ij}}\overline{H}_{ii}(-\lambda))N_{0j}(-\lambda,\mu)e^{-\lambda(2h_0-h_j)}+(H_{jj}(-\lambda)+\right.$$

$$+e^{-i\lambda L_{lj}} H_{lj}(-\lambda)) N_{1j}(-\lambda,\mu)\, e^{-\lambda(2h_0-h_j)} + \frac{(\nu+\lambda)\, e^{-\lambda h_0}}{2(\nu \operatorname{sh}\lambda h_0 - \lambda \operatorname{ch}\lambda h_0)} \times$$ [153

$$\begin{aligned}
&\times(\overline{H}_{lj}(-\lambda)\, \overline{N_{0j}(-\lambda,\mu)}\, e^{-\lambda(2h_0-h_j)} + H_{lj}(-\lambda)\, N_{1j}(-\lambda,\mu)\, e^{-\lambda(2h_0-h_j)} - \\
&- \overline{H}_{lj}(\lambda)\, \overline{N}_{0j}(\lambda,\mu)\, e^{\lambda(2h_0-h_j)} - H_{lj}(\lambda)\, \overline{N}_{1j}(\lambda,\mu)\, e^{\lambda(2h-h_j)} - \\
&- H_{lj}(\lambda)\, \overline{N}_{0j}(-\lambda,\mu)\, e^{\lambda h_j} - \overline{H}_{lj}(\lambda)\overline{N}_{1j}(-\lambda,\mu)\, e^{\lambda h_j} + H_{lj}(-\lambda)\, N_{0j}(-\lambda,\mu)\times \\
&\times e^{-\lambda h_j} + \overline{H}_{lj}(-\lambda)\, N_{1j}(-\lambda,\mu)\, e^{-\lambda h_j}\Big]\, d\lambda + \frac{1}{2}\int\limits_{L_j} \frac{(\nu+\lambda)e^{-\lambda h_0}e^{i\lambda L_{lj}}}{(\nu \operatorname{sh}\lambda h_0 - \lambda \operatorname{ch}\lambda h_0)} \times \\
&\times[(\overline{H}_{ll}(-\lambda)\, e^{-\lambda(2h_0-h_j)} - H_{ll}(\lambda)\, e^{\lambda h_j})\, N_{0j}(-\lambda,\mu) + \\
&+ (-H_{ll}(\lambda)\, e^{\lambda(2h_0-h_j)} + \overline{H}_{ll}(-\lambda)\, e^{-\lambda h_j})\, N_{1j}(-\lambda,\mu)]\, d\lambda + \\
&+ \frac{1}{2}\int\limits_{L_j} \frac{(\nu+\lambda)\, e^{-\lambda h_0}e^{-i\lambda L_{lj}}}{(\nu \operatorname{sh}\lambda h_0 - \lambda \operatorname{ch}\lambda h_0)}[(-\overline{H}_{ll}(\lambda)\, e^{\lambda(2h_0-h_j)} + H_{ll}(-\lambda)\, e^{-\lambda h_j})\, N_{0j} \times \\
&\times(\lambda,\mu) + (H_{ll}(-\lambda)\, e^{-\lambda(2h_0-h_j)} - \overline{H}_{ll}(\lambda)\, e^{\lambda h_j})\, N_{1j}(-\lambda,\mu)]\, d\lambda - \\
&- \frac{\nu}{8\pi(\nu h_0 - \operatorname{ch}^2\lambda_0 h_0)}[(\overline{H}_{lj}(-\lambda_0)\, N_{0j}(-\lambda_0,\mu) - H_{lj}(-\lambda_0)\, N_{1j}(-\lambda_0,\mu))\times \\
&\times e^{-\lambda_0(2h_0-h_j)} + (\overline{H}_{lj}(\lambda_0)\, N_{0j}(\lambda_0,\mu) - H_{lj}(\lambda_0)\, N_{1j}(\lambda_0,\mu))\, e^{\lambda_0(2h_0-h_j)} - \\
&- (H_{lj}(\lambda_0)\, N_{0j}(-\lambda_0,\mu) - \overline{H}_{lj}(\lambda_0) N_{1j}(-\lambda_0,\mu))\, e^{\lambda h_j} + (H_{lj}(-\lambda_0) N_{0j}(\lambda_0,\mu) - \\
&- \overline{H}_{lj}(-\lambda_0)\, N_{1j}(\lambda_0,\mu))\, e^{-\lambda h_j} + F_1\,(H_{ll}(\lambda_0))]\Big)\Big\}; \qquad \text{(IV.113)}
\end{aligned}$$

$$\begin{aligned}
F_1(H_{11}(\lambda_0)) &= 2e^{i\lambda_0 L_{21}}\, [\overline{H}_{11}(-\lambda_0)\, (N_{02}(-\lambda_0,\mu)\, e^{-\lambda_0(2h_0-h_2)} + \\
&+ N_{12}(\lambda_0,\mu)\, e^{-\lambda h_2}) - H_{11}(\lambda_0)\, (N_{12}(\lambda_0,\mu)\, e^{\lambda_0(2h_0-h_2)} + N_{02}(-\lambda_0,\mu)\, e^{\lambda h_2}) + \\
&+ 2e^{-i\lambda_0 L_{21}}\, [H_{11}(-\lambda_0)\, (N_{12}(-\lambda_0,\mu)\, e^{-\lambda_0(2h_0-h_2)} + N_{02}(\lambda_0,\mu)\, e^{-\lambda h_2}) + \\
&+ \overline{H}_{11}(\lambda)\, (N_{02}(\lambda_0,\mu)\, e^{\lambda_0(2h_0-h_2)} + N_{12}(-\lambda_0,\mu)\, e^{\lambda h_2})]. \qquad \text{(IV.114)}
\end{aligned}$$

$$F_1[H_{22}(\lambda_0)] = 0.$$

The final results may be obtained by the methods discussed earlier. In particular, functions $H_{ij}(\lambda)$ may, in the first approximation, be calculated by two methods and, depending on the method used, approximations of various degrees are obtained for P_i, Q_i and Γ_i.

Let us now determine circulation Γ_j.

In determining function $H_{ij}(\lambda)$ by using the complex velocity of the hydrofoil in an infinite flow, it turns out from system (IV.109) that [154

$$\Gamma_l = \Gamma_{l\infty}(1 - F_j - \overline{\Gamma}_j F_{j+2}). \tag{IV.115}$$

where F_j determines the effect of the free surface and finiteness of the depth on the characteristics of a hydrofoil.

The formula for calculating F_j was derived earlier (Chapter III). F_{j+2} determines the mutual effect of the hydrofoils in a system:

$$\begin{aligned} F_{j+2} = {} & -\operatorname{Im} 2R_l \left\{ \int_0^\infty J_0(2\lambda R_j) \left\{ \left[e^{-i\lambda L_{lj}} e^{\lambda h_j} (J_0(2\lambda R_l) + iJ_1(2\lambda R_l)) \times \right. \right. \right. \\ & \times e^{-\lambda(2h_0 - h_j)} + \frac{(\nu+\lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} [e^{-i\lambda L_{lj}} (J_0(2\lambda R_l) + iJ_1(2\lambda R_l) \times \\ & \times (e^{-\lambda(2h_0 - h_j)} - e^{-\lambda h_j}) e^{\lambda h_l} - e^{i\lambda L_{lj}} (J_0(2\lambda R_l) - iJ_1(2\lambda R_l) (e^{\lambda(2h_0 - h_j)} - \\ & \left. \left. - e^{\lambda h_j}) e^{-\lambda h_l} \right] \right\} d\lambda + \int_0^{\infty \operatorname{Sign}(h_j - h_l)} J_0(2\lambda R_j) e^{-\lambda(h_l - h_j)} e^{i\lambda L_{lj}} [J_0(2\lambda R_1) - \\ & - iJ_1(2\lambda R_1)] d\lambda + \frac{i\nu\pi}{2} J_0(2\lambda_0 R_j) [e^{-i\lambda_0 L_{lj}} (J_0(2\lambda_0 R_l) + iJ_1(2\lambda_0 R_l)) \times \\ & \times (e^{-\lambda(2h_0 - h_j)} - e^{-\lambda h_j}) e^{\lambda_0 h_l} + e^{i\lambda_0 L_{lj}} (J_0(2\lambda_0 R_l) - iJ_1(2\lambda_0 R_l) \times \\ & \left. \times (e^{\lambda(2h_0 - h_j)} - e^{\lambda h_j}) e^{-\lambda h_l}] \right\}, \end{aligned} \tag{IV.116}$$

$$\overline{\Gamma}_j = \frac{\Gamma_{l\infty}}{\Gamma_{j\infty}}.$$

If functions $H_{ij}(\lambda)$ are determined by the first two terms in the functional equations (IV.112), then the system (IV.109) will become a system of two algebraic equations for Γ_j

$$\Gamma_j(1 + F_j) + \Gamma_l F_{j+2} = \Gamma_{j\infty}, \quad (j = 1, 2), \tag{IV.117}$$

the solution of which is given by formulas (IV.63) and (IV.64).

For a tandem system, function F_{j+2} will be determined by the formula

$$F_{j+2} = -\operatorname{Im} 2R_j \left\{ \int_0^\infty J_0(2\lambda R_j) [e^{-i\lambda L_{lj}} (J_0(2\lambda R_l) + iJ_1(2\lambda R_l) \times \right.$$

$$\times e^{-\lambda(2h_0-h_i-h_j)}]\,d\lambda+\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_i)} J_0(2\lambda R_j)\times$$ [155

$$\times e^{-\lambda(h_j-h_i)}\,e^{-i\lambda L_{ij}}\,[J_0(2\lambda R_i)-iJ_1(2\lambda R_i)]\,d\lambda-$$

$$-\frac{1}{2}\int\limits_{L_i}\frac{(\nu+\lambda)e^{-\lambda(h_0+h_i)}e^{i\lambda L_{ij}}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\cdot(J_0(2\lambda R_i)-iJ_1(2\lambda R_i)\times(e^{\lambda(2h_0-h_j)}e^{-\lambda h_j})\,d\lambda+$$

$$+\frac{1}{2}\int\limits_{L_i}\frac{(\nu+\lambda)\,e^{-\lambda(h_0-h_i)}e^{-i\lambda L_{ij}}}{(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}[J_0(2\lambda R_i)+iJ_1(2\lambda R_i)\,(e^{-\lambda(2h_0-h_j)}-$$

$$\left.-e^{\lambda h_j})]\,d\lambda+i\nu\pi J_0(2\lambda R_j)\,\psi_j(\lambda_0)\right\};\qquad\text{(IV.118)}$$

$\psi_j(\lambda_0)=0$ with j = 1,

$$\psi_j(\lambda_0)=e^{-i\lambda_0 L_{12}}\,(J_0(2\lambda_0R_1)+J_1(2\lambda_0R_1))\,(e^{-\lambda_0(2h_0-h_2)}-e^{-\lambda h_2})\,e^{\lambda_0 h_1}+$$

$$+e^{i\lambda L_{12}}\,(J_0(2\lambda_0R_1)-J_1(2\lambda_0R_1))\,(e^{\lambda_0(2h_0-h_2)}-$$

$$-e^{\lambda h_2})\,e^{-\lambda_0 h_1}.\qquad (j=2)\qquad\text{(IV.119)}$$

As an example, let us examine the motion of a biplane without stagger with $\mathrm{Fr}_b\to\infty$, $\mathrm{Fr}_{\bar h}\to\infty$.

Function F_{j+2} will be determined from the formula

$$F_{j+2}=-2R_j\left\{\int\limits_0^{\infty}J_0(2\lambda R_j)\,J_1(2\lambda R_i)\left[e^{-2\lambda(h_0-h_{cp})}-\right.\right.$$

$$\left.-\frac{e^{-\lambda h_0}}{2\,\mathrm{ch}\,\lambda h_0}\,(e^{-2\lambda(h-h_{cp})}+e^{2\lambda(h_0-h_{cp})}-e^{-\lambda(h_j-h_i)}-e^{\lambda(h_j-h_i)})\right]d\lambda+$$

$$+\int\limits_0^{\infty\,\mathrm{Sign}(h_j-h_i)}J_0(2\lambda R_j)\,J_1(2\lambda R_i)\,e^{-\lambda(h_j-h_i)}\,d\lambda.\qquad\text{(IV.120)}$$

Using identity (III.68a) and the expression for $A_{nm}(x)$ for a biplane with identical foils we obtain

$$F_{j+2}=-2\sum_{k=0}^{\infty}\left\{A_{01}\left[-\bar h_{cp}+\left(k+\frac{1}{2}\right)\frac{t}{2}\right]-A_{01}\left[-\bar h_{cp}+\right.\right.$$

$$\left.+(k+1)\frac{t}{2}\right]-A_{01}\left(\bar h_{cp}+k\frac{t}{2}\right)+A_{01}\left[\bar h_{cp}+\left(k+\frac{1}{2}\right)\frac{t}{2}\right]+$$

$$+A_{01}\left[\frac{\bar a}{2}+\left(k+\frac{1}{2}\right)\frac{t}{2}\right]-A_{01}\left[\frac{\bar a}{2}+(k+1)\frac{t}{2}\right]+A_{01}\left[-\frac{\bar a}{2}+\right.$$ [156

$$+\left(k+\frac{1}{2}\right)\frac{t}{2}\Big]-A_{01}\Big[-\frac{\bar{a}}{2}+\left(k+\frac{1}{2}\right)\frac{t}{2}\Big]\Big\}+2A_{01}(\bar{a}), \qquad (IV.121)$$

where $\bar{a}=|\bar{h}_1-\bar{h}_2|$.

Functions $A_{nm}(x)$ may be determined by the expansion with respect to powers of parameter τ_x (see Ch. II).

Function $F_{j=2}$ we will determine approximately using only two functions $A_{01}(h_{cp})$ and keeping in the expansions only the first term:

$$F_{j+2}=\frac{1}{2}(\tau_3^2-\tau_5^2+\tau_4^2). \qquad (IV.122)$$

Then, formulas for γ_1 and γ_2, which correspond to those given in (IV.90) and (IV.91), will be in the form:

$$\gamma_1=\frac{1+\frac{1}{2}(\tau_2^2-\tau_{02}^2)-\frac{1}{2\bar{\Gamma}}(\tau_3^2-\tau_5^2+\tau_4^2)}{\left[1+\frac{1}{2}(\tau_2^2-\tau_{02}^2)\right]\left[1+\frac{1}{2}(\tau_1^2-\tau_{01}^2)\right]-\frac{1}{4}(\tau_3^2-\tau_5^2+\tau_4^2)}, \qquad (IV.123)$$

$$\gamma_2=\frac{1+\frac{1}{2}(\tau_1^2-\tau_{01}^2)-\frac{\bar{\Gamma}}{2}(\tau_3^2-\tau_5^2+\tau_4^2)}{\left[1+\frac{1}{2}(\tau_2^2-\tau_{02}^2)\right]\left[1+\frac{1}{2}(\tau_1^2-\tau_{01}^2)\right]-\frac{1}{4}(\tau_3^2-\tau_5^2+\tau_4^2)}, \qquad (IV.124)$$

where

$$\tau_{0i}=\sqrt{4(\bar{h}_0-\bar{h}_i)^2+1}-2(\bar{h}_0-\bar{h}_i);$$

$$\tau_5=\sqrt{4(\bar{h}_0-\bar{h}_{cp})^2+1}-2(\bar{h}_0-\bar{h}_{cp}).$$

All other designations are the same.

4.5. Optimum Relationships for Tandem System

Let us examine the simplest problems dealing with optimum relationships in the system consisting of two hydrofoils placed at a considerable distance from each other.

With $L\to\infty$, it follows from formulas (IV.103), (IV.104) and (IV.110) that

$$P_1=P_{1h};$$

$$P_2=\varrho v_0\Gamma_2-\frac{\varrho}{2\pi}\int_0^\infty\Big\{|H_{22}(-\lambda)|^2e^{-2\lambda h_0}+\frac{(\nu+\lambda)e^{-\lambda h_0}}{2(\nu\,\mathrm{sh}\,\lambda h_0-\lambda\,\mathrm{ch}\,\lambda h_0)}\times$$ [157

$$\times \left[|H_{22}(-\lambda)|^2 e^{-2\lambda h_0} - |H_{22}(\lambda)|^2 e^{2\lambda h_0} \right] \Big\} d\lambda + \frac{\nu\varrho \operatorname{Im}[H_{22}(\lambda_0) H_{22}(-\lambda_0)]}{2(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} +$$

$$+ \frac{\nu\varrho}{2(\nu h - \operatorname{ch}^2 \lambda_0 h_0)} \{ [H'_{22}(-\lambda_0) H'_{12}(-\lambda_0) - H''_{22}(-\lambda_0) H'_{12}(-\lambda_0)] \times$$

$$\times e^{-2\lambda_0 h_0} + [H'_{22}(\lambda_0) H'_{12}(\lambda_0) - H''_{22}(\lambda_0) H'_{12}(\lambda_0)] e^{2\lambda_0 h_0} + H'_{22}(-\lambda_0) H''_{12}(\lambda_0) +$$

$$+ H''_{22}(-\lambda_0) H'_{12}(\lambda_0) + H'_{22}(\lambda_0) H''_{12}(-\lambda_0) + H''_{22}(\lambda_0) H'_{12}(-\lambda_0) \}; \quad \text{(IV.125)}$$

$$Q_1 = Q_{1h};$$

$$Q_2 = -\frac{\varrho\nu |\overline{H}_{22}(\lambda_0) e^{\lambda_0 h_0} - H_{22}(-\lambda_0) e^{-\lambda_0 h_0}|^2}{4(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} - \frac{\varrho\nu}{2(\nu h_0 - \operatorname{ch}^2 \lambda_0 h_0)} \times$$

$$\times \{ [H'_{22}(-\lambda_0) H'_{12}(-\lambda_0) + H''_{22}(-\lambda_0) H''_{12}(-\lambda_0)] e^{-2\lambda_0 h_0} +$$

$$+ [H'_{22}(\lambda_0) H'_{12}(\lambda_0) + H''_{22}(\lambda_0) H''_{12}(\lambda_0)] e^{2\lambda_0 h_0} - H'_{22}(-\lambda_0) H'_{12}(\lambda_0) +$$

$$+ H''_{22}(-\lambda_0) H''_{12}(\lambda_0) - H'_{22}(\lambda_0) H'_{12}(\lambda_0) + H''_{22}(\lambda_0) H''_{12}(-\lambda_0) \}; \quad \text{(IV.126)}$$

$$\Gamma_1 = \Gamma_{1h};$$

$$\Gamma_2 = 4\pi R_2 \operatorname{Im} \Big\{ v_0 + \frac{1}{2\pi} \int_0^\infty J_0(2\lambda R_2) \Big[H_{22}(-\lambda) e^{-\lambda(2h_0 - h_2)} +$$

$$+ \frac{(\nu + \lambda) e^{-\lambda h_0}}{2(\nu \operatorname{sh} \lambda h_0 - \lambda \operatorname{ch} \lambda h_0)} (H_{22}(-\lambda) e^{-\lambda(2h_0 - h_2)} - H_{22}(\lambda) e^{\lambda(2h_0 - h_2)} -$$

$$- \overline{H}_{22}(\lambda) e^{\lambda h_2} + \overline{H}_{22}(-\lambda) e^{-\lambda h_2}) \Big] d\lambda + \frac{i\nu J_0(2\lambda_0 R_2)}{4(\nu h_0 - \operatorname{ch}^2 \lambda_0 v_0)} \times$$

$$[H_{22}(-\lambda_0) e^{-\lambda_0(2h_0 - h_2)} + H_{21}(\lambda_0) e^{\lambda_0(2h_0 - h_2)} -$$

$$- \overline{H}_{22}(\lambda_0) e^{\lambda h_2} + \overline{H}_{22}(-\lambda_0) e^{\lambda h_2} + F_2(\lambda_0)] \Big\}. \quad \text{(IV.127)}$$

P_{1h}, Q_{1h} and Γ_{1h} are the forces and circulation for an isolated hydrofoil submerged in a fluid of finite depth. Their values are determined from formulas (III.50), (III.51) and (III.64).

It is clear from formulas (IV.125)-(IV.127) that at large distances between hydrofoils the bow hydrofoil moves in a nonperturbed flow, while the stern hydrofoil moves in steady waves produced by the bow hydrofoil. [158

Let us determine the practical and impractical distances between hydrofoils from the point of view of the wave drag and lift of the system.

The problem of the minimum wave drag of the stern

hydrofoil at a steady circulation along the contour (i.e., constancy of the lift in the linear approximation) is reduced to the condition

$$Q_{2L} = 0 \text{ when } F_2 = \text{const.} \tag{IV.128}$$

This condition leads to a relationship

$$\operatorname{tg} \lambda_0 L = \frac{N}{M}; \tag{IV.129}$$

$$\begin{aligned}
N = {} & \{[H''_{11}(-\lambda_0) H'_{22}(-\lambda_0) - H'_{11}(-\lambda_0) H''_{22}(-\lambda_0)] e^{-2\lambda_0 h_0} + \\
& + [H''_{22}(\lambda_0) H'_{11}(\lambda_0) - H'_{22}(\lambda_0) H''_{11}(\lambda_0)] e^{2\lambda_0 h_0} + \\
& + H'_{22}(-\lambda_0) H''_{11}(\lambda_0) - H''_{22}(-\lambda_0) H'_{11}(\lambda_0) - H'_{22}(\lambda_0) H''_{11}(-\lambda_0) - \\
& - H''_{22}(\lambda_0) H'_{11}(-\lambda_0)\}; \\
M = {} & [H'_{22}(-\lambda_0) H'_{11}(-\lambda_0) + H''_{22}(-\lambda_0) H''_{11}(-\lambda_0)] e^{-2\lambda_0 h_0} + \\
& + [H'_{22}(\lambda_0) H'_{11}(\lambda_0) + H''_{22}(\lambda_0) H''_{11}(\lambda_0)] e^{2\lambda_0 h_0} - \\
& - H'_{22}(-\lambda_0) H'_{11}(\lambda_0) - H''_{22}(-\lambda_0) H''_{11}(\lambda_0) - H'_{22}(\lambda_0) H'_{11}(-\lambda_0) + \\
& + H''_{22}(\lambda_0) H''_{11}(-\lambda_0).
\end{aligned} \tag{IV.130}$$

For a fluid of infinite depth functions N and M are in the form

$$\begin{aligned}
N &= -[H''_{22}(\nu) H'_{11}(\nu) + H'_{22}(\nu) H''_{11}(\nu)], \\
M &= H'_{22}(\nu) H'_{11}(\nu) + H''_{22}(\nu) H''_{11}(\nu),
\end{aligned} \tag{IV.131}$$

and in determining functions $H(\lambda)$ by means of formulas, the relation (IV.129) will be written in the form

$$\operatorname{tg} \nu L = \frac{J_1(2\nu R_2) J_0(2\nu R_1) - J_0(2\nu R_2) J_1(2\nu R_1)}{J_0(2\nu R_1) J_0(2\nu R_2) + J_1(2\nu R_1) J_1(2\nu R_2)}. \tag{IV.132}$$

From the expression (IV.131) it follows that with identical hydrofoils the points of the minimax will be determined by the simple condition

$$\sin \nu L = 0. \tag{IV.133}$$

The minimax points of circulation along the contour [159
of the stern hydrofoil will be determined by the condition

$$\operatorname{Re} F_2(\lambda)_L = 0, \tag{IV.134}$$

which also results in relationship (IV.129), where

$$M = H_{11}'(-\lambda_0)\left(e^{-\lambda_0(2h_0-h_2)} + e^{-\lambda_0 h_2}\right) + H_{11}'(\lambda_0)\left(e^{\lambda_0(2h_0-h_2)} + e^{\lambda_0 h_2}\right); \qquad \text{(IV.135)}$$

$$N = -H_{11}''(-\lambda_0)\left(e^{-\lambda_0(2h_0-h_2)} + e^{-\lambda_0 h_2}\right) - H_{11}''(\lambda_0)\left(e^{\lambda_0(2h_0-h_2)} + e^{\lambda_0 h_2}\right).$$

With $e^{-\lambda_0 h_1} H_{11}'(-\lambda_0) = H_{11}''(\lambda_0) e^{\lambda_0 h_1}$, $H_{11}'(-\lambda_0) e^{-\lambda_0 h_1} = -H_{11}''(\lambda_0) e^{\lambda_0 h}$

$$tg\,\lambda_0 L = \frac{H_{11}''(\lambda_0)}{H_{11}'(\lambda_0)}. \qquad \text{(IV.136)}$$

Hence, in determining $H_{11}(\lambda_0)$ we obtain

$$\operatorname{tg}\lambda_0 L = -\frac{J_1(2\lambda_0 \cdot R_1)}{J_0(2\lambda_0 \cdot R_1)}. \qquad \text{(IV.137)}$$

For a fluid of finite depth formulas (IV.130) will be transformed into formulas (IV.131) in which λ_0 should be used in place of ν. Then, with the proper determination of function $H(\lambda)$ we will have

$$\operatorname{tg}\lambda_0 L = \frac{J_1(2\lambda_0 R_2) J_0(2\lambda_0 R_1) - J_0(2\lambda_0 R_2) J_1(2\lambda_0 R_1)}{J_0(2\lambda_0 R_1) J_0(2\lambda_0 R_2) + J_1(2\lambda_0 R_1) J_1(2\lambda_0 R_2)}. \qquad \text{(IV.138)}$$

From formulas (IV.137) and (IV.138) it follows that the minimax points of the wave drag and circulation do not coincide. For the fluids of both finite and infinite depths the position of these points is determined by the same formulas. Since the points for the wave drag minimum and circulation maximum do not coincide it is of interest to determine the position of points for the maximum-quality hydrofoil.

Let us determine the wave drag and circulation by the formulas

$$\begin{aligned} Q_2 &= \Gamma_2^2 A_1 + \Gamma_2 (A_2 \sin \nu L + A_3 \cos \nu L); \\ \Gamma_2 &= A_4 + A_5 \sin \nu L + A_6 \cos \nu L, \end{aligned} \qquad \text{(IV.139)}$$

The condition of the maximum (minimum) quality will be determined by the requirement

$$F(L)_L = 0, \quad F(L) = \frac{Q}{\Gamma}. \qquad \text{(IV.140)}$$

From (IV.139) and (IV.140) one obtains a formula for the extreme values [160

$$\operatorname{tg}\lambda_0 L = \frac{A_1A_5 + A_2}{A_1A_6 + A_3}. \qquad (\text{IV.141})$$

Functions A_i are determined by the expressions

$$\left.\begin{aligned}
A_1 &= -\frac{\varrho\nu[J_1^2(2\lambda R_2) + J_1^2(2\lambda_0 R_2)](e^{\lambda_0(h_0-h_2)} - e^{-\lambda_0(h_0-h_2)})^2}{4(\nu h_0 - \operatorname{ch}^2\lambda_0 h_0)}\\
A_2 &= \frac{\varrho\nu\Gamma_1(J_1(2\lambda_0R_1)J_0(2\lambda_0R_2) - J_1(2\lambda_0R_2)J_0(2\lambda_0R_1)]}{4(\nu h_0 - \operatorname{ch}^2\lambda_0 h_0)}B\\
A_3 &= -\frac{\varrho\nu\Gamma_1(J_0(2\lambda_0R_1)J_0(2\lambda_0R_2) + J_1(2\lambda_0R_1)J_1(2\lambda_0R_2)]}{4(\nu h_0 - \operatorname{ch}^2\lambda_0 h_0)}B\\
A_5 &= -\frac{\pi R_2\nu J_0(2\lambda_0R_2)J_1(2\lambda_0R_2)\Gamma_1}{(\nu h_0 - \operatorname{ch}^2\lambda_0 h_0)}B\\
A_6 &= \frac{\pi R_2\nu J_0(2\lambda_0R_1)J_1(2\lambda_0R_2)\Gamma_1}{(\nu h_0 - \operatorname{ch}^2\lambda_0 h_0)}\\
B &= e^{2\lambda_0(h-h_{\text{cp}})} + e^{-2\lambda_0(h-h_{\text{cp}})} - e^{-\lambda(h_1-h_2)} - e^{\lambda(h_1-h_2)}
\end{aligned}\right\}. \qquad (\text{IV.142})$$

Expressions (IV.141) and (IV.142) may be simplified:

$$\operatorname{tg}\lambda_0 L = \frac{A_1'A_5' + A_2'}{A_1'A_6' + A_3'}; \qquad (\text{IV.143})$$

$$\left.\begin{aligned}
A_1' &= -[J_0^2(2\lambda_0R_2) + J_1^2(2\lambda_0R_2)]\frac{(e^{\lambda_0(h_0-h_2)} - e^{-\lambda_0(h_0-h_2)})^2}{(\nu h_0 - \operatorname{ch}^2\lambda_0 h_0)}\\
A_2' &= -J_1(2\lambda_0R_2)J_0(2\lambda_0R_1) + J_1(2\lambda_0R_1)J_0(2\lambda_0R_2)\\
A_3' &= -J_0(2\lambda_0R_1)J_0(2\lambda_0R_2) + J_1(2\lambda_0R_1)J_1(2\lambda_0R_2)\\
A_5' &= -\pi R_2\nu J_0(2\lambda_0R_2)J_1(2\lambda_0R_1)\\
A_6' &= \pi R_2\nu J_0(2\lambda_0R_1)J_0(2\lambda_0R_2)
\end{aligned}\right\}. \qquad (\text{IV.144})$$

Formulas (IV.142) and (IV.144) are obtained under the condition that $H(\lambda)$ was determined by using the complex velocity of the hydrofoil moving in an infinite flow. When using a different method of determining $H(\lambda)$ one should utilize functions $A_5'' = \frac{A_5}{1+F_2}$, $A_6'' = \frac{A_6}{1+F_2}$ in place of A_5 and A_6.

The extreme points of the quality are in different positions in respect to the wave profile for fluids of both finite and infinite depth. This is due to the fact that the values of the hydrofoil wave drag are different for fluids of finite and infinite depth. [161

In [114] and [215] problems of the hydrofoil motion

in steady waves are studied. The results obtained in this chapter also include this problem for fluids of finite and infinite depth. Therefore, the discussion will be limited to certain results of this problem examined in [114].

For a hydrofoil moving in steady waves the complex velocity of the flow may be determined in the following way:

$$v(z) = -v_0 + i\nu v_0 a e^{-i\nu z + i\beta} + v_1(z) + v_2(z), \qquad \text{(IV.145)}$$

where $v_1(z)$ - an analytical function in the entire plane of the complex variable outside the C_1 contour, which includes the hydrofoil profile;

$v_2(z)$ - an analytical function inside the C_1 contour;

a - the amplitude of steady waves.

Let us examine the value of circulation Γ_b for a flow about a hydrofoil at a complex velocity

$$v(z) = -v_0 + i\nu v_0 a e^{-i\nu z + i\beta}.$$

The complex velocity of the flow is

$$v_\infty(z) = -v_0 + i\nu v_0 a e^{-i\nu z + i\beta} + v_{\infty 1}(z), \qquad \text{(IV.146)}$$

where $v_{\infty 1}(z)$ is an analytical function in the entire plane of the complex variable outside the C_1 contour.

The boundary conditions on the contour of a cylinder which corresponds to the appropriate profile of the hydrofoil will be written in the form

$$\mathrm{Re}\left[v_{\infty 1}(z)\, u \frac{dz}{du}\right] = \mathrm{Re}\left[v_0 - i\nu v_0 a e^{-i\nu z + i\beta}\right] u \frac{dz}{du}. \qquad \text{(IV.147)}$$

Solving the Dirichlet problem for the outside shape of the cylinder we obtain

$$v_{\infty 1}(z) = \frac{du}{dz}\left\{-v_0 + \frac{1}{2\pi}\nu v_0 a e^{i\beta} + G(\nu, u) + \right.$$
$$\left. + \frac{R^2}{u^2}\bar{v}_0\left[1 + \frac{\nu a e^{-i\beta}}{2\pi}\overline{G\left(\nu\frac{R^2}{\bar{u}}\right)}\right] + \frac{C}{2\pi i u}\right\}. \qquad \text{(IV.148)}$$

For the N. Ye. Zhukovskiy hydrofoil

$$v_{\infty 1}(z) = \frac{du}{dz}\left\{-v_0[1 - i\nu a e^{-i\beta} J_0(2\nu R)e^{-\nu h}] + \right.$$

$$\left. + \frac{\bar{v}_0 R^2}{u^2}[1 + i\nu a e^{i\beta} J_0(2\nu R)e^{-\nu h}] + \frac{\Gamma}{2\pi i u}\right\} +$$

$$+ \frac{\nu v_0 a e^{-\nu h}}{2\pi}\left[B(\nu, u)e^{i\beta} + \overline{B\left(\nu, \frac{R_2}{u}\right)}\frac{du}{d\frac{R^2}{u}}\frac{R^2}{u^2}\right], \qquad \text{(IV.149)}$$ [162

hence, according to condition (II.33), we obtain

$$\Gamma = \Gamma_\infty\left(1 - \frac{\nu a \cos\beta J_0(2\nu R)e^{-\nu h}}{\sin\alpha_i}\right). \qquad \text{(IV.150)}$$

The problem concerning the interaction of thin hydrofoils may be solved by the methods discussed in Ch. II.

Let us assume that a system of n thin hydrofoils moves under the free surface of a liquid.

The complex velocity of the perturbed flow will be determined in the following way:

$$v(z) = \sum_{i=1}^{n}\frac{1}{2\pi i}\int_{C_i}\gamma_i(s)\left[\frac{1}{z - s} + K_i(z, s)\right]ds. \qquad \text{(IV.151)}$$

The boundary condition on the line C_1 is in the form

$$\varphi_y = -v_0 f_l'(x) = F_l(x) \text{ on } C_l. \qquad \text{(IV.152)}$$

By introducing function $\Phi(z)$ expressed in terms of $\Phi(z) = iv(z)$ we arrive at the Dirichlet problem for a plane cut along sections C_i:

$$\mathrm{Re}\,\Phi_+(x) = \mathrm{Re}\,\Phi_-(x) = F_l(x) \text{ on } C_l,$$

where $F_j(x)$ is a function given on C_j, which satisfies Gel'der's condition.

Examining the limit value of the $\Phi(z)$ function when approaching the line of the cut C_j from both above and below we arrive at a system of integral equations

$$\frac{1}{2\pi}\sum_{i=1}^{n} \mathrm{Re}\int_{C_i} \gamma_i(s)\left[\frac{1}{z_j - s} + K_i(z_j, s)\right] ds = F_j(x) \quad \text{on } C_j, \qquad (j = 1, 2 \ldots n) \tag{IV.153}$$

where the nucleus will contain a singular part when i = j, and when i ≠ j the nuclei will be regular.

For a system consisting of two hydrofoils, the system of integral equations may be written in the form

$$\int_{-1}^{+2} \bar{\gamma}_j(s)\left[\frac{1}{\bar{x} - \bar{s}} + K_j(\bar{x} - \bar{s})\right] d\bar{s} + \int_{-1}^{+1} \bar{\gamma}_i(s) K_{ij}(\bar{x} - \bar{s})\, ds =$$

$$= -2\pi \bar{f}_j'(x). \qquad (j = 1, 2;\ i = 1, 2;\ i \neq j) \tag{IV.154}$$

For a tandem system the effect of the stern hydrofoil on the bow hydrofoil may be neglected and the problem of interaction will then be reduced to the solution of two integral equations: [163

$$\int_{-1}^{+1} \bar{\gamma}_1(s)\left[\frac{1}{\bar{x} - \bar{s}} + K_1(\bar{x} - \bar{s})\right] d\bar{s} = -2\pi \bar{f}_1'(x),$$

$$\int_{-1}^{+1} \bar{\gamma}_2(s)\left[\frac{1}{\bar{x} - \bar{s}} + K_2(\bar{x} - \bar{s})\right] d\bar{s} + \int_{-1}^{+1} \bar{\gamma}_1(\bar{s}) K_{12}(\bar{x} - \bar{s})\, d\bar{s} =$$

$$= -2\pi \bar{f}_2'(x).$$

Such a formulation of the problem of the tandem system interaction was used in the study carried out by W. H. Isay [194].

CHAPTER V. THEORY OF THE SUBMERGED HYDROFOIL IN A PLANE-PARALLEL UNSTEADY FLOW [164

5.1. Formulation of the Problem. General Aspects

Numerous practical problems related to the motion of hydrofoil craft are reduced to the study of the unsteady phenomena on the hydrofoil; however, very few final results concerning the unsteady hydrodynamics of a submerged hydrofoil have been obtained so far. Perhaps the most significant results in this field belong to I. T. Yegorov [29, 30].

Yegorov analyzed the problem of the unsteady motion of a thin hydrofoil with the linear boundary conditions on the free surface and $Fr \rightarrow \infty$. By using such a formulation, the problem of the hydrofoil motion is reduced to that dealing with the motion of a biplane in an infinite fluid for which the aerodynamic methods are used exclusively. Later on Yegorov extended this solution to cover the unsteady motion of an array of profiles [32].

The problem of the unsteady motion of a two-dimensional profile, in the absence of a vortical trail, was studied by A. N. Shebalov [170-172]. Some of his results will be given below.

The general problem of the unsteady motion of a hydrofoil under the arbitrary conditions is analyzed in this chapter.

The periodic oscillating motion of a submerged hydrofoil is considered to be the basic unsteady motion under study. This case lends itself to mathematical analysis easier and more fully; it covers directly a series of important practical problems (motion of a hydrofoil in a seaway, rolling and pitching of hydrofoil ships, vibrational loads and flutter of hydrofoils, and the theory of automatic control of hydrofoils). In addition, this case hardly makes this problem less general, since the other cases of
unsteady motion may be studied with the aid of the Fourier [165
integral.

Let us consider a hydrofoil submerged to a depth h under the free surface of a fluid. The hydrofoil moves at a certain constant velocity v_0 with very small changes in this velocity. The profile of the hydrofoil is deformed.

Because of the fact that the problem of hydrofoils

with arbitrary thicknesses and profiles becomes more complex we made the assumptions which are usually used in the linear theory of thin hydrofoils.

The mathematical formulation of this problem was presented by means of formulas (29)-(33) in the preface.

Let us express the velocity potential (x, y, t) in the form

$$\varphi(x, y, t) = \varphi_1(x, y) + \varphi_2(x, y, t), \tag{V.1}$$

where $\varphi_1(x, y)$ - velocity potential corresponding to the steady motion of the hydrofoil at a constant velocity v_0;

$\varphi_2(x, y, t)$ - velocity potential of the unsteady motion of the fluid.

Let us write $\varphi_2(x, y, t)$ in the following form:

$$\varphi_2(x, y, t) = \bar{\varphi}(x, y)e^{i\omega t}. \tag{V.2}$$

Here and in the discussion below only the real part of the expressions, containing the exponentially temporary multiplier, should be used.

The relations (V.1) and (V.2) provide two boundary problems for φ_1 and $\bar{\varphi}$:

$$\begin{gathered} \text{I. } \nabla^2\varphi_1 = 0, \\ \varphi_{1xx} + \nu\varphi_{1y} = 0 \text{ with } y = 0, \\ \varphi_{1y} = -v_0 s_{Ly} \text{ on } s \\ \nabla\varphi_1 \underset{x\to\infty}{=} 0, \end{gathered} \tag{V.3}$$

$$\text{II. } \nabla^2\bar{\varphi} = 0, \tag{V.4}$$

$$\bar{\varphi}_y - 2i\tau'\bar{\varphi}_x - \nu_1\bar{\varphi} + \frac{\tau'^2}{\nu_1}\bar{\varphi}_{xx} = 0 \text{ with } y = 0, \tag{V.5}$$

$$\bar{\varphi}_{y+} = \bar{\varphi}_{y-}; \qquad (ip\bar{\varphi} - \bar{\varphi}_x)_+ = (ip\bar{\varphi} - \bar{\varphi}_x)_- \text{ on } \Sigma \tag{V.6}$$

$$\bar{\varphi}_y = \bar{B} \text{ on } s \tag{V.7}$$

$$\nabla\bar{\varphi} \underset{x\to\infty}{=} 0, \tag{V.8}$$

where $\tau' = \frac{v_0\omega}{g}$; $\nu_1 = \frac{\omega^2}{g}$.

If the boundary condition (36) is considered with the coefficient of scattering, then instead of expressions [166

(V.5) and (V.8), we obtain the following:

$$\bar{\varphi}_y - 2i\tau'[1 - i\beta]\bar{\varphi}_x - \nu_1(1 - 2i\beta)\bar{\varphi} + \frac{\tau'^2}{\nu_1}\varphi_{xx} = 0, \qquad \text{(V.5a)}$$

where $\beta = \frac{\mu}{2\omega}$.

The problem (V.3) and its various generalizations have been investigated thoroughly in Chapters I-IV, while the problem (V.4)-(V.8) will be studied in this chapter and partially in Chapter XI.

As seen from conditions (V.4)-(V.8), the use of the potential $\varphi_2(x, y, t)$ in the form (V.2) has excluded from the problem the time (t) and has reduced the problem to a stationary problem for the Laplacian equation with conditions (V.5)-(V.8).

The first condition (V.6) is a natural condition which does not complicate the problem. However, the second condition (V.6) is connected with the special structure of the line, which is present only in the unsteady motion of the profile and the finite circulation along the contour. This condition complicates the problem considerably. In order to solve it, it is necessary to use the physical picture to consider the nature of motion and to depart from purely mathematical forms, or to use more effective methods as compared with those used in Chapters II and III.

In this chapter the investigation is based on certain physical presentations developed in the field of unsteady aerodynamics.

Let us consider the motion of a vortical layer with a length of 2a, which simulates the motion of a submerged hydrofoil. During the unsteady motion the intensity of this layer and its general circulation along the contour depends on time. According to the Helmholtz's theorem there are vortices moving in the opposite direction somewhere in the flow so that the total circulation of the entire vortical system, including the hydrofoil and the initial vortex (located at infinity behind the hydrofoil) must be equal to zero.

The vortices in the flow form a vortical trail which is moving together with the fluid during the forward motion of the profile with the infinitely small unsteady additions; it can be assumed that the vortical trail will also be rectilinear.

The assumption about the rectilinearity of the vortical trail, the Helmholtz theorem and the additional hypothesis concerning the nature of the vortical layer formation completely define the time dependent structure of the vortical trail which depends on the structure of vortical layer simulating the unsteady motion of the hydrofoil. Thus, such an approach appears to bypass the second boundary condition (V.6) and does not require satisfaction of this condition. Naturally, the success of the solution will depend on how close to reality will be this or that hypothesis concerning the formation of the vortex layer.

There are two aerodynamic phenomena which accompany the unsteady motion of a hydrofoil. They were developed by Carman and Sears, Birnbaum and Kussner. These hypotheses are described in detail in the monograph by A. I. Nekrasov [92]. [167

In further discussions we will follow the physical presentation developed by Birnbaum and Kussner. They assume that with a change in the vortical layer intensity on the hydrofoil by $\Delta\gamma$, the elementary free vortices with an intensity of $\Delta\gamma$ will appear at a point on the profile. All the free elementary vortices in the relative flow will be driven away from the various areas of the profile. When approaching a certain point on the profile at a certain instant of time they will form a single free vortex which, after leaving the profile, will supplement the vortical trail behind the hydrofoil.

Thus, we will assume that, during the unsteady motion of a submerged hydrofoil, there exist a steady vortical layer γ_0 on the hydrofoil with a potential $\varphi_1(x, y)$ which satisfies the boundary problem (V.3) which deals with the quasi-steady vortical layer, a combination of free vortices on the profile, and a vortical trail behind the hydrofoil with the potentials satisfying conditions (V.5) and (V.8).

A general and very effective method of solving hydrodynamic problems, which is based on the Prandtl conception of acceleration potential and the equations of hydrodynamics and which does not require any specific physical knowledge of the nature of the flow and structure of the surface Σ, will be discussed in Chapter VIII. This method will be used in Chapter XI in considering the three-dimensional and, partly, two-dimensional problems of the unsteady motion of the hydrofoil. It was shown by Nekrasov that when considering motion of a hydrofoil of infinite dimensions in an infinite fluid, the method of the

acceleration potential and the method based on the physical pictured offered by Birnbaum and Kussner lead to the same integral equation. For a submerged hydrofoil both of these methods give the same results.

Let us derive the basic integral equation of the problem. Let us consider that the duration of the unsteady flow is so long that the vortical trail extends from the trailing edge of the hydrofoil to infinity.

The potential of velocities may be expressed as follows:

$$\varphi_2(x, y, t) = \frac{\mathrm{Re}}{2\pi i}\left\{\int_{-a}^{+a} \gamma(s, t)\,[\ln(z-s) + G_1(z, s)]\,ds + \right.$$
$$+\int_{-a}^{+a} \varepsilon(s, t)\,[\ln(z-s) + G_1(z, s)]\,ds +$$
$$\left. + \int_{-\infty}^{-a} \varepsilon(s, t)\,[\ln(z-s) + G_1(z, s)]\,ds\right\}, \qquad (IV.9)$$

where $G_1(z, s)$ is a function which is analytical in the lower half-plane and satisfies conditions (V.5) and (V.8). [168

The motion is considered in the xOy coordinate system in which the origin of coordinates O coincides with the middle of the profile (-a, +a) and axis Ox is directed along the vector of the forward velocity v_0.

According to condition (V.6) we obtain

$$2\pi v_y(x, t) = \int_{-a}^{+a} \gamma(s, t)\left[\frac{1}{x-s} + G_1'(x, s)\right]ds + \int_{-a}^{+a} \varepsilon(s, t)\left[\frac{1}{x-s} + \right.$$
$$\left. + G_1'(x, s)\right]ds + \int_{+a}^{+\infty} \varepsilon(\xi, t)\left[\frac{1}{x-\xi} + G_1'(x, s)\right]d\xi. \qquad (IV.10)$$

To obtain the unknown integral equation of the problem it is necessary to establish the relation between functions $\gamma(s, t)$ and $\varepsilon(s, t)$ and determine the $G_1'(x, t)$ functions. The relation between the $\varphi(s, t)$ and $\varepsilon(s, t)$ functions is determined from the physical picture of the flow [92].

Let us consider a stationary point A in space. The

intensity of the vortical layer located near the stationary point at the time $t' + \Delta t'$ will be

$$\gamma(x', t' + \Delta t') = \gamma(x', t') + \gamma_{t'}\Delta t' + \frac{1}{2}\gamma_{t't'}\Delta t'^2.$$

Formation of the free vortices will correspond to the increment of the intensity within the same length of time. The intensity of these vortices will be

$$-\gamma_{t'}\Delta t' - \frac{1}{2}\gamma_{t't'}\Delta t'^2 + \ldots,$$

$$\Delta t' = \frac{x - x'}{v_0},$$

$$t' = t - \frac{x - x'}{v_0},$$

where x is the abscissa of the point A in the mobile coordinate system at time t.

Then, the intensity of the free vortex may be written in the form

$$\gamma_t\left(x', t - \frac{x - x'}{v_0}\right)\left(\frac{\Delta x'}{v_0}\right) - \frac{1}{2}\gamma_{tt}\left(x', t - \frac{x - x'}{v_0}\right)\left(\frac{\Delta x'}{v_0}\right)^2.$$

Consequently, from the moment the point A passes from the leading to the trailing edge of the hydrofoil the free vortices will enter the infinitely small region near the point A; when the profile will reach point A with its [169
trailing edge, these vortices will supplement the vortical trail.

All free vortices, formed at point A before time t, will by this time acquire an intensity $\xi(x, t)$ which is equal to the integral

$$\varepsilon(x_1 t) = -\frac{1}{v_0}\int_{-c}^{x}\gamma_t\left(x', t - \frac{x - x'}{W}\right)dx', \qquad (V.11)$$

and the intensity of vortices $\varepsilon(\xi, t)$ in the vortical trail will be

$$\varepsilon(\xi, t) = -\frac{1}{v_0}\int_{-a}^{a}\gamma_t\left(x', t - \frac{\xi - x}{v_0}\right)dx. \qquad (V.12)$$

For the periodic motion

$$\left.\begin{aligned}&\gamma_{st}=\gamma(s)e^{i\omega},\ \xi(s,t)=\varepsilon(s)e^{i\omega t},\quad \varepsilon(\xi,t)=\varepsilon(\xi)e^{i\omega t}\\&\varepsilon(x)=-ipe^{-ipx}\int_{-a}^{x}\gamma(s')e^{ips'}ds'\\&\varepsilon(\xi)=-ipe^{-ip\xi}\int_{-a}^{+a}\gamma(s')e^{ips'}ds'\qquad p=\frac{\omega}{v_0}\end{aligned}\right\}\tag{V.13}$$

and equation (V.10) will be in the form

$$\begin{aligned}2\pi v_y(x)=&\int_{-a}^{+a}\gamma(s)\left[\frac{1}{x-s}+G_1'(x',s)\right]ds-\\&-ip\left\{\int_{-a}^{+a}\left[e^{-ips}\int_{-a}^{s}\gamma(s')e^{ips'}ds'\left[\frac{1}{x-s}+G_1'(x,s)\right]\right\}ds+\right.\\&+\int_{+a}^{\infty}e^{-ip\xi}\int_{-a}^{+a}\gamma(s')e^{ips'}ds'\left[\frac{1}{x-\xi}-G_1'(x,\xi)\right]d\xi.\end{aligned}\tag{V.14}$$

With $h\to\infty$, equation (V.14) will transform into the known Birnbaum equation.

5.2. Motion of the Fluctuating Strength Vortex Under the Free Surface of a Fluid [170

To determine the function $G_1'(x, s)$ in the integral equation it is necessary to consider a problem of a fluctuating strength vortex moving under the free surface of a fluid, with the boundary conditions given in (V.5a).

To solve this problem let us use the M. D. Khaskind method [160]. We are looking for the potential φ in the form

$$\varphi=\mathrm{Re}\left[i\ln\frac{1}{z-\xi}+i\ln\frac{1}{z-\bar{\xi}}\right]+F,\tag{V.15}$$

where F is a function which is harmonic in the entire half-plane; when $y < 0$ it will satisfy the condition

$$\begin{aligned}&\frac{\tau'^2}{\nu_1}F_{xx}-2i\tau'(1-i\beta)F_x+F_y-\nu_1(1-2i\beta)F=\\&=-\frac{1}{z-\xi}-\frac{1}{\bar{z}-\bar{\xi}}.\end{aligned}\tag{V.16}$$

F can be written in the form of a sum of two functions:

$$F(z, \bar{z}) = F_1(z) + F_2(\bar{z});$$

these functions are determined from the following differential equations

$$\frac{\tau^2}{\nu_1} F_{1zz} - i[2\tau'(1 - i\beta) - 1] F_{1z} - \nu_1(1 - 2i\beta) F_1 = -\frac{1}{z - \xi}; \quad \text{(V.17)}$$

$$\frac{\tau'^2}{\nu_1} F_{2zz} - i\nu[2\tau(1 - i\beta) + 1] F_{2z} - \nu_1(1 - 2i\beta) F_2 = -\frac{1}{\bar{z} - \xi}. \quad \text{(V.18)}$$

Let us analyze the roots of the corresponding characteristic equations.

For $\beta = 0$, the roots of the characteristic equation (V.17) are $i\lambda_1$ and $-i\lambda_2$ and those of equation (V.18) are $i\lambda_3$ and $i\lambda_4$:

$$\lambda_{1,2} = \nu \frac{1 - 2\tau' \pm \sqrt{1 - 4\tau'}}{2\tau'^2}; \qquad \lambda_{3,4} = \nu \frac{1 + 2\tau' \pm \sqrt{1 + 4\tau'}}{2\tau'^2}.$$

For $\beta \neq 0$ the roots of the characteristic equations are: $-i\lambda_1'$, $-i\lambda_2'$, $i\lambda_3'$ and $i\lambda_4^1$, where λ_5^1 is determined (to an accuracy involving terms containing β^2) with the aid of the following formulas:

$$\lambda_{1,2}' = \lambda_{12} + i\beta \frac{\nu}{\tau'} \left(1 \pm \frac{1}{\sqrt{1 - 4\tau'}}\right); \quad \text{(V.19)}$$

$$\lambda_{3,4}' = \lambda_{3,4} - i\beta \frac{\nu}{\tau'} \left(1 \pm \frac{1}{\sqrt{1 + 4\tau'}}\right).$$

With $\tau' < \frac{1}{4}$, $\mathrm{Im}\,\lambda_1' > 0$, and $\mathrm{Im}\,\lambda_2' < 0$. It then follows that [171

with $z \leqslant 0$ the lower limits of the integration should be assumed to be $+\infty$ and $-\infty$. With $\tau' < \frac{1}{4}$, λ_1 and λ_2 will be complex; it can be immediately assumed that $\beta = 0$ and that the limited solution will be with the same lower limits $\mathrm{Im}\lambda_{3,4}' < 0$ and for any values of τ'. Therefore, to obtain a limited solution of the equation (V.18) for $z \leqslant 0$, the lower limits of integration should be assumed to be $\pm\infty$.

Taking this into consideration the solutions of equations (V.17) and (V.18) for β = 0 may be written as follows:

$$F_1(z) = -\frac{i}{\sqrt{1-4\tau}}\left(e^{-i\lambda_1 z}\int\limits_{-\infty}^{z}\frac{e^{i\lambda_1 t}}{t-\bar{\xi}}\,dt - e^{-i\lambda_2 z}\int\limits_{+\infty}^{z}\frac{e^{i\lambda_2 t}}{t-\bar{\xi}}\,dt\right); \qquad (V.20)$$

$$F_2(\bar{z}) = \frac{i}{\sqrt{1+4\tau'}}\left(e^{i\lambda_3\bar{z}}\int\limits_{-\infty}^{\bar{z}}\frac{e^{-i\lambda_3\bar{t}}}{\bar{t}-\xi}\,d\bar{t} - e^{i\lambda_4\bar{z}}\int\limits_{-\infty}^{\bar{z}}\frac{e^{-i\lambda_4\bar{t}}}{\bar{t}-\xi}\,d\bar{t}\right). \qquad (V.21)$$

From the formulas (V.20) and (V.21) it is easy to obtain the asymptotic expressions for the potential when $x \to \pm\infty$:

$$\varphi = -\frac{2\pi}{\sqrt{1-4\tau'}}e^{-i\lambda_2(z-\bar{\xi})}; \qquad (x\to-\infty). \qquad (V.22)$$

$$\varphi = -\frac{2\pi}{\sqrt{1-4\tau'}}e^{-\lambda_1(z-\bar{\xi})} - \frac{2\pi}{\sqrt{1+4\tau'}}[e^{i\lambda_3(z-\bar{\xi})} -$$
$$-e^{i\lambda_4(\bar{z}-\xi)}]. \qquad (x\to+\infty). \qquad (V.23)$$

The expressions (V.22) indicate that a moving fluctuating vortex creates four systems of waves with the wave number λ_5.

With $\tau' > \frac{1}{4}$, the numbers λ_1 and λ_2 will be complex and the corresponding waves will attenuate while the asymptotic form of the function φ will be as follows:

$$\varphi = 0; \qquad (x\to-\infty)$$
$$\varphi = -\frac{2\pi}{\sqrt{1+4\tau'}}(e^{-i\lambda_3(\bar{z}-\xi)} - e^{i\lambda_4(\bar{z}-\xi)}). \qquad (x\to+\infty).$$

After differentiating the expression (V.15) we obtain [172

$$\varphi_y = \mathrm{Re}\left(\frac{1}{z-\zeta}+\frac{1}{z-\bar{\zeta}}\right) + i\left[\frac{1}{\sqrt{1-4\tau'}}\left(-\lambda_1 e^{-i\lambda_1 z}\int\limits_{-\infty}^{z}\frac{e^{i\lambda_1 t}}{t-\bar{\zeta}}\,dt + \right.\right.$$
$$\left. + \lambda_2 e^{-i\lambda_2 z}\int\limits_{\infty}^{z}\frac{e^{i\lambda_2 t}}{t-\bar{\zeta}}\,dt\right) + \frac{1}{\sqrt{1+4\tau'}}\left(\lambda_3 e^{-i\lambda_3 z}\int\limits_{-\infty}^{z}\frac{e^{i\lambda_3 t}}{t-\bar{\zeta}}\,dt - \right.$$

$$\overline{-\lambda_4 e^{-i\lambda_4} \int\limits_{-\infty}^{z} \frac{e^{i\lambda_4 t}}{t-\zeta}\, dt\Bigg)\Bigg]}. \qquad (V.23)$$

The appropriate formulas for the moving and the fluctuating source were obtained by M. D. Khaskind [160].

Following his method we obtain

$$\varphi = \ln\frac{1}{r} - \ln\frac{1}{r'} + F, \qquad (V.24)$$

where $r = \sqrt{(x-\zeta)^2 + (y-\eta)^2}$; $r' = \sqrt{(x-\xi)^2 + (y+\eta)^2}$;

$F = F_1 + F_2$.

$$F_1 = \frac{1}{\sqrt{1-4\tau'}}\left(e^{-i\lambda_1 z} \int\limits_{-\infty}^{z} \frac{e^{i\lambda t}}{t-\bar{\zeta}}\, dt - e^{-i\lambda_2 t} \int\limits_{+\infty}^{z} \frac{e^{i\lambda_2 t}}{t-\xi}\, dt\right),$$

$$F_2 = \frac{1}{\sqrt{1+4\tau'}}\left(e^{i\lambda_3 \bar{z}} \int\limits_{-\infty}^{\bar{z}} \frac{e^{-i\lambda_3 t'}}{t-\xi}\, d\bar{t} - e^{i\lambda_4 \bar{z}} \int\limits_{-\infty}^{\bar{z}} \frac{e^{-i\lambda_4 \bar{t}}}{\bar{t}-\zeta}\, dt\right).$$

$$\varphi_y = \mathrm{Re}\left(\frac{i}{z-\zeta} - \frac{i}{z-\bar{\zeta}}\right) + i\left[\frac{1}{\sqrt{1-4\tau'}}\left(-i\lambda_1 e^{-i\lambda_1 z} \int\limits_{-\infty}^{z} \frac{e^{i\lambda_1 t}}{t-\bar{\zeta}}\, dt + \right.\right.$$

$$\left. + i\lambda_2 e^{-i\lambda_2 z} \int\limits_{+\infty}^{z} \frac{e^{i\lambda_2 t}}{t-\bar{\zeta}}\, dt\right) + \frac{1}{\sqrt{1+1\tau'}}\overline{\left(i\lambda_3 e^{-i\lambda_3 z} \int\limits_{-\infty}^{z} \frac{e^{i\lambda_3 t}}{t-\zeta}\, dt - \right.}$$

$$\left.\left.\overline{- i\lambda_4 e^{-i\lambda_4 z} \int\limits_{-\infty}^{z} \frac{e^{i\lambda_4 t}}{z-\zeta}\, d\zeta}\right)\right]. \qquad (V.25)$$

From the expressions (V.23) and (V.25), ony may obtain the values of φ_y for the vortex source with an intensity of $B = \Gamma + iQ$: [173

$$\varphi_y = \mathrm{Re}\left(\frac{B}{z-\zeta} + \frac{B}{z-\zeta}\right) + i\left[\frac{\bar{B}}{\sqrt{1-4\tau_0'}}\left(-\lambda_1 e^{-i\lambda_1 z} \int\limits_{-\infty}^{z} \frac{e^{i\lambda_1 t}}{t-\zeta}\, dt + \right.\right.$$

$$\left. + \lambda_2 e^{-i\lambda_2 z} \int\limits_{+\infty}^{z} \frac{e^{i\lambda_2 t}}{t-\bar{\zeta}}\, dt\right) + \frac{B}{\sqrt{1+4\tau'}}\overline{\left(\lambda_3 e^{-i\lambda_3 z} \int\limits_{-\infty}^{z} \frac{e^{i\lambda_3 t}}{t-\zeta}\, dt - \right.}$$

$$\left.\left.\overline{- \lambda_4 e^{-i\lambda_1 z} \int\limits_{-\infty}^{z} \frac{e^{i\lambda_4 t}}{t-\bar{\zeta}}\, dt}\right)\right]. \qquad (V.26)$$

From this formula one may easily obtain the known results: for example, with $\lambda = \lambda_3 = \nu$, $\lambda_2 = \lambda_4 = 0$ and $\tau' = 0$, (V.26) transforms into the following expression:

$$\varphi_\nu = \mathrm{Re}\left(\frac{B}{z-\zeta} + \frac{\bar{B}}{z-\bar{\zeta}} - i\nu\bar{B}e^{-i\nu z}\int\limits_{-\infty}^{z}\frac{e^{i\nu t}}{t-\bar{\zeta}}\,dt\right).$$

For $v_0 = 0$, $\lambda_1 = \lambda_2 = \infty$, $\lambda_2 = \lambda_4 = \nu'$, and $\tau' = 0$, we obtain the formula for the fluctuating strength vortex source:

$$\varphi_\nu = \mathrm{Re}\left(\frac{B}{z-\zeta} - \frac{\bar{B}}{z-\bar{\zeta}}\right) + i\left(\bar{B}\nu' e^{-i\nu' z}\int\limits_{+\infty}^{z}\frac{e^{i\nu' t}}{t-\zeta}\,dt - \right.$$

$$\left. - B\nu' e^{-i\nu' z}\int\limits_{-\infty}^{z}\frac{e^{i\nu' t}}{t-\bar{\zeta}}\,dt\right). \qquad \text{(V.27)}$$

This formula can be easily transformed into the formula given by N. Ye. Kochin [59].

5.3. Singular and Regular Equations of Periodic Motion of the Hydrofoil

Formula (V.23) solves the problem of determining the function $G_1(x, s)$. Directly from this formula we obtain

$$G_1'(x) = \mathrm{Re}\frac{1}{x-2ih} + i\left[\frac{1}{\sqrt{1-4\tau'}}\left(-\lambda_1 e^{-i\lambda_1 x}\int\limits_{-\infty}^{x}\frac{e^{i\lambda_1 t}}{t-2ih}\,dt + \right.\right.$$ [174

$$\left. + \lambda_2 e^{-i\lambda_2 x}\int\limits_{+\infty}^{x}\frac{e^{i\lambda_2 t}}{t-2ih}\,dt\right] + \frac{1}{\sqrt{1+4\tau'}}\left[\lambda_3 e^{-i\lambda_3 x}\int\limits_{-\infty}^{x}\frac{e^{i\lambda_3 t}}{t-2ih}\,dt - \right.$$

$$\left.\left. - \lambda_4 e^{-i\lambda_4 x}\int\limits_{-\infty}^{x}\frac{e^{i\lambda_4 t}}{t-2ih}\,dt\right)\right]. \qquad \text{(V.28)}$$

Formulas (V.14) and (V.28) result in the basic singular integral equation for the periodic motion of the hydrofoil.

In the future it will be more convenient to use these equations in dimensionless form:

$$2\pi v_y(\bar{x}) = \int_{-1}^{+1} \gamma(\bar{s})\left[\frac{1}{\bar{x}-\bar{s}} + G_1'(\bar{x}-\bar{s})\right] d\bar{s} -$$

$$-ik\left\{\int_{-1}^{+1} e^{-ik\bar{s}}\left[\frac{1}{\bar{x}-\bar{s}} + G_1'(\bar{x}-\bar{s})\right]\int_{-1}^{\bar{s}} \gamma(\bar{s}')\, e^{ik\bar{s}'}\, ds'ds +\right.$$

$$\left.+\int_{+1}^{+\infty} e^{-ik\xi}\left[\frac{1}{\bar{x}-\bar{\xi}} + G_1'(\bar{x}-\bar{\xi})\right]\int_{-1}^{+1} \gamma(\bar{s}')\, e^{ik\bar{s}'} d\bar{s}'\, d\xi\right\}, \qquad \text{(V.29)}$$

$$G_1'(\bar{x}) = \operatorname{Re}\frac{1}{\bar{x}-4ih} + i\left\{\frac{1}{\sqrt{1-4\tau'}}\left[-\lambda_1\int_{+\infty}^{0}\frac{e^{-i\lambda_1 u}}{\bar{x}-4\bar{h}(u+i)}du +\right.\right.$$

$$\left.+\bar{\lambda}_2\int_{-\infty}^{0}\frac{e^{-i\lambda_2 u}}{x-4\bar{h}(u+i)}du\right] + \overline{\frac{1}{\sqrt{1+4\tau'}}\left[\bar{\lambda}_3\int_{+\infty}^{0}\frac{e^{-i\bar{\lambda}_3 u}}{\bar{x}-4\bar{h}(u+i)}du -\right.}$$

$$\left.\left.\overline{-\bar{\lambda}_4\int_{+\infty}^{0}\frac{e^{-i\lambda_4 u}}{x-4\bar{h}(u+i)}du}\right]\right\}, \qquad \text{(V.30)}$$

where $v_y(\bar{x}) = \frac{v_y(x)}{v_0}$, $\bar{x} = \frac{x}{a}$, $h = \frac{h}{2a}$, $\lambda_i = \lambda_i 2h$,

$k = \frac{wa}{v_0}$ — the Strouhal number.

There is no general method of solving the singular equation obtained for the simple case of steady motion and, therefore, it is important to regularize this equation, since, after reducing it to the Fredholm or quasi-Fredholm type, one may analyze it and solve it by using the methods employed in the ordinary integral equations theory. [175

The Birnbaum integral equation that describes the unsteady motion of a thin hydrofoil in an infinite flow, has a closed solution and, consequently, we can use the Vekua-Karleman method for regularizing our equation. By carrying over terms containing the regular function $G_1(x)$ to the other side of the equation, the equation may be written in the form of Birnbaum's equation

$$2\pi f(\bar{x}) = \int_{-1}^{+1}\frac{\gamma(\bar{s})}{\bar{x}-\bar{s}}d\bar{s} - ik\left[\int_{-1}^{+1}\frac{e^{-ik\bar{s}}}{\bar{x}-\bar{s}}d\bar{s}\int_{-1}^{\bar{s}}\gamma(\bar{s}')\, e^{ik\bar{s}'}d\bar{s}' +\right.$$

$$+\int_{+1}^{\infty}\frac{e^{ik\bar{\zeta}}}{\bar{x}-\bar{\zeta}}\,d\bar{\zeta}\int_{-1}^{+1}\gamma(\bar{s}')\,e^{ik\bar{s}'}\,d\bar{s}'\Bigg],$$

the solution of which, obtained by Kussner and Schwarz [92], will have the form

$$\gamma(x)=\frac{2}{\pi}\sqrt{\frac{1-\bar{x}}{1+\bar{x}}}\int_{-1}^{+1}\left\{\sqrt{\frac{1+\bar{s}}{1-\bar{s}}}\left[\frac{1}{\bar{s}-\bar{x}}+(C(k)-1)\right]+\right.$$

$$\left.+\frac{ik}{2}\sqrt{\frac{1+x}{1-x}}\lg\left(\frac{1-\bar{s}\,\bar{x}+\sqrt{1-\bar{s}^2}\sqrt{1-\bar{x}^2}}{1-\bar{s}\,\bar{x}-\sqrt{1-\bar{s}^2}\sqrt{1-\bar{x}^2}}\right)\right\}f(\bar{s})\,d\bar{s}, \qquad (V.31)$$

where C(k) is the Theodorsen function [92].

After transformations, from the expression (V.31) we obtain the following regular integral equation:

$$\gamma(\bar{x})=\gamma_0(\bar{x})+\frac{1}{\pi^2}\sqrt{\frac{1-\bar{x}}{1+x}}\left\{\int_{-1}^{+1}\gamma(\bar{\sigma})\int_{-1}^{+1}\sqrt{\frac{1+s}{1-\bar{s}}}\left[\frac{1}{\bar{s}-x}+\right.\right.$$

$$\left.+(C(k)-1)+\frac{ik}{2}\sqrt{\frac{1+\bar{x}}{1-\bar{x}}}\lg\left(\frac{1-\bar{s}\bar{x}+\sqrt{1-\bar{s}^2}\sqrt{1-\bar{x}^2}}{1-\bar{s}\bar{x}-\sqrt{1-\bar{s}^2}\sqrt{1-\bar{x}^2}}\right)\right]\times$$

$$\left.\times\left[G_1^{\bar{p}}(s-\bar{\sigma})-ike^{ik\bar{\sigma}}\int_{\bar{\sigma}}^{\infty}e^{-ik\bar{p}}G_1^{\bar{p}}(s-p)\,d\bar{p}\right]ds\,d\bar{\sigma}\right\}, \qquad (V.32)$$

where [176

$$\gamma_0(\bar{x})=\frac{2}{\pi}\sqrt{\frac{1-\bar{x}}{1-\bar{x}}}\int_{-1}^{+1}\left\{\sqrt{\frac{1+\bar{s}}{1-\bar{s}}}\left[\frac{1}{\bar{s}-\bar{x}}+(C(k)-1)\right]+\right.$$

$$\left.+\frac{ik}{2}\sqrt{\frac{1+\bar{x}}{1-x}}\lg\left(\frac{1-\bar{s}\bar{x}+\sqrt{1-\bar{s}^2}\sqrt{1-\bar{x}^2}}{1-\bar{s}\bar{x}-\sqrt{1-s^2}\sqrt{1-\bar{x}^2}}\right)\right\}v_y(\bar{s})\,d\bar{s}.$$

This equation can be written in a more convenient form by means of substitution

$$\varphi(x)=\gamma(x)\sqrt{\frac{1+x}{1-x}}, \qquad (V.33)$$

$$\varphi(x)=\varphi_0(x)-\frac{1}{\pi}\int_{-1}^{+1}\varphi(\bar{\sigma})\,F_2(\bar{x},\bar{\sigma})\,d\bar{\sigma},$$

where

$$F_2(\bar{x}, \bar{\sigma}) = \frac{1}{\pi}\sqrt{\frac{1-\bar{\sigma}}{1+\bar{\sigma}}}\int_{-1}^{+1}\sqrt{\frac{1+\bar{s}}{1-\bar{s}}}\left\{\left[\frac{1}{\bar{s}-\bar{x}}+(C(k)-1)\right]+\right.$$

$$\left.+\frac{ik}{2}\sqrt{\frac{1+\bar{\sigma}}{1-\bar{\sigma}}}\lg\left(\frac{1-\bar{s}\bar{x}+\sqrt{1-\bar{s}^2}\sqrt{1-\bar{x}^2}}{1-\bar{s}\bar{x}-\sqrt{1-\bar{s}^2}\sqrt{1-\bar{x}^2}}\right)\right\}K(\bar{s}-\bar{\sigma})\,d\bar{s} \tag{V.34}$$

$$K(\bar{s}-\bar{\sigma}) = G_1(\bar{s}-\sigma) - ike^{ik\bar{\sigma}}\int_{\bar{\sigma}}^{\infty} e^{-ik\bar{p}}G_1(\bar{s}-\bar{p})\,dp.$$

The lifting force and the moment of the hydrofoil will also be determined by formulas (I.29) and (I.30):

$$P = \varrho a v_0^2 \int_{-1}^{+1} \gamma(\bar{s}, t)\,d\bar{s},$$

$$M = -\varrho a^2 v_0^2 \int_{-1}^{+1} s\gamma(s, t)\,ds.$$

Let us assume that

$$P = \bar{P}e^{i\omega t}, \qquad M = \bar{M}e^{i\omega t},$$

$$\bar{P} = \varrho a v_0^2 \int_{-1}^{+1} \bar{\gamma}(\bar{s})\,d\bar{s}, \quad \bar{M} = -\varrho a^2 v_0^2 \int_{-1}^{+1} \bar{s}\,\bar{\gamma}(\bar{s})\,d\bar{s}. \tag{V.35}$$

Let us determine $\gamma(\bar{s})$ from equation (V.32). Then formulas for P and M will acquire the form: [177

$$\bar{P} = \bar{P}_0 - \frac{\varrho v_0^2 a}{\pi}\int_{-1}^{+1}\gamma(\bar{\sigma})\sqrt{\frac{1+\sigma}{1-\sigma}}\,d\sigma\int_{-1}^{+1}\sqrt{\frac{1-x}{1+x}}F_1(\bar{x}, \bar{\sigma})\,d\bar{x}, \tag{V.36}$$

$$M = M_0 + \frac{\varrho v_0^2 a^2}{\pi}\int_{-1}^{+1}\bar{\gamma}(\bar{\sigma})\sqrt{\frac{1+\bar{\sigma}}{1-\bar{\sigma}}}\,d\bar{\sigma}\int_{-1}^{+1}\sqrt{\frac{1-\bar{x}}{1+\bar{x}}}F_1(\bar{x}, \bar{\sigma})\,\bar{x}\,d\bar{x}, \tag{V.37}$$

where P_0 and $\bar{M}_0$ are the lifting force and the moment acting on the hydrofoil in an infinite flow.

Let us express P_0 and $\bar{M}_0$ through the normal velocity:

$$\bar{P}_0 = -2\varrho v_0^2 a\int_{-1}^{+1}\left[C(\bar{x})\sqrt{\frac{1+\bar{x}}{1-\bar{x}}} + ik\sqrt{1-\bar{x}^2}\right]v_y(\bar{x})\,d\bar{x},$$

$$M_0 = 2\varrho v_0^2 a^2 \int_{-1}^{+1} \left\{ \sqrt{1-\bar{x}^2} - \frac{ik}{2}\bar{x}\sqrt{1-\bar{x}^2} + \frac{1}{2}[C(k)-1] \right\} v_y(\bar{x})\, d\bar{x}. \quad \text{(V.38)}$$

Now, let us transform formulas (V.36) and (V.37) to another new form. Integrating, we obtain the relations

$$\sqrt{\frac{1+\bar{\sigma}}{1-\bar{\sigma}}} \int_{-1}^{+1} \sqrt{\frac{1-\bar{x}}{1+\bar{x}}} F(\bar{x}\bar{\sigma})\, d\bar{x} = \int_{-1}^{+1} \left[C(\bar{x}) \sqrt{\frac{1+\bar{s}}{1-\bar{s}}} + \right.$$
$$\left. + ik\sqrt{1-\bar{s}^2}) \right] K(\bar{s}-\bar{\sigma})\, d\bar{s},$$
$$\sqrt{\frac{1+\bar{\sigma}}{1-\bar{\sigma}}} \int_{-1}^{+1} \sqrt{\frac{1-\bar{x}}{1+\bar{x}}} F(\bar{x},\bar{\sigma})\bar{x}\, d\bar{x} = \int_{-1}^{+1} \left[\sqrt{1-\bar{s}^2} - \frac{ik}{2}\bar{s}\sqrt{1-\bar{s}^2} + \right.$$
$$\left. + \frac{1}{2}(C(k)-1)\sqrt{\frac{1+\bar{s}}{1-\bar{s}}} \right] K(\bar{s}-\bar{\sigma})\, d\bar{s}.$$

By taking these formulas into account, formulas for forces will be in the form

$$\bar{P} = -2\varrho v_0^2 a \int_{-1}^{+1} \left[C(k)\sqrt{\frac{1+\bar{x}}{1-\bar{x}}} + ik\sqrt{1-\bar{x}^2} \right] [v_y(\bar{x}) - N_1(\bar{x})]\, dx, \quad \text{(V.39)}$$

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$$M = -2\varrho v_0^2 a^2 \int_{-1}^{+1} \left\{ \sqrt{1-\bar{x}^2} - \frac{ik}{2}\bar{x}\sqrt{1-\bar{x}^2} + \right.$$
$$\left. + \frac{1}{2}[C(k)-1]\sqrt{\frac{1+x}{1-x}} \right\} [v_y(\bar{x}) - N_1(\bar{x})]\, d\bar{x}, \quad \text{(V.40}$$

where

$$N_1(x) = \frac{1}{2\pi} \int_{-1}^{+1} \gamma(\sigma)\, K(x-\sigma)\, d\sigma. \quad \text{(V.41)}$$

Formulas (V.39) and (V.40) show that the influence of the free surface causes changes in normal velocities on the profile and that the hydromechanical characteristics of the hydrofoil are equal to those of a certain hydrofoil in an infinite fluid which has a normal velocity $v_y'(x) = v_1(x) - N_1(x)$.

In order to determine $\bar{L}$ and M it is necessary to find the values of $N_1(x)$. The exact value of $N_1(x)$ may be found after solving the integral equation (V.32), but the theory of the deeply submerged hydrofoil may be developed using

$\gamma_0(\sigma)$ instead of $\gamma(\sigma)$ in formula (V.41).

5.4. Expansion of the K(s - σ) Function in Powers of Parameter τ

For solving equation (V.33) we will need the expansion of the K(s - σ) function with respect to the positive powers of the paremeter $\tau = \sqrt{4\bar{h}^2+1} - 2\bar{h}$. Let us write function K(s - σ) in the form

$$K(s-\sigma) = G_1'(s-\sigma) - i\bar{k}\int_0^{\infty} e^{-i\bar{k}u} G_1'(s-\sigma-4\bar{h}u)\,du, \qquad \bar{k} = 4\bar{h}k.$$

The function $\dfrac{1}{x-4\bar{h}(u+i)}$ expands into the following series:

$$\frac{1}{x-4\bar{h}(u+i)} = -\sum_{n=1,2,3}^{\infty} \tau^n \sum_{p=1,2,3}^{n} \frac{\left(\frac{n+p}{2}-1\right)!}{(p-1)!\left(\frac{n-p}{2}\right)!} x^{p-1} \frac{1}{(u+i)^p}.$$

Let us search for the expansion of function K(s - σ) in the form of (I.81):

$$K(s-\sigma) = \sum_{n=1,2,3}^{\infty} \tau^n \sum_{p=1,2,3}^{n} (s-\sigma)^{p-1} K_{n,p}, \qquad \text{(V.42)}$$

[179

where $K_{n,p} = \begin{cases} \dfrac{\left(\frac{n+p}{2}-1\right)!}{(p-1)!\left(\frac{n-p}{2}\right)!}(1+\Sigma F_{p-1}(\lambda_l)) & \text{(n, p - even)}, \\ \dfrac{\left(\frac{n+p}{2}-1\right)!}{(p-1)!\left(\frac{n-p}{2}\right)!}\Sigma F_{p-1}(\lambda_l) & \text{(n, p - odd)}. \end{cases}$

Let us now determine function $\Sigma F_{p-1}(\lambda_l)$. For the integrals in the expression (V.30) we have the expansion

$$\lambda_l \int_{\pm\infty}^{0} \frac{e^{-i\lambda_l u}du}{x-4\bar{h}(u+i)} = \sum_{n=1,2,3}^{\infty} \tau^n \sum_{p=1,2,3}^{n} \frac{\left(\frac{n+p}{2}-1\right)! \, x^{p-1}}{(p-1)!\left(\frac{n-p}{2}\right)!} \times$$

$$\times \begin{bmatrix} -F'_{p-1}(\lambda_l) \\ -P'_{p-1}(\lambda_l) \end{bmatrix}, \qquad \text{(V.43)}$$

where

$$F'_n(\lambda_l) = \lambda_l \int\limits_{+\infty}^{0} \frac{e^{-i\lambda u}}{(u+i)^{n+1}}\, du,$$

$$P'_n(\lambda_l) = \lambda_l \int\limits_{-\infty}^{0} \frac{e^{-i\lambda u}}{(u+i)^{u+1}}\, du,$$

$$F'_n(\lambda) = -\,i^{-n} F_n(\lambda),$$

$$P'_n(\lambda) = -\,i^{-n} F_n(\lambda),$$

and where $F_n(\lambda)$ is the function derived in Ch. II.

Let us establish one equality for the expansion of the function

$$k\lambda \int\limits_0^\infty \int\limits_0^\infty \frac{e^{-iku} e^{-i\lambda u_1}}{(\rho-\sigma) - 4\bar{n}(u+u_1+i)}\, du du_1$$

Let us integrate by parts:

$$\int\limits_0^\infty \int\limits_0^\infty \frac{e^{-ik_1 v} e^{-ik_2 u}}{(u+v+i)^{n+1}}\, dudv = \frac{1}{n} \int\limits_0^\infty \frac{e^{-ik_1 c}}{(v+i)^n}\, dv -$$

$$-\frac{ik_2}{n} \int\limits_0^\infty \int\limits_0^\infty \frac{e^{-ik_1 v} e^{-ik_2 u}}{(u+v+i)^n}\, dudv, \qquad [180$$

$$\int\limits_0^\infty \int\limits_0^\infty \frac{e^{-ik_1 v} e^{-ik_2 u}}{(u+v+i)^{n+1}}\, dudv = \frac{1}{n} \int\limits_0^\infty \frac{e^{-ik_2 u}}{(v+i)^n}\, dv -$$

$$-\frac{ik_1}{n} \int\limits_0^\infty \int\limits_0^\infty \frac{e^{-ik_1 v} e^{-ik_2 u}}{(u+v+i)^n}\, dudv.$$

From the above relations we obtain

$$\int\limits_0^\infty du \int\limits_0^\infty \frac{e^{-ik_1 v} e^{-ik_2 u}}{(u+v+i)^n}\, dv = -\,i \frac{1}{(k_2-k_1)} \left[\int\limits_0^\infty \frac{e^{-ik_1 v}}{(v+i)^n}\, dv - \int\limits_0^\infty \frac{e^{-ik_2 u}}{(u+i)^n}\, du \right]. \qquad \text{(V.44)}$$

These relations are sufficient for writing the general expansion. By combining these results we obtain for $\Sigma F_n(\lambda_l)$

$$\sum F_n(\lambda_i) = i\left\{\frac{1}{\sqrt{1-4\tau'}}\left[-F'_n(\lambda_1) + P'_n(\lambda_2) + \frac{k\lambda_1}{\lambda_1 - k}\left(\frac{F'_n(k)}{k} - \right.\right.\right.$$

$$\left.\left. - \frac{F'_n(\lambda_1)}{\lambda_1}\right) - \frac{k\lambda_2}{\lambda_2 - k}\left(\frac{F'_n(k)}{k} - \frac{p'_n(\lambda_2)}{\lambda_2}\right)\right] + \frac{1}{\sqrt{1+4\tau'}}\left[\overline{F'_n(\lambda_3)} - \right.$$

$$- \overline{F'_n(\lambda_4)} + \frac{k\lambda_3}{\lambda_3 + k}\left(-\frac{F'_n(-k)}{k} - \frac{\overline{F'_n(\lambda_3)}}{\lambda_3}\right) - \frac{k\lambda_4}{\lambda_4 + k}\left(-\frac{F'_n(-k)}{k} - \right.$$

$$\left.\left.\left. - \frac{\overline{F'_n(\lambda_4)}}{\lambda_4}\right)\right] - F_n(k)\right\}. \qquad (V.45)$$

With the change in direction of the Ox axis the functions $F'_n(\lambda_i)$ in the expression (V.45) should be replaced by $P'_n(\lambda_i)$.

For k = 0:

$$\Sigma F_n(\lambda_i) = i\{(-F'_n(\lambda) + \overline{F'_n(\lambda)}) = 2\,\mathrm{Im}\,F'_n(\lambda),$$

Since

$$F'_n(\lambda) = -i^{-n}\,\mathrm{Re}\,F_n(\lambda) - i^{-(n+1)}\,\mathrm{Im}\,F_n(\lambda),$$

and then

$$\sum F_n(\lambda_i) = \begin{cases} 2(-1)^{\frac{n-1}{2}}\,\mathrm{Re}\,F_n(\lambda) & (n\text{ - odd}), \\ 2(-1)^{\frac{n}{2}+1}\,\mathrm{Im}\,F_n(\lambda) & (n\text{ - even}). \end{cases}$$

Substituting these values in the expression (V.42) we again obtain the expansion (I.81).

5.5. Solution of the Regular Integral Equation [181

If only a finite number of terms is retained in the expansion (V.42), the kernel of the equation (V.33) will degenerate and the equation will become equivalent to a certain system of algebraic equations which may easily be obtained.

Let us present an approximate solution of the problem (V.33) using the method discussed in Ch. II. Solving the equation (V.33) by the iteration method we may write formulas for forces in the following way:

$$\overline{P} = \varrho a v_0^2 \int_{-1}^{+1} \gamma_0(x)\left[1 - \sum_{n=0}^{\infty} \overline{N}_n(x)(-1)^n\right] dx, \qquad (V.46)$$

$$\overline{N}_n(\sigma) = \frac{1}{n^2}\int_{-1}^{+1} K(\rho - \sigma)\int_{-1}^{+1}\left\{\sqrt{\frac{1+\rho}{1-\rho}}\sqrt{\frac{1-t}{1+t}}\left[\frac{1}{\rho - t} + (C(k) - 1)\right] + \right.$$

$$+\frac{ik}{2}\lg\frac{1-pt+\sqrt{1-p^2}\sqrt{1-t^2}}{1-pt-\sqrt{1-p^2}\sqrt{1-t^2}}\Bigg\}\overline{N}_{n-1}(t)\,dt\,dp, \qquad (V.47)$$

$$\overline{N}_0(\sigma)=\frac{1}{\pi}\int_{-1}^{+1}\left[C(k)\sqrt{\frac{1+p}{1-p}}+ik\sqrt{1-p^2}\right]K(p-\sigma)\,dp$$

or

$$\overline{P}=\frac{2\varrho a v_0^2}{\pi}\int_{-1}^{+1}v_n(s)\,ds\int_{-1}^{+1}\Bigg\{\sqrt{\frac{1+s}{1-s}}\sqrt{\frac{1-x}{1+x}}\left[\frac{1}{s-x}+(C(x)-1)\right]+$$

$$+\frac{ik}{2}\lg\frac{1-sx+\sqrt{1-s^2}\sqrt{1-x^2}}{1-sx-\sqrt{1-s^2}\sqrt{1-x^2}}\Bigg\}\left[1-\sum_{n=0}^{\infty}\overline{N}_n(x)(-1)^n\right]dx. \qquad (V.48)$$

Similarly, for the moment:

$$\overline{M}=-\varrho a^2 v_0^2\int_{-1}^{+1}\gamma_0(x)\left[x-\sum_{n=0}^{\infty}\overline{M}_n(x)(-1)^n\right]dx,$$

$$\overline{M}_n(\sigma)=\frac{1}{\pi^2}\int_{-1}^{+1}K(p-\sigma)\int_{-1}^{+1}\Bigg\{\sqrt{\frac{1+p}{1-p}}\sqrt{\frac{1-t}{1+t}}\left[\frac{1}{p-t}+(C(k)-1)\right]+$$

$$+\frac{ik}{2}\lg\frac{1-pt+\sqrt{1-p^2}\sqrt{1-t^2}}{1-pt-\sqrt{1-p^2}\sqrt{1-t^2}}\Bigg\}\overline{M}_{n-1}(t)\,dt\,dp, \qquad (V.49)$$

$$\overline{M}_0=\frac{1}{\pi^2}\int_{-1}^{+1}K(p-\sigma)\int_{-1}^{+1}\Bigg\{\sqrt{\frac{1+p}{1-p}}\sqrt{\frac{1-t}{1+t}}\left[\frac{1}{p-t}+(C(k)-1)+\right.$$ [182

$$\left.+\frac{ik}{2}\lg\frac{1-pt+\sqrt{1-p^2}\sqrt{1-t^2}}{1-pt-\sqrt{1-p^2}\sqrt{1-t^2}}\right\}dt\,dp, \qquad (V.50)$$

$$\overline{M}=-\frac{2\varrho a v_0^2}{\pi}\int_{-1}^{+1}v_n(s)\,ds\int_{-1}^{+1}\Bigg\{\sqrt{\frac{1+s}{1-s}}\sqrt{\frac{1-x}{1+x}}\left[\frac{1}{s-x}+(C(x)-1)+\right.$$

$$\left.+\frac{ik}{2}\lg\frac{1-sx+\sqrt{1-s^2}\sqrt{1-x^2}}{1-sx-\sqrt{1-s^2}\sqrt{1-x^2}}\right\}\left[x-\sum_{n=0}^{\infty}\overline{M}_n(x)(-1)^n\right]dx. \qquad (V.51)$$

Let us write the expansion of the kernel K(p - σ) in the form of (I.70). Then

$$\left.\begin{aligned}K(p-\sigma)&=\xi\overline{\varphi}_1(p-\sigma)+\xi^2\overline{\varphi}_2(p-\sigma)+\xi^3\overline{\varphi}_3(p-\sigma)+\ldots\\ \overline{N}_0(\sigma)&=\xi\psi_{01}(\sigma)+\xi^2\psi_{02}(\sigma)+\xi^3\psi_{03}(\sigma)+\ldots\\ M(\sigma)&=\xi f_{01}(\sigma)+\xi^2 f_{02}(\sigma)+\xi^3 f_{03}(\sigma)+\ldots\end{aligned}\right\}. \qquad (V.52)$$

$$\psi_{0n}(\sigma) = -\frac{1}{\pi}\int_{-1}^{+1}\bar{\varphi}_n(p-\sigma)\left(C(k)\sqrt{\frac{1+p}{1-p}}+ik\sqrt{1-p^2}\right)dp, \qquad (V.53)$$

$$f_{0n}(\sigma) = -\frac{1}{\pi}\int_{-1}^{+1}\bar{\varphi}_n(p-\sigma)\left\{\sqrt{1-p^2}-\frac{ik}{2}p\sqrt{1-p^2}+\right.$$

$$\left.+\frac{1}{2}[C(k)-1]\sqrt{\frac{1+p}{1-p}}\right\}dp, \qquad (V.54)$$

$$\begin{aligned}\bar{N}_k(\sigma) &= \xi^{k+1}\psi_{k,k+1}(\sigma)+\ldots+\xi^n\psi_{k,n}(\sigma),\\ \bar{M}_k(\sigma) &= \xi^{k+1}f_{k,k+1}(\sigma)+\ldots+\xi^n f_{k,n}(\sigma),\end{aligned} \qquad (V.55)$$

$$\psi_{k,s}(\sigma) = \frac{1}{\pi^2}\int_{-1}^{+1}(\bar{\varphi}_1(p-\sigma)\bar{\psi}_{k-1,s-1}+\bar{\varphi}_2(p-\sigma)\psi_{k-1,s-2}+\ldots+$$

$$+\bar{\varphi}_m(p-\sigma)\psi_{k-1,s-m})\,d\sigma,$$

$$\bar{\psi}_{k-1,s}(p) = \int_{-1}^{+1}\left\{\sqrt{\frac{1+p}{1-p}}\sqrt{\frac{1-t}{1+t}}\left[\frac{1}{p-t}+(C(k)-1)\right]+\right. \qquad [183$$

$$\left.+\frac{ik}{2}\lg\frac{1-pt+\sqrt{1-p^2}\sqrt{1-t^2}}{1-pt-\sqrt{1-p^2}\sqrt{1-t^2}}\right\}\psi_{k-1,s}dt. \qquad (V.56)$$

$$f_{k,s}(\sigma) = \frac{1}{\pi}\int_{-1}^{+1}(\varphi_1(p-\sigma)\bar{f}_{k-1,s-1}+\varphi_2(p-\sigma)\bar{f}_{k-1,s-2}+\ldots+$$

$$+\varphi_m(p-\sigma)\bar{f}_{k-1,s-m})\,dp,$$

$$\bar{f}_{k-1,s} = \int_{-1}^{+1}\left\{\sqrt{\frac{1+p}{1-p}}\sqrt{\frac{1-t}{1+t}}\left[\frac{1}{p-t}+(C(k)-1)\right]+\right.$$

$$\left.+\frac{ik}{2}\lg\frac{1-pt+\sqrt{1-p^2}\sqrt{1-t^2}}{1-pt-\sqrt{1-p^2}\sqrt{1-t^2}}f_{k-1,s}\right\}dt. \qquad (V.57)$$

By assuming functions $\varphi_1(p - \sigma)$ in the form of (I.75), functions $\psi_{k,s}$ and $f_{k,s}$ will be determined by formulas (I.77) and (I.79), in which the coefficients $G_{p,k-m}$ are found as follows:

$$C^{\bullet}_{0,n,m} = \frac{1}{\pi}\int_{-1}^{+1}B_{0,n,m}(p)\left[C(k)\sqrt{\frac{1+p}{1-p}}+ik\sqrt{1-p^2}\right]dp;$$

$$C^*_{p,k,m} = \frac{1}{\pi}\int_{-1}^{+1}\sqrt{\frac{1+p}{1-p}}B_{p,k,m}(p) + \frac{ik}{2}B'_{p,k,m}(p)\sqrt{1-p^2}\Big]dp; \quad (V.58)$$

$$B_{p,k,m} = B_{0,m+1,m}d_{p-1,k-m-1} + B_{0,m+2,m}d_{p-1,k-m-2} + B_{0,k-1,m}d_{p-1,p},$$

$$d_{p,k}(s) = \frac{1}{\pi}\int_{-1}^{+1}\sqrt{\frac{1-\sigma}{1+\sigma}}\left[\frac{1}{s-\sigma} + C(k) - 1\right][C^*_{p,k,0} +$$

$$+ C^*_{p,k,1}\sigma + C^*_{p,k,k-1}\sigma^{k-p-1}]d\sigma; \quad (V.59)$$

$$B'_{p,k,m} = B_{0,m+1,n}g_{p-1,k-m-1} + B_{0,m+2,n}g_{p-1,k-m-2} + \ldots + B_{0,k-1,m}g_{p-1,p},$$

$$g_{p,k}(s) = \frac{1}{\pi}\int_{-1}^{+1}\lg\frac{1-s\sigma+\sqrt{1-s^2}\sqrt{1-\sigma^2}}{1+s\sigma+\sqrt{1-s^2}\sqrt{1+\sigma^2}}[C^*_{p,k,0} + C^*_{p,k,1}\sigma + \ldots +$$

$$+ C^*_{p,k,k-1}\sigma^{k-p-1}]d\sigma;$$

$$\bar{C}^*_{0,n,m} = -\frac{1}{\pi}\int_{-1}^{+1}B_{0,n,m}(p)\Big\{\sqrt{1-p^2} + \frac{ik}{2}p\sqrt{1-p^2} +$$ [184

$$+ \frac{1}{2}[C(k) - 1]\sqrt{\frac{1+p}{1-p}}\Big\}dp, \quad (V.60)$$

$$\bar{G}^*_{p,k,m} = \frac{1}{\pi}\int_{-1}^{+1}\left[\sqrt{\frac{1+p}{1-p}}B_{p,k,m}(p) + \frac{ik}{2}\bar{B}'_{p,k,m}(p)\sqrt{1-p^2}\right]dp,$$

$$\bar{B}_{p,k,m} = B_{0,m+1,m}\bar{d}_{p-k,k-m-1} + B_{0,m+2}\bar{d}_{p-1,k-m-2} + \ldots +$$

$$+ B_{0,k-1,m}\bar{d}_{p-1,p},$$

$$\bar{B}'_{p,k,m} = B_{0,m+1,m}\bar{g}_{p-1,k-m-1} + B_{0,m+2,m}\bar{g}_{p-1,k-m-2} + B_{0,k-1,m}\bar{g}_{k-1}. \quad (V.61)$$

Functions $\bar{d}_{n,k}$ and $\bar{g}_{n,k}$ are determined from formulas (V.59) and (V.60), in which $\bar{C}^*_{p,k,m}$ is used instead of $C^*_{p,k,m}$.

Taking into consideration all these relations let us write the formulas for forces as follows:

$$\bar{P} = -2\varrho a v_0^2\int_{-1}^{+1}v_n(s)\left[\sqrt{\frac{1+s}{1-s}}A(s) + \frac{ik}{2}\sqrt{1-s^2}B(s)\right]ds, \quad (V.62)$$

$$\bar{M} = 2\varrho a^2 v_0^2\int_{-1}^{+1}v_n(s)\left[\sqrt{\frac{1+s}{1-s}}\bar{A}(s) + \frac{ik}{2}\sqrt{1-s^2}\bar{B}(s)\right]ds, \quad (V.63)$$

where

$$A(s)=d_{00}-\sum_{n=1}^{\infty}\varepsilon^n\sum_{k=0}(-1)^k d_{k,n}(s),$$
(V.64)
$$B(s)=g_{00}-\sum_{n=1}^{\infty}\varepsilon^n\sum_{k=0}^{n-1}(-1)\,g_{k,n}(s),$$

$$\bar{A}(s)=\bar{d}_{00}-\sum_{n=1}^{\infty}\varepsilon^n\sum(-1)^k\bar{d}_{kn}(\delta),$$
(V.65)
$$\bar{B}(s)=g_{00}-\sum_{n=1}^{\infty}\varepsilon^n\sum_{k=0}^{n-1}(-1)^k\bar{g}_{k,n}(s).$$

Let us determine coefficients $C^*_{p,k,m}$ and $\bar{C}^*_{p,k,m}$. [185

These coefficients can be easily found from the expansion (V.42).

Below is a list of coefficients $C^*_{p,k,m}$ and $\bar{C}^*_{p,k,m}$:

$$C^{\bullet}_{010}=K_{11}\left[C(k)+\frac{ik}{2}\right],\qquad C^{\bullet}_{020}=\frac{1}{2}C(k)K_{22},$$

$$C^{\bullet}_{021}=-K_{22}\left[C(k)+\frac{ik}{2}\right],\qquad C^{\bullet}_{030}=\frac{K_{31}}{K_{11}}C^{\bullet}_{010}+K_{33}\left[C(k)+\frac{ik}{2}\right],$$

$$C^{\bullet}_{031}=-K_{33}C(k),\qquad C^{\bullet}_{032}=K_{33}\left[C(k)+\frac{ik}{2}\right],$$

$$C^{\bullet}_{040}=C(k)\left[\frac{1}{2}K_{42}+\frac{3}{8}K_{44}\right],\qquad C^{\bullet}_{042}=\frac{3}{2}K_{44}C(k),$$

$$C^{\bullet}_{041}=-K_{42}\left[C(k)+\frac{ik}{2}\right]-K_{44}\left[\frac{3}{2}C(k)+\frac{3}{8}ik\right],$$

$$C^{\bullet}_{043}=-K_{44}\left[C(k)+\frac{ik}{2}\right],\qquad C^{\bullet}_{120}=C^{\bullet 2}_{010},$$

$$C^{\bullet}_{230}=C^{\bullet 3}_{010},\qquad C^{\bullet}_{340}=C^{4}_{010},$$

$$C^{\bullet}_{130}=2C^{\bullet}_{010}C^{\bullet}_{020}-K_{11}\frac{C^{\bullet}_{021}C(k)}{2},$$

$$C^{\bullet}_{131}=C^{\bullet}_{021}C^{\bullet}_{010},$$

$$C^{\bullet}_{240}=3C^{\bullet 2}_{010}C^{\bullet}_{020}-K_{11}C(k)\,C^{\bullet}_{131},$$

$$C^{\bullet}_{240}=C^{\bullet}_{021}C^{\bullet 2}_{010},$$

$$C^{\bullet}_{140}=K_{11}\left[C^{\bullet}_{030}C(k)-C^{\bullet}_{031}\frac{C(k)}{2}+\frac{C^{\bullet}_{032}}{2}C(k)\right]+K_{22}\left\{C^{\bullet}_{020}\frac{C(k)}{2}+\right.$$

$$+\frac{C^*_{021}}{4}[1-C(k)]\Big\}+\Big[K_{31}+\frac{K_{33}}{2}\Big]C^*_{010}C(k)+\frac{ik}{2}\Big[K_{11}\Big(C^*_{030}+\frac{C^*_{032}}{4}\Big)+$$

$$+\frac{K_{22}}{8}C^*_{021}+C^*_{010}\Big(K_{31}+\frac{K_{33}}{2}\Big)\Big]. \qquad (V.66)$$

$$C^*_{141}=-K_{22}C(k)\left[C^*_{020}-\frac{1}{2}C^*_{021}\right]-K_{33}C^*_{010}C(k)+\frac{ik}{2}K_{22}C^*_{020},$$

$$C^*_{142}=K_{33}C^*_{010}\left(C(k)+\frac{ik}{2}\right).$$

$$\bar{C}^*_{010}=-\frac{1}{2}K_{11}C(k),$$ [186

$$\bar{C}^*_{020}=K_{22}\Big\{-\frac{1}{4}[C(k)-1[+\frac{ik}{16}\Big\},\quad \bar{C}^*_{021}=\frac{1}{2}K_{22}C(k),$$

$$\bar{C}^*_{030}=\frac{K_{31}}{K_{11}}\bar{C}^*_{010}-K_{33}\left\{\frac{1}{4}[C(k)-1]+\frac{1}{8}\right\},$$

$$\bar{C}^*_{031}=-K_{33}\left\{\frac{1}{2}[C(k)-1]+\frac{ik}{8}\right\},\qquad \bar{C}^*_{032}=-\frac{1}{2}K_{33}C(k),$$

$$\bar{C}^*_{040}=K_{42}\left\{-\frac{1}{2}[C(k)-1]+\frac{ik}{16}\right\}-K_{44}\left\{\frac{3}{16}[C(k)-1]-\frac{ik}{32}\right\},$$

$$\bar{C}^*_{041}=K_{42}\frac{C(k)}{2}-K_{44}\left\{-\frac{3}{4}[C(k)-1]-\frac{3}{8}\right\},$$

$$\bar{C}^*_{042}=-K_{44}\left\{\frac{3}{4}[C(k)-1]-\frac{3ik_3}{16}\right\},\quad \bar{C}^*_{043}=K_{44}\frac{C(k)}{2},$$

$$\bar{C}^*_{120}=\bar{C}^*_{010}\bar{C}^*_{010},\quad \bar{C}^*_{230}=\bar{C}^*_{010}C^{*2}_{010},$$

$$\bar{C}^*_{340}=\bar{C}^*_{010}C^{*3}_{010},$$

$$\bar{C}^*_{130}=\bar{C}^*_{020}C^*_{010}+\bar{C}^*_{010}C^*_{020}-K_{11}\frac{\bar{C}^*_{021}}{2}C(k),$$

$$\bar{C}^*_{131}=\bar{C}^*_{010}C^*_{021},$$

$$\bar{C}^*_{240}=\bar{C}^*_{130}C^*_{010}+\bar{C}^*_{010}C^*_{010}C^*_{020}-\frac{\bar{C}^*_{130}K_{11}}{2}C(k),$$

$$\bar{C}^*_{241}=C^*_{021}C^*_{010}\bar{C}^*_{010}, \qquad (V.67)$$

$$\bar{C}^*_{140}=K_{11}\Big[\bar{C}^*_{030}C(k)-\bar{C}^*_{031}\frac{C(k)}{2}+\bar{C}^*_{032}\frac{C(k)}{2}\Big]+K_{22}\Big\{\bar{C}^*_{020}\frac{C(k)}{2}+$$

$$+\frac{\bar{C}^*_{021}}{4}[1-C(k)]\Big\}+\Big(K_{31}+\frac{K_{33}}{2}\Big)\bar{C}^*_{010}C(k)+\frac{ik}{2}\Big[K_{11}\Big(\bar{C}^*_{030}+$$

$$+\frac{\bar{C}^*_{032}}{4}\Big)+\frac{K_{22}}{8}\bar{C}^*_{021}+C_{010}\Big(K_{31}+\frac{K_{33}}{4}\Big)\Big].$$

$$\bar{C}^*_{141}=-K_{22}\left[C(k)\bar{C}^*_{020}-\bar{C}^*_{021}\frac{C(k)}{2}\right]-K_{33}\bar{C}^*_{010}C(k)-\frac{ik}{2}K_{22}\bar{C}^*_{020},$$

$$C^{*}_{142} = K_{33} C^{*}_{010} \left[C(k) + \frac{ik}{2} \right].$$

Functions $d_{n,m}$ and $g_{n,m}$ are found in the form [187

$$d_{00} = C(k), \quad \alpha_{01} = C^{*}_{010} C(k),$$

$$d_{03} = C^{*}_{030} C(k) + C^{*}_{031} \left\{ s - \frac{1}{2} [C(k) + 1] \right\} + C^{*}_{032} \left[s^2 - s + \frac{1}{2} C(k) \right], \quad (V.68)$$

$$d_{02} = C^{*}_{020} C(k) + C^{*}_{021} \left\{ s - \frac{1}{2} [C(k) + 1] \right\},$$

$$d_{04} = C^{*}_{040} C(k) + C_{041} \left\{ s - \frac{1}{2} [C(k) + 1] \right\} + C^{*}_{043} \left\{ s^3 - s^2 + \frac{1}{2} s - \right.$$

$$\left. - \frac{3}{8} \left[C(k) + \frac{1}{3} \right] \right\},$$

$$d_{12} = C^{*}_{120} C(k),$$

$$d_{13} = C^{*}_{130} C(k) + C_{131} \left\{ s - \frac{1}{2} [C(k) + 1] \right\},$$

$$d_{14} = C^{*}_{140} C(k) + C^{*}_{141} \left\{ s - \frac{1}{2} [C(k) + 1] \right\} + C^{*}_{142} \left[s^2 - s + \frac{1}{2} C(k) \right], \quad (V.69)$$

$$d_{23} = C^{*}_{230} C(k),$$

$$d_{24} = C^{*}_{240} C(k) + C^{*}_{241} \left\{ s - \frac{1}{2} [C(k) + 1] \right\},$$

$$d_{34} = C^{*}_{340} C(k),$$

$$g_{00} = 2, \quad g_{01} = 2C^{*}_{010}, \quad g_{02} = 2C^{*}_{020} + C^{*}_{021} s,$$

$$g_{03} = 2C^{*}_{030} + C^{*}_{031} s + \frac{C^{*}_{032}}{2} (1 + 2s^2),$$

$$g_{04} = 2C^{*}_{040} + C_{041} s + \frac{C^{*}_{042}}{3} (1 + 2s^2) + \frac{C^{*}_{043}}{2} \left(\frac{1}{2} s + s^3 \right),$$

$$g_{12} = 2C^{*}_{120}, \quad g_{13} = 2C^{*}_{130} + C^{*}_{131} s,$$

$$g_{14} = 2C^{*}_{140} + C^{*}_{141} s + \frac{C^{*}_{142}}{3} (1 + 2s^2),$$

$$g_{23} = 2C^{*}_{230}, \quad g_{24} = 2C^{*}_{240} + C^{*}_{241} s,$$

$$g_{34} = 2C^{*}_{340}, \quad \bar{d}_{00} = s - \frac{1}{2} [C(k) + 1], \quad \bar{g}_{00} = s. \quad (V.70)$$

Functions $\bar{d}_{nm}$ and $\bar{g}_{nm}$ are determined from formulas [188
(V.68) and (V.69), in which $\bar{C}^{*}_{p,k,m}$ is substituted for

$C^*_{p,k,m}$. Having the values of functions d_{nm} and g_{nm} and coefficients $C^*_{p,k,m}$, one may determine the values of the lifting force and moment for the various special cases of motion.

Let us examine a particular case of reciprocating oscillations of a plate. For this type of oscillation the normal velocity is given by the formula

$$v_n(s) = ik\bar{\eta},$$

where $\bar{\eta} = \frac{\eta}{a}$ - the relative amplitude of oscillations.

The dimensionless lifting force and moment may be written as follows:

$$\bar{P} = ik\bar{\eta}\left[C(k)\gamma_1 + \frac{ik}{2}\gamma_2\right], \tag{V.71}$$

$$\bar{M} = ik\bar{\eta}\left[-\frac{1}{2}C(k)\gamma_3 + \frac{ik}{2}\gamma_4\right]; \tag{V.72}$$

for functions γ_j the following formulas were obtained:

$$\gamma_1 = 1 - \tau C^*_{010} - \tau^2\left(C^*_{020} - C^*_{120} - \frac{C^*_{021}}{2}\right) - \tau^3\left(C^*_{030} - C^*_{130} - C^*_{230} - \right.$$
$$\left. - \frac{C^*_{031} - C^*_{131} - C^*_{032}}{2}\right) - \tau^4\left(C^*_{040} - C^*_{140} + C^*_{240} - C^*_{340} - \right.$$
$$\left. - \frac{C^*_{041} - C^*_{141} + C^*_{241} - C^*_{042} + C^*_{142}}{2} - \frac{3}{8}C^*_{043}\right), \tag{V.73}$$

$$\gamma_2 = 1 - \tau C^*_{010} - \tau^2(C^*_{020} - C^*_{120}) - \tau^3\left(C^*_{030} - C^*_{130} + C^*_{230} + \frac{1}{4}C^*_{031}\right) -$$
$$- \tau^4\left[C^*_{040} - C^*_{140} + C^*_{240} - C^*_{340} + \frac{1}{4}(\bar{C}^*_{042} - \bar{C}^*_{142})\right], \tag{V.74}$$

$$\gamma_3 = 1 + 2\tau\bar{C}^*_{010} + 2\tau^2\left(\bar{C}^*_{020} - \bar{C}^*_{120} - \frac{\bar{C}^*_{021}}{2}\right) + 2\tau^3\left(\bar{C}^*_{030} - \bar{C}^*_{130} + \right.$$
$$\left. + \bar{C}^*_{230} - \frac{\bar{C}^*_{031}}{2} + \frac{\bar{C}^*_{131}}{2} + \frac{\bar{C}^*_{032}}{2}\right) + 2\tau^4\left(\bar{C}^*_{040} - \bar{C}^*_{140} + \bar{C}^*_{240} - \bar{C}^*_{340} + \right.$$
$$\left. + \frac{\bar{C}^*_{041} - \bar{C}^*_{141} + \bar{C}^*_{241}}{2} + \frac{\bar{C}^*_{042} - \bar{C}^*_{142}}{2} - \frac{3}{8}\bar{C}^*_{043}\right), \tag{V.75}$$

$$\gamma_4 = -\bar{C}^*_{010}\tau - \tau^2(\bar{C}^*_{020} - \bar{C}^*_{120}) - \tau^3\left(\bar{C}^*_{030} - \bar{C}^*_{130} + \bar{C}^*_{230} + \frac{1}{4}\bar{C}^*_{032}\right) -$$
$$- \tau^4\left[\bar{C}^*_{040} - \bar{C}^*_{140} + \bar{C}^*_{240} - \bar{C}^*_{340} - \frac{1}{4}(\bar{C}^*_{042} - \bar{C}^*_{142})\right]. \tag{V.76}$$

For the submerged hydrofoil, it is difficult to separate the overall forces into the quasi-steady part and the part which depends on the vortical trail. [189

One may attempt to isolate the lifting force which depends on the entrained mass. This force is determined through the intensity of the vortical layer from the formula

$$P_1 = \varrho \frac{d}{dt} \int_{-a}^{+a} x \gamma_0(x, t)\, dx,$$

where $\gamma_0(x, t)$ is the quasi-steady intensity of the layer.

The quasi-steady moment is determined by the formula

$$M_0 = -\varrho v_0 \int_{-a}^{+a} x \gamma_0(x, t)\, dx.$$

By comparing these expressions we obtain

$$P_1 = -\frac{d}{dt}\frac{M_0}{v_0} \qquad (V.77)$$

and for the periodic oscillations

$$P_1 = -ik\frac{M_0}{a}. \qquad (V.78)$$

The expression for the quasi-steady moment will be obtained if we assume that k = 0 in the brackets of formula (V.72). Then, the force caused by the entrained mass will be expressed in the form

$$\bar{P}_1 = -\frac{ik}{2}(ik\eta)\gamma_{30}. \qquad (V.79)$$

In determining γ_{30} it should be kept in mind that functions k_{nm} in the coefficients $\bar{C}^*_{nmp}$ also depend on the Strouhal number. These dependences should be retained, since they determine the nature of wave formation on the free surface during the oscillation of a body with a certain frequency and k = 0 should be used only in those terms which depend on the vortical layer.

For the infinite fluid, force $\bar{P}_1$ will be determined by the second term in the expression (V.71). It is of interest to note that for the case of reciprocating

oscillations this term is the same as the formula (V.79). Comparing the coefficients with the same powers in the expressions (V.75) and (V.74) we obtain:

$$|2\bar{C}^{\bullet}_{010}|_0 = -|C^{\bullet}_{010}|_0$$

$$|\bar{C}^{\bullet}_{021}| = |C^{\bullet}_{020}|, \; -2|\bar{C}^{\bullet}_{120}|_0 = |C^{\bullet}_{120}|_0,$$

. . . .

. . . .

from which it follows that $\gamma_{20} = \gamma_{30}$.

This conformity is at the same time used as a tool for checking the correctness of determining coefficients C^{*}_{nmp}, $\bar{C}^{*}_{nmp}$, γ_2 and γ_3. [190

Function γ_{30} which determines the relative change of the force due to the entrained mass is in the following form:

$$\gamma_{30} = 1 - \tau K^0_{11} - \tau^2\left(\frac{1}{2}K^0_{22} - K^{02}_{11}\right) - \tau^3\left(K^0_{11} + \frac{K^0_{33}}{2} - \right.$$
$$\left. - \frac{3}{2}K^0_{11}K^0_{22} + K^{03}_{11}\right) - \tau^4\left(\frac{3}{4}K^0_{44} + K^0_{22} - \frac{9}{4}K^0_{33}K^0_{11} - \frac{1}{4}K^{02}_{22} - \right.$$
$$\left. - 2K^{02}_{11} + \frac{5}{2}K^{02}_{11}K^0_{22} - K^{04}_{11}\right). \qquad (V.80)$$

At high and low velocities of motion, γ_{30} is determined by the formulas

$$\mathrm{Fr} \to 0 \quad \gamma_{30} = 1 + \frac{1}{2}\tau^2 + \frac{1}{2}\tau^4 + 0(\tau^6), \qquad (V.81)$$

$$\mathrm{Fr} \to \infty \quad \gamma_{30} = 1 - \frac{1}{2}\tau^2 + 0(\tau^4). \qquad (V.82)$$

By assuming that k = 0 everywhere in formula (V.73), then the latter will transform into formula (I.82).

In conclusion let us examine the rotational and oscillation motions.

The normal velocity is expressed by the formula

$$v_y(s) = ik(\bar{\eta} + \bar{\varepsilon}\bar{\beta}) - \bar{\beta} + ik\bar{\beta}s,$$

where $\bar{\xi} = \frac{\xi}{a}$ - the relative distance between the center of rotation and the origin of the coordinate axes;

$\bar{\beta}$ - the amplitude of the rotational oscillations.

Let us determine the lifting force and the moment by the formulas

$$\bar{P} = [ik(\bar{\eta} + \bar{\xi\beta}) - \beta] C(k) \gamma_1 - ik\beta \frac{C(k)}{2} \gamma_5 + + \frac{ik}{2} \{[ik(\bar{\eta} + \xi\bar{\beta}) - \bar{\beta}] \gamma_2 - ik\beta\gamma_6\}, \tag{V.83}$$

$$\bar{M} = \frac{1}{2} C(k) [ik(\bar{\eta} + \bar{\xi\beta}) - \bar{\beta}] \gamma_3 - ik\beta\gamma_7 + \frac{ik}{2} \Big\{[ik(\bar{\eta} + \bar{\xi\beta}) - - \bar{\beta}] \gamma_4 - \frac{ik}{8} \beta\gamma_8\Big\}, \tag{V.84}$$

[191

$$\gamma_5 = 1 - \tau \overset{\bullet}{C}_{010} - \tau^2 \left[\overset{\bullet}{C}_{020} - \overset{\bullet}{C}_{120} + \frac{1}{2} \overset{\bullet}{C}_{021} (1 - C(k)\right] - - \tau^3 \Big\{\overset{\bullet}{C}_{030} - \overset{\bullet}{C}_{130} + \overset{\bullet}{C}_{230} + \frac{1}{2} (\overset{\bullet}{C}_{031} - \overset{\bullet}{C}_{131})[1 - C(k)] + + \frac{1}{2} \overset{\bullet}{C}_{032} \left[C(k) - \frac{1}{2}\right]\Big\} - \tau^4 \Big\{\overset{\bullet}{C}_{040} - \overset{\bullet}{C}_{140} + \overset{\bullet}{C}_{240} - \overset{\bullet}{C}_{340} + + \frac{1}{2} (\overset{\bullet}{C}_{041} - \overset{\bullet}{C}_{141} + \overset{\bullet}{C}_{241}) [1 - C(k)] + \left(\frac{1}{2} C(k) - \frac{1}{4}\right) (\overset{\bullet}{C}_{042} - \overset{\bullet}{C}_{142}) + + \frac{3}{8} \overset{\bullet}{C}_{043} [1 - C(k)]\Big\}, \tag{V.85}$$

$$\gamma_6 = -\frac{\tau^2}{8} \overset{\bullet}{C}_{021} - \frac{\tau^3}{8} (\overset{\bullet}{C}_{031} - \overset{\bullet}{C}_{131}) - \frac{\tau^4}{8} \left(\overset{\bullet}{C}_{041} - \overset{\bullet}{C}_{141} + \overset{\bullet}{C}_{241} + \frac{1}{2} \overset{\bullet}{C}_{041}\right), \tag{V.86}$$

$$\gamma_7 = \frac{1}{4} [1 - C(k)] - \frac{\tau}{2} \bar{\overset{\bullet}{C}}_{010} C(k) - \tau^2 \Big\{\frac{C(k)}{2} (\bar{\overset{\bullet}{C}}_{020} - \bar{\overset{\bullet}{C}}_{120}) + + \frac{\bar{\overset{\bullet}{C}}_{021}}{4} [1 - C(k)]\Big\} - \tau^3 \Big\{\frac{C(k)}{2} (\bar{\overset{\bullet}{C}}_{030} - \bar{\overset{\bullet}{C}}_{130} + \bar{\overset{\bullet}{C}}_{230}) + + \frac{\bar{\overset{\bullet}{C}}_{031} - \bar{\overset{\bullet}{C}}_{131}}{4} [1 - C(k)] + \frac{1}{4} \bar{\overset{\bullet}{C}}_{032} \left[C(k) - \frac{1}{2}\right]\Big\} - \tau^4 \Big\{C(k) (\bar{\overset{\bullet}{C}}_{040} - - \bar{\overset{\bullet}{C}}_{140} + \bar{\overset{\bullet}{C}}_{240} - \bar{\overset{\bullet}{C}}_{340}) + \frac{\overset{\bullet}{C}_{041} - \overset{\bullet}{C}_{141} + \overset{\bullet}{C}_{241}}{4} [1 - C(k)] + + \frac{\bar{\overset{\bullet}{C}}_{042} - \bar{\overset{\bullet}{C}}_{142}}{4} \left[C(k) - \frac{1}{2}\right] + \frac{3}{16} \bar{\overset{\bullet}{C}}_{043} [1 - C(k)]\Big\}, \tag{V.87}$$

$$\gamma_7 = 1 - \bar{\overset{\bullet}{C}}_{021} \tau^2 - \tau^3 (\bar{\overset{\bullet}{C}}_{031} - \bar{\overset{\bullet}{C}}_{131}) - \tau^4 \left(\bar{\overset{\bullet}{C}}_{041} - \bar{\overset{\bullet}{C}}_{141} + \bar{\overset{\bullet}{C}}_{241} + \frac{1}{2} \bar{\overset{\bullet}{C}}_{043}\right). \tag{V.88}$$

One could study the problem dealing with the determination of the suction force during the unsteady motion of a profile as well as the related problem concerning the thrust and efficiency of an oscillating hydrofoil in the same way as it is done in aerodynamics. However, these problems are not discussed here. Also, the problem of the profile wave drag during the unsteady motion will not be discussed here.

5.6. The Unsteady Motion of a Two-Dimensional Profile of an Arbitrary Shape Under the Free Surface of a Fluid [192

A problem concerning the motion of an arbitrary-shape two-dimensional profile is discussed by A. N. Shebalov [170-172]. Shebalov examined a special case of the profile motion in which the circulation of the velocity along the profile remains constant and there is no vortical trail. By using such a formulation, the problem is simplified considerably and can be solved by Kochin's method even if the profile has an arbitrary shape.

The problem of the unsteady motion of a profile with a constant circulation requires an investigation of conditions which are necessary for ensuring constancy of circulation. Such conditions may become impossible to achieve for all possible types of hydrofoil motion. In addition, these possible types of hydrofoil motion are limited considerably by the assumptions made in the derivation of the linear boundary conditions on the free surface.

In deriving boundary conditions it is assumed that only the horizontal velocity of the incident flow has a finite value. The vertical velocity is very small and is disregarded. Consequently, it follows that under the assumed boundary conditions one should consider such motions which have very low vertical velocities. However, the case of motion with a constant circulation along a circular trajectory, which is well known in aerodynamics, cannot be considered here.

Let us discuss briefly certain results obtained by Shebalov.

The boundary conditions on the free surface are assumed to be in the form

$$\varphi_{tt} + g\varphi_y + v^2(t)\,\varphi_{xx} - 2v(t)\,\varphi_{xt} - v_t(t)\,\varphi_x = 0; \quad (y = 0) \qquad \text{(V.58) [sic]}$$

conditions on the profile

$$\varphi_n = v_n(t);$$

at infinity

$$\varphi(x, y, t) \to 0 \qquad x \to +\infty,$$
$$\varphi(x, y, t) \to 0 \qquad y \to -\infty;$$

initial conditions

$$(\varphi_t)_{y=0} = 0, \quad (\varphi)_{y=0} = 0, \qquad (t = 0). \tag{V.90}$$

The potential of the vortex under the free flow is determined in the form

$$\varphi = -\frac{\Gamma}{2\pi}\left(\operatorname{arctg}\frac{y+h}{x} + \operatorname{arctg}\frac{y-h}{x}\right) + \varphi_1, \tag{V.91}$$

where φ_1 is a function which is harmonic in the half-plane and which is determined from the equation

$$\varphi_{1tt} = -2v\varphi_{1xt} + v^2\varphi_{1xx} + g\varphi_{1y} - v_t\varphi_{1x} =$$
$$= -g\frac{\Gamma}{\pi}\int_0^\infty e^{\lambda(y-h)}\sin\lambda x d\lambda, \tag{V.92}$$ [193

from which

$$\varphi_1 = -\frac{g\Gamma}{\pi}\int_0^\infty\int_0^t \frac{e^{\lambda(y-h)}}{\sqrt{gx}}\sin\lambda\left[x + \int_\tau^t v dt\right]\sin\sqrt{g\lambda}(t-\tau)\, d\tau d\lambda. \tag{V.93}$$

According to the Cauchy-Rieman conditions for the function of current ψ_1 we obtain

$$\psi_1 = -\int\frac{\partial\varphi_1}{\partial y}dx - \theta(y),$$

where $\theta(y)$ is an unknown function of y, which may be assumed to be equal to zero. Then

$$\psi_1 = -\frac{g\Gamma}{2\pi}\int_0^\infty\int_0^t \frac{e^{\lambda(y-h)}}{\sqrt{g\lambda}}\cos\lambda\left[x + \int_\tau^t v d\tau\right]\sin\sqrt{g\lambda}\,(t-\tau)\, d\tau d\lambda. \tag{V.94}$$

The complex potential of the vortex during the unsteady motion was obtained in the form

$$w(z, t) = \frac{\Gamma}{2\pi i}\ln\frac{z+ih}{z-ih} +$$
$$+ \frac{g}{\pi i}\int_0^\infty\int_0^t \frac{\Gamma e^{-i\lambda(z-ih)}}{\sqrt{g\lambda}} e^{-i\lambda\int_\tau^t v d\tau}\sin\sqrt{g\lambda}(t-\tau)\, d\lambda d\tau. \tag{V.95}$$

Similarly, for the source

$$\varphi = \frac{Q(t)}{2\pi}[\ln \sqrt{x^2 + (y+h)^2} - \ln \sqrt{x^2 + (y-h)^2}] + \varphi_1, \tag{V.96}$$

$$\varphi_{1tt} - 2v_0\varphi_{1xt} + v_{1xx}^2 + g\varphi_{1y} - v_t\varphi_{1x} = \frac{gQ(t)}{\pi}\int_0^\infty e^{\lambda(y-h)} \cos kx dk \tag{V.97}$$

and the complex potential of the source

$$W(z, t) = \frac{Q}{2\pi} \ln (z + ih)(z - ih) +$$

$$+ \frac{g}{\pi}\int_0^\infty \int_0^t \frac{Q(t)}{\sqrt{g\lambda}} e^{-i\lambda(z-ih)} e^{-i\lambda\int_\tau^t v d\tau} \sin\sqrt{g\lambda}(t-\tau) d\lambda d\tau. \tag{V.98}$$

From the expressions (V.95) and (V.98) we obtain the expression for the complex potential of the vortex source: [194

$$W(z, t) = \frac{B}{2\pi i} \ln (z - z_1) - \frac{\bar{B}}{2\pi i} \ln (z - \bar{z}_1) +$$

$$+ \frac{g}{\pi i}\int_0^\infty \int_0^t \frac{e^{-i\lambda(z-\bar{z}_1)}}{\sqrt{g\lambda}} \bar{B} e^{-i\lambda\int_\tau^t v d\tau} \sin\sqrt{g\lambda}(t-\tau) d\lambda d\tau. \tag{V.99}$$

Using Cauchy's formula for the doubly-connected region between the C_1 and C_2 contours, which envelop the hydrofoil profile, we obtain the for complex velocity of a two-dimensional profile the following expression:

$$v(z, t) = \frac{1}{2\pi i}\int_{C_1} \frac{v(\xi, t)}{z - \xi} d\xi + \frac{1}{2\pi i}\int_{C_1} \frac{\bar{v}(\xi, t)}{z - \xi} d\xi +$$

$$+ \frac{\nu}{\pi}\int_{C_1}\int_0^\infty \frac{e^{-i\lambda(z-\bar{\xi})}}{\lambda - \nu} \bar{v}(\xi, t) d\lambda d\xi +$$

$$+ \frac{\nu}{\pi}\int_{C_1}\int_0^\infty \frac{e^{-i\lambda(z-\xi)}}{\lambda - \nu} v(\xi, t) e^{-i\lambda vt}[-i\lambda v \sin\sqrt{g\lambda t} - \sqrt{g\lambda} \cos\sqrt{g\lambda} t] d\lambda d\xi_1. \tag{V.100}$$

Forces acting on the profile may be determined from the L. I. Sedov formula [128]. Let us limit ourselves to the calculation of the first term of Sedov's formula only:

$$Y - iX = \frac{\varrho}{2}\int_{C_1} |v(z)|_0^2 dz.$$

After transformations we obtain the expressions for y and x in the following form:

$$Y = \varrho v(t)\Gamma - \frac{\varrho}{2\pi}\int_0^\infty |H(\lambda, t)|^2 d\lambda +$$

$$+ \frac{\varrho}{2}\int_0^\infty |H(\lambda, t)|^2 \sqrt{g\lambda}\, d\lambda \int_0^t \cos\left(\lambda \int_0^t v d\tau\right) \cos\sqrt{g\lambda}\,(t-\tau)\, d\tau + \varrho g s, \quad (V.101)$$

$$X = -\frac{\varrho}{2}\int_0^\infty |H(\lambda, t)|^2 \sqrt{g\lambda}\, d\lambda \int_0^t \sin\left(\lambda \int_\tau^t v d\tau\right) \sin\sqrt{g\lambda}\,(t-\tau)\, d\tau, \quad (V.102)$$ [195

where $H(\lambda, t)$ is the Kochin function.

For a cylinder with a radius R and circulation along the contour Γ the force X is determined from the formula

$$X = -\frac{\varrho}{\pi}\int_0^\infty e^{-2\lambda h}\sqrt{g\lambda}\,[\Gamma + 2\pi v(\tau)\,\lambda R^2]^2 d\lambda \times$$

$$\times \int_0^t \sin\left(\lambda \int_\tau^t v d\tau\right) \sin\sqrt{g\lambda}\,(t-\tau)\, d\tau. \quad (V.103)$$

With Γ = 0 formula (V.103) transforms into the formula for determining the wave drag of the cylinder during the unsteady motion, which was obtained by L. N. Sretenskiy in a somewhat different form:

$$R_{By} = -X = 4\pi\varrho g R_0^4 \int_0^\infty e^{-2\lambda h}\lambda^2 \sqrt{g\lambda}\, d\lambda \times$$

$$\times \int_0^t v^2(\tau) \sin\left(\lambda \int_\tau^t v(\tau)\, d\tau\right) \sin\sqrt{g\lambda}\,(t-\tau)\, d\tau. \quad (V.104)$$

For any given moment of time after which the velocity becomes constant, the wave drag of a cylinder with circulation will be determined by the formula

$$R_B = \mathrm{Im}\, \frac{\varrho}{2\pi v^2}\int_0^\infty \frac{\sqrt{g\lambda}}{\lambda}(\Gamma + 2\pi v\lambda R^2)^2 \times$$

$$\times \frac{(\lambda v + \sqrt{g\lambda})\, e^{i(\lambda v - \sqrt{g\lambda})} - (\lambda v - \sqrt{g\lambda})\, e^{i(\lambda v + \sqrt{g\lambda})t}}{\lambda - v}. \quad (V.105)$$

For hydrofoils having a small relative thickness the principal effect of the free surface will be determined by the variation in circulation along the profile. However, Shebalov did not examine this problem.

CHAPTER VI. THEORY OF THE HYDROFOIL IN A PLANE-PARALLEL FLOW OF FLUIDS OF VARIOUS DENSITIES [196

6.1. General Aspects. Formulation of the Problem

A number of technical problems are reduced to the study of the hydrofoil motion near the interface of fluids with different densities. The problem of the hydrofoil motion in a three-dimensional flow of fluids of different densities will be discussed in Chapter XII, while this chapter will be devoted to the hydrofoil motion in a plane-parallel flow.

There are literature sources discussing the investigations of motion of fluids with various densities, i.e., motion caused by a body submerged in one of these fluids. The problem of waves on the interface of fluids with different densities, caused by the irregularities of the Earth, is analyzed in two publications by Kochin [53, 54]. His two other publications [62, 63] are devoted to the accurate analysis of the steady waves of finite amplitude on the interface of two fluids of finite depth. V. S. Voytsenya [12, 13] studied a two-dimensional problem concerning the motion and oscillation of a body near the interface between two fluids of different densities. In problems considered by Voytsenya the upper layer of the fluid is of finite depth. Very recently a study was published by A. B. Lotov [75] dealing with the problem of the hydrofoil motion above the interface between two fluids.

Let us first consider the stationary problems of motion of the hydrofoil in fluids of different densities. In analyzing such problems two half-planes of complex variable are introduced: an upper half-plane where the fluid has a density ρ_1 and a lower half-plane in which the density of the fluid is ρ_2. In the discussion below all parameters related to the upper half-plane will be designated by an index 1 and those related to the lower half-plane by an index 2.

Let us denote the basic velocities of both flows as [197
V_1 and V_2. Let $-V_1 z + W_1(z)$ and $-V_2 z + W_2(z)$ be the complex potentials of the flow of the upper and lower fluids.

The components of the velocities along the Ox and Oy axes will be in the form

$$u_k = v_k + \varphi_{kx}; \quad v_k = \varphi_{ky}. \tag{VI.1}$$

If η is the equation for the interface between the fluids, then, disregarding the small values of the second order, we obtain an expression

$$\varphi_{ky} = -v_k \frac{d\eta}{dx}, \quad (y = 0). \tag{VI.2}$$

According to Bernoulli's formula the pressure in the flow is

$$p_k = C_k - \frac{\varrho_k}{2}(v_k^2 - 2v_k\varphi_{kx} + \varphi_{kx}^2 + \varphi_{ky}^2) - gp_k y.$$

Disregarding the small values of the second order, we can write the following for the points on the interface between two fluids:

$$p_k = C_k' + \varrho_k v_k \zeta_k - g\varrho_k \eta.$$

However, since the pressure is continuous along the L line and since $p_1 = p_2$, then

$$g\eta(\varrho_1 - \varrho_2) = \varrho_2 v_2 \varphi_{2x} - \varrho_1 v_1 \varphi_{1x}, \quad (y = 0). \tag{VI.3}$$

If we denote

$$W_k(z) = v_k \widetilde{W}_k(z), \quad \frac{\varrho_k v_k^2}{\varrho_1 v_1^2 + \varrho_2 v_2^2} = m_k, \quad \frac{g(\varrho_1 - \varrho_2)}{\varrho_1 v_1^2 + \varrho_2 v_2^2} = \bar{v},$$

then conditions (VI.2) and (VI.3) may be written in the form

$$\widetilde{\varphi}_{1y} = \widetilde{\varphi}_{2y}, \quad (y = 0) \tag{VI.4}$$

$$m_2\varphi_{2x} - m_1\widetilde{\varphi}_{1x} = -\bar{v}\eta. \tag{VI.5}$$

After differentiating (VI.3) with respect to x and using (VI.2) in the linear approximation, the boundary conditions may also be written in the form

$$\varphi_{1y} = \frac{v_1}{v_2}\varphi_{2y}, \tag{VI.6}$$

$$\frac{\varrho_1 v_1}{\varrho_2 v_2}(\varphi_{xx} + \nu_1 \varphi_{1y}) - (\varphi_{2xx} + \nu_2 \varphi_{2y}) = 0, \quad (y = 0) \tag{VI.7}$$

where $\nu_k = \frac{g}{v_k^2}$.

Let us formulate the boundary problem for the Laplacian equation for the hydrofoil motion in a plane-parallel flow of fluids of different densities. [198

Let us find the solutions of equations

$$\nabla^2\varphi_k = 0, \qquad k = 1, 2$$

for the upper and lower half-planes, excluding that part which corresponds to the hydrofoil profile, using the conditions

$$\varphi_{1y} = \frac{v_1}{v_2}\varphi_{2y}, \qquad (y = 0) \tag{VI.8}$$

$$\frac{\varrho_1 v_1}{\varrho_2 v_2}(\varphi_{1xx} + \nu_1\varphi_{1y}) - (\varphi_{2xx} + \nu_2\varphi_{2y}) = 0,$$

$$\varphi_{jn} = + v_{nj}\cos(n, x) \tag{VI.9}$$

on the profile of the hydrofoil C in the j-th half-plane.

$$\nabla\varphi_k = 0, \tag{VI.10}$$

for

$$x \to \infty$$

$$y \to \pm\infty$$

$\nabla\varphi$ is finite at the trailing edge of the hydrofoil C. (VI.11)

With the appropriate selection of the new unknown function φ_3, the nonuniform conditions (VI.8) may be reduced to the already known uniform condition.

During the motion of a singularity located in the lower half-plane, the expression for the complex velocity may be written in the following way:

$$W_1(z) = \omega(z) + w_1(z), \tag{VI.12}$$

where $W_1(z)$ is an analytical function in the lower half-plane.

Let us assume that

$$W_2(z) = \frac{v_2}{v_1}\omega(z) + \frac{v_2}{v_1}w_2(z). \tag{VI.13}$$

Naturally, the function $W_2(z)$ should be analytical in the upper half-plane. In addition, the function $\omega(z)$ is also analytical in the upper half-plane.

If we denote

$$W_k(z) = \Phi_k(x, y) + i\psi_k(x, y), \qquad (k = 1, 2)$$

then the condition (VI.6)

$$\Phi_{1y} = \Phi_{2y}$$

with y = 0 will be satisfied if [199

$$\Phi_2(x, y) = -\Phi_1(x; -y), \qquad \text{(VI.14)}$$

from which

$$\Phi_{2xx} = -\Phi_{1xx}.$$

Therefore, from the expressions (VI.12) and (VI.13) we find

$$\varphi_{1xx} = \varphi_{xx} + \Phi_{1xx},$$

$$\varphi_{2xx} = \frac{v_2}{v_1}\varphi_{xx} - \frac{v_2}{v_1}\Phi_{1xx} = 2\frac{v_2}{v_1}\varphi_{xx} - \frac{v_2}{v_1}\varphi_{1xx}, \qquad (y=0)$$

and the condition (VI.7) will acquire the form

$$\varphi_{1xx} + \bar{v}\varphi_{1y} = 2m_2\varphi_{xx}. \qquad (y=0), \qquad \text{(VI.15)}$$

Let us examine the function

$$\omega_1(z) = \bar{\omega}(z), \qquad \text{(VI.16)}$$

which will be analytical in the lower half-plane.

In addition, let us assume that

$$\tilde{\omega}(z) = \omega(z) + \omega_1(z). \qquad \text{(VI.17)}$$

With y = 0

$$\tilde{\varphi} = 2\varphi, \qquad \tilde{\psi} = 0,$$

and then

$$\tilde{\varphi}_x = 2\varphi_x, \quad \tilde{\varphi}_y = \tilde{\psi}_x = 0. \qquad (y=0)$$

If we assume that

$$w_1(z) = m_2\tilde{\omega}(z) + w_3(z), \qquad \text{(VI.18)}$$

the condition (VI.15) will yield the following known expression for the function φ_3:

$$\varphi_{3xx} + \bar{v}\varphi_{3y} = 0. \qquad (y=0). \qquad \text{(VI.19)}$$

The above approach was used by N. Ye. Kochin in [53].

The advantages of the boundary conditions (VI.19) consist in the fact that they coincide with the boundary conditions on the free surface in problems which were

discussed in the preceding chapters. Also, one may use the results concerning the motion of isolated peculiarities under the free surface when analyzing motions considered here and utilize the theoretical methods of motion of bodies under the free surface.

Utilization of the boundary conditions in the form given in (VI.8) also does not present any particular difficulties so that the problems can be easily solved under these boundary conditions as well with the aid of the Fourier method.

Given below is a solution of the problem of motion of singularities under the conditions (VI.19). It was obtained by Kochin. In the discussion below the boundary conditions (VI.8) and the more general conditions for the case of unsteady motion will be used. [200

6.2. Motion of Isolated Singularities Near the Interface Between the Fluids of Different Densities

Let us examine the solution of the problem of motion of singularities under the interface which was obtained by Kochin [53]. First let us present certain general results and obtain the expression for $W_2(z)$. Condition (VI.19) may be written in the complex form in the same way as was done in Chapter I.

$$\operatorname{Im}[iW_{3zz}(z) - \bar{\nu}W_{3z}(z)] = 0. \qquad (y = 0) \qquad \text{(VI.20)}$$

Let us introduce a function of the complex variable z:

$$f(z) = W_{3zz}(z) - i\bar{\nu}W_{3z}(z).$$

Function f(z) is a single-valued and analytical function in the entire lower half-plane with the exception of the point $z = \alpha$. Since both $W_1(z)$ and $\tilde{\omega}(z)$ have a singularity $\omega(z)$ at point $z = \alpha$, it follows from (IV.18) that $W_3(z)$ has a singularity $\omega(z)m_j$ and the function f(z) has a singularity $F(z) = m_1(\omega_{zz} + i\bar{\nu}\omega_z)$ at this point. However, according to condition (VI.20) the function continues analytically into the upper half-plane in correspondence with the formula $f(z) = -\overline{f(\bar{z})}$.

Considering that the function f(z) vanishes at an

infinitely far point, we obtain

$$f(z) = W_{3zz}(z) - i\bar{\nu} W_{3z}(z) = F(z) - F(z). \qquad \text{(VI.21)}$$

After integrating the equation obtained let us determine the function $\omega_3(z)$ with an accuracy of up to and including functions $ae^{-i\bar{\nu}z} + b$.

The constant b does not affect motion of the fluid. The constant a is determined from the condition (VI.10).

Taking into consideration that

$$W_2(z) = -\bar{W}_1(\bar{z}),$$

$$W_1(z) = W_1(z) - \omega(z) = \omega(z)\, m_2 - \omega(z)\, m_1 + W_3(z),$$

we will find

$$W_2(z) = \frac{v_2}{v_1}[m_1 \tilde{\omega}(z) - \bar{W}_3(\bar{z})]. \qquad \text{(VI.22)}$$

The uniqueness of the solution obtained may be proven in the same way as was done in Ch. II.

Now let us examine motion of the vortex under the interface between two fluids. In this case [201

$$\omega(z) = \frac{\Gamma}{2\pi i} \ln(z - \zeta),$$

$$\bar{\omega}(\bar{z}) = -\frac{\Gamma}{2\pi i} \ln(z - \bar{\zeta}),$$

$$F(z) = m_1 \frac{\Gamma}{2\pi i} \left(-\frac{1}{(z-\zeta)^2} + \frac{i\bar{\nu}}{z-\zeta} \right).$$

The solution of equation (VI.22) is given by formula (I.15), Chapter I, in which $m_1\Gamma$ should be used instead of Γ.

From formula (I.15) we obtain the formulas for $W_1(z)$ and $W_2(z)$:

$$W_1(z) = \frac{\Gamma}{2\pi i} \left(m \frac{(z-\zeta)}{(z-\bar{\zeta})} + 2m_1 e^{-i\bar{\nu}z} \int_{+\infty}^{z} \frac{e^{i\bar{\nu}t}}{t-\bar{\zeta}}\, dt \right), \qquad \text{(VI.23)}$$

$$W_2(z) = \frac{\Gamma v_2}{2\pi i v_1} \left(2m_1 e^{+i\bar{\nu}z} \int_{+\infty}^{z} \frac{e^{-i\bar{\nu}t}}{t-\zeta}\, dt \right). \qquad \text{(VI.24)}$$

For the source with the intensity Q the calculations

are similar. Formulas for $W_1(z)$ and $W_2(z)$ are in the form

$$W_1(z) = \frac{Q}{2\Gamma}\left[\ln(z-\zeta)(z-\bar{\zeta}) - 2m_1 e^{-i\bar{\nu}z}\int_{+\infty}^{z}\frac{e^{i\bar{\nu}t}}{t-\bar{\zeta}}dt\right]. \qquad \text{(VI.26)}$$

$$W_2(z) = \frac{Q}{2\Gamma}\frac{v_2}{v_1}\left(2m_1 e^{i\bar{\nu}z}\int_{+\infty}^{z}\frac{e^{-i\bar{\nu}t}}{t-\zeta}dt\right). \qquad \text{(VI.27)}$$

Formulas for the dipole, obtained in the same way, are

$$W_1(z) = \frac{B}{z-\zeta} + (m_2 - m_1)\frac{\bar{B}}{z-\bar{\zeta}} + 2i\bar{\nu}\bar{B}m_1 e^{-i\bar{\nu}z}\int_{+\infty}^{z}\frac{e^{i\bar{\nu}t}}{t-\bar{\zeta}}dt, \qquad \text{(VI.28)}$$

$$W_2(z) = \frac{v_1}{v_2}\left[2m_1 B\left(\frac{1}{z-\zeta} + i\bar{\nu}e^{i\bar{\nu}z}\int_{+\infty}^{z}\frac{e^{-i\bar{\nu}t}}{t-\zeta}dt\right)\right]. \qquad \text{(VI.29)}$$

Let us now solve a number of problems by the Fourier method.

For the purpose of simplification, let us consider a case of motion when $V_1 = V_2$. This case is of the highest interest in the hydrodynamics of high-speed ships.

In the complex form the boundary conditions (VI.8) will be [202

$$\operatorname{Im}\{\bar{\varrho}[iW_{1zz}(z) - \nu W_{1z}(z)] - [iW_{2zz}(z) - \nu W_{2z}(z)]\} = 0, \qquad \text{(VI.30)}$$

$$\operatorname{Im} W_{1z}(z) = \operatorname{Im} W_{2z}(z), \quad \bar{\varrho} = \frac{\varrho_1}{\varrho_2} < 1. \qquad \text{(VI.31)}$$

As in the preceding chapters the following integral representations are used in the solution:

$$\frac{1}{z} = i\int_0^\infty e^{-i\lambda z}d\lambda, \qquad \operatorname{Im} z < 0,$$

$$\frac{1}{z} = -i\int_0^\infty e^{+i\lambda z}d\lambda, \qquad \operatorname{Im} z > 0.$$

In problems dealing with the hydrofoil motion in the upper half-plane we will look for the complex potentials for the respective motion in the following form

$$\begin{aligned} W_1(z) &= W_1^0(z) + F_1(z), \\ W_2(z) &= W_2(z), \end{aligned} \qquad \text{(VI.32)}$$

where $W_1^0(z)$ is a function analytical in the upper half-plane.

Then, from the boundary conditions (VI.30) and (VI.31) we obtain the relationships which are satisfied when y = 0:

$$\operatorname{Im}\{\bar{\varrho}[iW^0_{1zz}(z)-\nu W^0_{1z}(z)]-[iW_{2zz}(z)-\nu W_{2z}(z)]\}=\bar{\varrho}N_1(z), \qquad \text{(VI.33)}$$

$$\operatorname{Im} W^0_{1z}(z)+\operatorname{Im} F_{1z}(z)=\operatorname{Im} W_{2z}(z),$$

$$N_1(z)=-\operatorname{Im}[iF_{1zz}(z)-\nu F_{1z}(z)]. \qquad \text{(VI.34)}$$

Function $F_1(z)$ is determined according to one condition:

$$\operatorname{Re} F_{1zz}(z)=0, \quad \operatorname{Im} F_{1z}(z)=0. \quad (y=0) \qquad \text{(VI.35)}$$

The complex potentials for a hydrofoil moving under an interface may be sought in the following form:

$$W_1(z)=W_1(z),$$

$$W_2(z)=W_2^0(z)+F_2(z),$$

where $W_2^0(z)$ is a function which is analytical in the lower half-plane.

And again, from conditions (VI.30) and (VI.31) we obtain the relationships

$$\operatorname{Im}\{\bar{\varrho}[iW_{1zz}(z)-\nu W_{1z}(z)]-(iW^0_{2zz}(z)-\nu W^0_{2zz}(z))\}=N_2(z), \qquad \text{(VI.36)}$$

$$\operatorname{Im} W^0_{2z}(z)+\operatorname{Im} F_{2z}(z)=\operatorname{Im} W_{1z}(z),$$

$$N_2(z)=\operatorname{Im}[iF_{2zz}(z)-\nu F_{2z}(z)]. \qquad \text{(VI.37)}$$

The function $F_2(z)$ should also be determined from one of the conditions (VI.35). [203

Let us examine in greater detail the problem of motion of a vortex located at point ζ in the upper half-plane. We will determine $F_1(z)$ from the condition $\operatorname{Im} F_{1z}(z)$ = = 0 for y = 0:

$$F_1(z)=\frac{\Gamma}{2\pi i}\ln\left(\frac{z-\zeta}{z-\bar{\zeta}}\right).$$

Using the integral representations we obtain

$$N_1 = \frac{\Gamma}{\pi} \int_0^\infty e^{-\lambda h} \lambda \sin(x - \xi) \, d\lambda$$

and $W_1^0(z)$ and $W_2(z)$ will be sought in the form

$$W_1(z) = \frac{\Gamma}{\pi} \int_0^\infty [A_1(\lambda) + iB_1(\lambda)] \frac{e^{i\lambda(z-\zeta)}}{\lambda} \, d\lambda,$$

$$W_2(z) = \frac{\Gamma}{\pi} \int_0^\infty [A_2(\lambda) + iB_2(\lambda)] \frac{e^{-i\lambda(z-\zeta)}}{\lambda} \, d\lambda.$$

From the relations (VI.33) and (VI.34) we obtain

$$A_1(\lambda) = A_2(\lambda) = 0,$$

$$B_1(\lambda) = B_2(\lambda) = \frac{\bar{\varrho}}{1-\bar{\varrho}} \frac{\lambda}{\lambda - \bar{\nu}}, \quad \bar{\nu} = \nu a, \quad a = \frac{1-\bar{\varrho}}{1+\bar{\varrho}}.$$

Then, the solution which satisfies the condition in which there are no disturbances ahead of the hydrofoil at infinity will be in the form

$$W_1(z) = \frac{\Gamma}{2\pi i} \left[\ln\left(\frac{z-\zeta}{z-\bar{\zeta}}\right) - \frac{2\bar{\varrho}}{1+\bar{\varrho}} \int_0^\infty \frac{e^{i\lambda(z-\bar{\zeta})} d\lambda}{\lambda - \bar{\nu}} - \frac{2\bar{\varrho}}{1+\bar{\varrho}} \pi i e^{i\nu(z-\bar{\zeta})} \right], \quad \text{(VI.38)}$$

$$W_2(z) = \frac{\Gamma}{\pi i} \frac{\bar{\varrho}}{1+\bar{\varrho}} \left(-\int_0^\infty \frac{e^{-i\lambda(z-\zeta)}}{\lambda - \bar{\nu}} - \pi i e^{-i\nu(z-\zeta)} \right). \quad \text{(VI.39)}$$

Functions $W_1(z)$ for the source are determined in a similar way:

$$W_1(z) = \frac{Q}{2\pi} \left[\ln(z-\zeta) - a \int_0^\infty e^{i\lambda(z-\bar{\zeta})} \frac{\lambda - \nu}{\lambda(\lambda - \bar{\nu})} d\lambda - \frac{2\bar{\varrho}}{1+\bar{\varrho}} \pi i e^{i\bar{\nu}(z-\bar{\zeta})} \right]. \quad \text{(VI.40)}$$

$$W_2(z) = \frac{Q}{\pi} \frac{\bar{\varrho}}{1+\bar{\varrho}} \left(-\int_0^\infty \frac{e^{-i\lambda(z-\zeta)}}{\lambda - \bar{\nu}} d\lambda + \pi i e^{-i\bar{\nu}(z-\zeta)} \right). \quad \text{(VI.41)}$$

Combining (VI.38)-(VI.41) we obtain for the vortex source [204

$$W_1(z) = \frac{B}{2\pi i} \ln(z-\zeta) + \frac{Ba}{2\pi i} \int_0^\infty \frac{e^{i\lambda(z-\bar{\zeta})}\lambda - \nu}{\lambda(\lambda - \bar{\nu})} d\lambda + \frac{\bar{B}\bar{\varrho} e^{i\nu(z-\bar{\zeta})}}{1+\bar{\varrho}}, \quad \text{(VI.42)}$$

$$W_2(z) = -\frac{B\bar{\varrho}}{\pi i(1+\bar{\varrho})} \left(+ \int_0^\infty \frac{e^{-i\lambda(z-\zeta)}}{\lambda - \bar{\nu}} d\lambda + \pi i e^{-i\bar{\nu}(z-\zeta)} \right). \quad \text{(VI.43)}$$

Let us introduce the expressions for the complex potentials of individual singularities located at point ζ in the lower half-plane, which were obtained by the Fourier method:

<u>for the vortex</u>

$$W_1(z) = -\frac{\Gamma}{\pi i(1+\bar{\varrho})}\left(\int_0^\infty \frac{e^{i\lambda(z-\zeta)}}{\lambda-\bar{\nu}}\,d\lambda + \pi i e^{i\nu(z-\zeta)}\right), \qquad \text{(VI.44)}$$

$$W_2(z) = -\frac{\Gamma}{2\pi i}\left(\ln\frac{(z-\zeta)}{(z-\bar{\zeta})} - \frac{2}{1+\bar{\varrho}}\int_0^\infty \frac{e^{-i\lambda(z-\bar{\zeta})}}{\lambda-\bar{\nu}}\,d\lambda - \frac{2}{1+\bar{\varrho}}\pi i e^{-i\nu(z-\bar{\zeta})}\right); \qquad \text{(VI.45)}$$

<u>for the source</u>

$$W_1(z) = -\frac{Q}{\pi(1+\bar{\varrho})}\left[\int_0^\infty \frac{e^{i\lambda(z-\zeta)}}{\lambda-\bar{\nu}} - i\pi e^{i\bar{\nu}(z-\zeta)}\right], \qquad \text{(VI.46)}$$

$$W_2(z) = \frac{Q}{2\pi}\left[\ln(z-\zeta) + a\int_0^\infty e^{-i\lambda(z-\bar{\zeta})}\frac{\lambda+\nu}{\lambda(\lambda-\bar{\nu})}\,a\lambda + \frac{2\pi i}{1+\bar{\varrho}}e^{-i\bar{\nu}(z-\bar{\zeta})}\right]; \qquad \text{(VI.47)}$$

<u>for the vortex source</u>

$$W_1(z) = -\frac{B}{i\pi(1+\bar{\varrho})}\left(\int_0^\infty \frac{e^{i\lambda(z-\zeta)}}{\lambda-\bar{\nu}}\,d\lambda - i\pi e^{-i\bar{\nu}(z-\zeta)}\right), \qquad \text{(VI.48)}$$

$$W_2(z) = \frac{B}{2\pi i}\ln(z-\zeta) - \frac{\bar{B}a}{2\pi i}\int_0^\infty e^{-i\lambda(z-\bar{\zeta})}\frac{\lambda+\nu}{\lambda(\lambda-\bar{\nu})}\,d\lambda - \frac{\bar{B}}{1+\bar{\varrho}}e^{-i\bar{\nu}(z-\bar{\zeta})}. \qquad \text{(VI.49)}$$

6.3. Motion of a Two-Dimensional Profile Above the Interface Between the Fluids of Different Densities [205

The problem of motion of a two-dimensional profile arbitrary in shape is solved by the methods discussed in Ch. II. Let us take a point z in the upper half-plane and draw two contours C_1 and C_2 in such a way that the point z would be outside the contour C_1 and within the contour C_2. Using Cauchy's integral for the doubly-connected area and expressions (VI.42) and (VI.43) for the complex velocity $v_1(z)$ we obtain the expression for the complex velocity of the flow:

$$\left.\begin{aligned}
v_1(z) &= -v_0 + v_{11}(z) + v_{12}(z)\\
v_{11}(z) &= \frac{1}{2\pi i}\int_C \frac{v(\zeta_2)}{z-\zeta_2}\,d\zeta_2\\
v_{12}(z) &= \frac{a}{2\pi}\int_0^\infty e^{i\lambda z}\bar{H}(-\lambda)\frac{\lambda-\nu}{\lambda-\bar{\nu}}\,d\lambda + i\frac{\bar{\varrho}_1}{1+\bar{\varrho}_1}\,\bar{\nu}e^{i\bar{\nu}z}\bar{H}(-\bar{\nu})
\end{aligned}\right\}, \tag{VI.50}$$

where $H(-\lambda)=\int_C e^{i\lambda z}v_1(z)\,dz$ - the Kochin function;

$v_{11}(z)$ - analytical function in the entire plane outside C_1;

$v_{12}(z)$ - analytical function within C_2.

Let us restrict ourselves to the calculation of the lifting force and wave drag acting on the body. To calculate the lifting force P and the wave drag Q we will use the first S. A. Chaplygin formula.

After transformation we obtain:

$$P = \varrho_1 v_0\Gamma - \frac{\varrho_1 a}{2\pi}\int_0^\infty |H(-\lambda)|^2\frac{\lambda-\nu}{\lambda-\bar{\nu}}\,d\lambda, \tag{VI.51}$$

$$Q = +\bar{\nu}\frac{\bar{\varrho}}{1+\bar{\varrho}_1}\varrho_1|H_1(-\lambda)|^2. \tag{VI.52}$$

When a = 1 and Q = 0, formula (VI.51) will give the value of the hydrofoil lifting force near a solid wall. With a = 0 formula (VI.51) will become the Zhukovskiy formula.

To determine circulation Γ it is necessary to express $v_1(z)$ through the function $H(\lambda)$.

Let us map conformally the shape of the contour C on a circle with a radius R so that an infinitely distant point z in a plane would become an infinitely distant point u and $\left(\frac{du}{dz}\right)_{z=\infty} = 1.$

The boundary conditions on the contour may be written in the form [206

$$\operatorname{Re} v_{11}(z)\frac{dz}{d\sigma}\sigma = -\operatorname{Re}[v_{12}(z)-v_0]\frac{dz}{d\sigma}\sigma \quad \text{on K.}$$

Using the Schwarz formula which determines the function analytical outside of the circle in terms of its real

part on the circle, we obtain

$$v_{12}(z)=\frac{du}{dz}\left\{-v_0+\frac{\bar{v}_0R^2}{u^2}+\frac{Ci}{u}+\frac{a}{4\pi^2 i}\int_0^\infty\left[G_1(-\lambda,u)\bar{H}(-\lambda)+\right.\right.$$

$$\left.+\frac{R^2}{u^2}G\left(\overline{-\lambda,\frac{R^2}{\bar{u}}}\right)H(-\lambda)\right]\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda+\frac{\bar{\nu}\bar{Q}_1}{2\pi(1+\bar{Q}_1)}\times$$

$$\left.\times\left[G(\bar{\nu},u)\bar{H}(-\bar{\nu})-\frac{R^2}{u^2}G\left(\overline{-\bar{\nu}\frac{R^2}{\bar{u}}}\right)H(-\bar{\nu})\right]\right\}. \qquad \text{(VI.53)}$$

In order to determine the value of circulation Γ it is necessary to gather the terms containing $\frac{1}{u}$ in the expression (VI.58) and to use the Zhukovskiy-Chaplygin postulate.

Let us examine a special case in which the hydrofoils were obtained with the aid of the Zhukovskiy transforming function. Function $G(\lambda, u)$ will be determined by formula (II.35). Then, from the expression (VI.53) it follows

$$v_{11}(z)=\frac{du}{dz}\left\{-\left[v_0+\frac{a}{2\pi}\int_0^\infty e^{-\lambda h}J_0(2\lambda R)H(-\lambda)\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda-\right.\right.$$

$$\left.-\frac{\bar{\nu}i\bar{Q}_1}{1+\bar{Q}_1}e^{-\bar{\nu}h}H(-\bar{\nu})J_0(2\lambda R)\right]+\frac{R^2}{u^2}\left[\bar{v}_0+\frac{a}{2\pi}\int_0^\infty e^{-\lambda h}J_0(2\lambda R)\times\right.$$

$$\left.\left.\times\bar{H}(-\lambda)\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda+\frac{\bar{\nu}i\bar{Q}_1}{1+\bar{Q}_1}J_0(2\lambda R)e^{-\lambda h}\bar{H}(-\lambda)\right]+\frac{\Gamma}{2\pi iu}\right\}+$$

$$+\frac{a}{2\pi^2 i}\int_0^\infty\left[B_1(\lambda)e^{-\lambda h}\bar{H}(-\lambda)+\frac{R^2}{u^2}\bar{B}_1(\lambda)e^{-\lambda h}H(-\lambda)\times\right.$$

$$\left.\times\frac{du}{d\frac{\bar{R}^2}{\bar{u}}}\right]\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda+\frac{\bar{\nu}\bar{Q}_1}{2\pi(1+\bar{Q}_1)}\left[B_1(\lambda)\bar{H}(-\bar{\nu})-\frac{R^2}{u^2}B_1(\lambda)H(-\nu)-\frac{du}{d\frac{\bar{R}^2}{\bar{u}}}\right]. \qquad \text{(VI.54)}$$

Satisfying conditions of the Zhukovskiy-Chaplygin postulate at the point u_0 = -R we obtain [207

$$\Gamma=4\pi R\,\mathrm{Im}\left[v_0+\frac{a}{2\pi}\int_0^\infty H(-\lambda)e^{-\lambda h}J_0(2\lambda R)\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda-\right.$$

$$\left.-\frac{\bar{\nu}i\bar{Q}_1}{1+\bar{Q}_1}H(-\bar{\nu})e^{-\nu h}J_1(2\lambda R)\right]. \qquad \text{(VI.55)}$$

The function H(-λ) will be determined in the first approximation from the complex velocity of the hydrofoil motion in an infinite flow. Then, after calculations we obtain

$$\left.\begin{aligned} &P = \varrho_1 v_0 \Gamma - \frac{\varrho \Gamma_\infty^2 a}{2\pi}\left[\frac{\bar{A}_{00}+\bar{A}_{11}}{R} - 2\bar{\nu}\left(1-\frac{1}{a}\right)(B_{00}+B_{11})\right] \\ &\bar{\gamma} = \frac{\Gamma}{\Gamma_\infty} = 1 + 2a\left[A_{01} - \bar{\nu}\left(1-\frac{1}{a}\right)B_{01}\right] - \pi\bar{\omega}J_0^2\left(\frac{\bar{\omega}}{2}\right)e^{-2\bar{\omega}\bar{h}}\frac{\bar{\varrho}_1}{1+\bar{\varrho}_1} \\ &A_{nm} = R\int_0^\infty e^{-2h\lambda} J_n(2\lambda R)\, J_m(2\lambda R)\, d\lambda \\ &B_{nm} = \int_{-\infty}^0 e^{-2\nu(1-\lambda)h} J_n[2\nu(1-\lambda)R]\, J_m[2\nu(1-\nu)R]\, d\lambda \\ &\bar{h} = \frac{h}{4R}, \quad \bar{\omega} = 4\bar{\nu}R. \end{aligned}\right\} \qquad \text{(VI.56)}$$

The combination $A_{nm} - \frac{\omega}{4}\left(1-\frac{1}{a}\right)B_{01}$ may be determined in the form of a series in powers of the parameter $\tau = \sqrt{4\bar{h}^2+1} - 2\bar{h}$:

$$\left.\begin{aligned} &A_{00} - \frac{\bar{\omega}}{4}\left(1-\frac{1}{a}\right)B_{00} = \\ &= \frac{1}{a}\sum_{n=0}^{\infty}\tau^{2n+1}\sum_{s=0}^{n}\frac{(-1)^s s!\,(s+n)!}{s!s!s!s!\,(n-s)!\,2^{2s+1}}\left[a+(a-1)\operatorname{Re}F_{n+s}\left(\frac{\bar{\omega}}{2\tau}\right)\right] \\ &A_{01} - \frac{\bar{\omega}}{4}\left(1-\frac{1}{a}\right)B_{01} = \\ &= \frac{1}{a}\sum_{n=0}^{\infty}\tau^{2n+2}\sum_{s=0}^{n}\frac{(-1)^s(2s+1)!\,(s+n+1)!}{s!\,s!\,(s+1)!\,(s+1)!\,(n-s)!\,2^{2s+2}} \times \\ &\times\left[a+(a-1)\operatorname{Re}F_{n+s+1}\left(\frac{\bar{\omega}}{2\tau}\right)\right] \\ &A_{11} - \frac{\bar{\omega}}{4}\left(1-\frac{1}{a}\right)B_{11} = \\ &= \frac{1}{a}\sum_{n=0}^{\infty}\tau^{2n+3}\sum_{s=0}^{n}\frac{(-1)^s(2s+2)!\,(s+n+2)!}{s!\,(s+1)!\,(s+2)!\,(n-s)!\,2^{2s+3}} \times \\ &\times\left[a+(a-1)\operatorname{Re}F_{n+s+2}\left(\frac{\bar{\omega}}{2\tau}\right)\right] \end{aligned}\right\} \qquad \text{(VI.57)}$$

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When $\omega \to \infty$, formulas (15) and (16) provide the characteristics of the hydrofoil near a solid wall.

If only the first terms are retained, the formula (VI.7) will acquire the form

$$\gamma = 1 + \frac{a}{2}\tau^2\left[1 - \frac{(1-a)}{a}\operatorname{Re}F_1\left(\frac{\bar{\omega}}{2\tau}\right)\right] - \pi\bar{\omega}J_0^2\left(\frac{\bar{\omega}}{2}\right)e^{-2\bar{\omega}h}\frac{\bar{\varrho}_1}{1+\bar{\varrho}_1}. \quad \text{(VI.58)}$$

For the hydrofoil moving in air near the surface of water $\bar{\varrho}_1 = \frac{1}{800}$, $(1-a) = 2\bar{\varrho}$. For $\frac{\bar{\omega}}{2\tau} \approx 4{,}5$ function $\operatorname{Re}F_1\left(\frac{\bar{\omega}}{2\tau}\right)$ has a minimum which is equal to 1.8. It follows from formula (VI.7) that deviation γ in this case as compared with the deviation obtained during motion near a solid wall is less than 0.3%.

Thus, for the hydrofoil moving in air near a water surface, the lifting force differs little from that of the hydrofoil moving near a solid wall.

6.4. Motion of a Two-Dimensional Profile Under the Interface Between the Fluids of Different Densities

This problem is solved in the similar way as the preceding problem. Using the same approach we obtain the following expression for the complex velocity $v_2(z)$:

$$\left.\begin{aligned} v_2(z) &= -v_0 + v_{21}(z) + v_{22}(z) \\ v_{21} &= \frac{1}{2\pi i}\int_{C_1}\frac{v_2(\zeta)d\zeta}{z-\zeta} \\ v_{22}(z) &= \frac{a}{2\pi}\int_0^\infty e^{-i\lambda z}\overline{H}(\lambda)\frac{\lambda+\nu}{\lambda-\bar{\nu}}d\lambda + \frac{i\bar{\nu}}{i+\bar{\varrho}}e^{-i\bar{\nu}z}\overline{H}(\bar{\nu}) \end{aligned}\right\}. \quad \text{(VI.59)}$$

The complex velocity in the upper half-plane will be determined by the expression [209

$$v_1(z) = \frac{a}{\pi(1+\bar{\varrho})}\int_{C_1}v_2(\zeta)\left(-\int_0^\infty e^{i\lambda(z-\zeta)}\frac{\lambda}{\lambda-\bar{\nu}}d\lambda + \frac{i\bar{\nu}}{1+\bar{\varrho}_{11}}e^{i\bar{\nu}(z-\zeta)}\right)d\zeta. \quad \text{(VI.60)}$$

The lifting force and wave drag will be determined from the formulas

$$P = \varrho_2 v_0\Gamma - \frac{\varrho_2 a}{2\pi}\int_0^\infty |H(\lambda)|^2\frac{\lambda+\nu}{\lambda-\bar{\nu}}d\lambda, \quad \text{(VI.61)}$$

$$Q = \frac{\varrho_2}{1+\varrho} \bar{\nu} |H(\bar{\nu})|^2. \qquad \text{(VI.62)}$$

When a = 1 formulas (VI.61) and (VI.62) acquire the form of formulas (II.12) and (II.13).

Values of $v_{21}(z)$ and Γ are determined in the same way.

$$v_{21}(z) = \frac{du}{dz}\Big\{ - v_0 + \frac{\bar{v}_0 R^2}{u^2} + \frac{Ci}{u} +$$

$$+ \frac{a}{4\pi^2 i} \int_0^\infty \left[G(\lambda, u)\, \overline{H}(\lambda) + \frac{R^2}{u^2} G\left(\lambda, \overline{\frac{R^2}{\bar{u}}}\right) H(\lambda) \right] \frac{\lambda + \nu}{\lambda - \bar{\nu}} d\lambda +$$

$$+ \frac{\bar{\nu}}{2\pi(1+\bar{\varrho}_1)} \left[G(\bar{\nu}, u)\, \overline{H(\bar{\nu})} - \frac{R^2}{u^2} G\left(\overline{\bar{\nu}, \frac{R^2}{\bar{u}}}\right) H(\bar{\nu}) \right] \Big\} \qquad \text{(VI.63)}$$

and for the hydrofoil described by the Zhukovskiy function,

$$v_{21}(z) = \frac{du}{dz}\Big\{ - \Big[v_0 + \frac{a}{2\pi} \int_0^\infty e^{-\lambda h} J_0(2\lambda R)\, H(\lambda) \frac{\lambda + \nu}{\lambda - \bar{\nu}} d\lambda -$$

$$- \frac{\bar{\nu}}{1+\varrho} iH(\bar{\nu})\, e^{-\bar{\nu}h} J_0(2\bar{\nu}R) \Big] + \frac{R^2}{\pi^2} \Big[\bar{v}_0 + \frac{a}{2\pi} \int_0^\infty e^{-\lambda h} J_0(2\lambda R) \overline{H}(\lambda) \times$$

$$\times \frac{\lambda + \nu}{\lambda - \bar{\nu}} d\lambda + \frac{\bar{\nu} i}{1+\varrho} e^{-\bar{\nu}h} J_0(2\bar{\nu}R) \overline{H}(\bar{\nu}) \Big] + \frac{\Gamma}{2\pi i u} \Big\} +$$

$$+ \frac{a}{4\pi^2 i} \int_0^\infty \left[B_1(\lambda)\, e^{-\lambda h} \overline{H}(\lambda) + \frac{R^2}{\pi^2} \overline{B_1(\lambda)}\, H(\lambda) \frac{du}{d\frac{R^2}{u}} \right] \frac{\lambda + \nu}{\lambda - \bar{\nu}} d\lambda +$$ [210

$$+ \frac{\bar{\nu}}{2\pi(1+\bar{\varrho})} e^{-\bar{\nu}h} \left[B_1(\bar{\nu}) \overline{H}_2(\nu) - \frac{R^2}{u^2} \bar{B}_1(\bar{\nu})\, H(\nu) \frac{du}{d\frac{R^2}{u}} \right], \qquad \text{(VI.64)}$$

$$\Gamma = 4\pi R\, \mathrm{Im} \Big[v_0 + \frac{a}{2\pi} \int_0^\infty H(\lambda)\, e^{-\lambda h} J_0(2\lambda R) \frac{\lambda + \nu}{\lambda - \bar{\nu}} d\lambda -$$

$$- \frac{\bar{\nu} i}{1+\varrho} H(\bar{\nu})\, J_0(2\bar{\nu}R)\, e^{-\bar{\nu}h} \Big\}. \qquad \text{(VI.65)}$$

When determining function $H(\lambda)$ from the complex velocity of the hydrofoil in an infinite flow the lifting force and the relative circulation will be determined by the following formulas:

$$P = \varrho_2 v_0 \Gamma - \frac{\varrho \Gamma_\infty^2}{2\pi} a \left[\frac{A_{00} + A_{11}}{2R} - 2\bar{v} \left(\frac{1}{a} + 1 \right) (B_{00} + B_{11}) \right]. \qquad \text{(VI.66)}$$

$$\gamma = 1 - 2a \left[A_{01} - \frac{\omega}{4} \left(\frac{1}{a} + 1 \right) B_{01} \right] - \frac{\pi \bar{\omega}}{1 + \bar{\varrho}} J_0^2 \left(\frac{\bar{\omega}}{2} \right) e^{-2\bar{\omega} h}. \qquad \text{(VI.67)}$$

The combination $A_{nm} - \frac{\bar{\omega}}{u} \left(\frac{1}{a} + 1 \right) B_{nm}$ may also be determined by formulas (VI.57), in which a + 1 should be used instead of a - 1.

Taking into account the first terms in the expansion (VI.57) the relative ciruclation will be determined by the formula

$$\gamma = 1 - \frac{a}{2} \tau^2 \left[1 - \frac{(1+a)}{a} \operatorname{Re} F_1 \left(\frac{\bar{\omega}}{2\tau} \right) \right] - \pi \bar{\omega} J_0^2 \left(\frac{\bar{\omega}}{2} \right) e^{-2\bar{\omega} h} \frac{1}{1 + \bar{\varrho}} \qquad \text{(VI.68)}$$

when $\bar{\omega} \to 0 \ \gamma = 1 - \frac{a}{2} \tau^2$.

During the motion of the hydrofoil under the interface between the air and water the lifting force differs slightly from that of the hydrofoil moving near the free surface.

Let us compare the values of the wave drag of the hydrofoil above and below the interface. Let us consider two identical hydrofoils located at the same distance from the interface. In determining function H(λ) according to approximations discussed in Chapter II [211

$$H_1(-\lambda) = \frac{\Gamma_1}{\Gamma_2} \overline{H_2(\lambda)},$$

then

$$Q_1 = \frac{\bar{v} \bar{\varrho} \varrho_1}{1 + \bar{\varrho}} \left(\frac{\Gamma_1}{\Gamma_2} \right)^2 |H_2(\lambda)|^2,$$

$$Q_2 = \bar{v} \frac{\varrho_2}{1 + \bar{\varrho}} |H_2(\lambda)|^2 \qquad \text{(VI.69)}$$

and

$$\zeta = \frac{Q_1}{Q_2} = \bar{\varrho}^2 \left(\frac{\Gamma_1}{\Gamma_2} \right)^2.$$

With equal lifting forces on the hydrofoils $\frac{\Gamma_1}{\Gamma_2} = \frac{1}{\bar{\varrho}}$ and hence $\zeta = 1$. Thus, hydrofoils which have the same

lifting force, geometrical dimensions and which are located at the same distance from the interface have the same wave drag.

6.5. Motion of a Two-Dimensional Profile Under the Interface Between the Fluids of Different Densities and with the Upper Layer Having a Free Surface

The two-dimensional problem of motion of a body under the interface with the upper layer having a free surface was investigatted by V. S. Voytsenya [13]. Given below are the principal results of his study.

Let us denote the thickness of the upper layer of fluid by symbol d and use the following dimensionless parameters in the analysis:

$$z' = zd, \quad W'_f = W_f v_0 d, \quad \frac{gd}{v_0^2} = \nu_a, \quad \frac{\varrho_1}{\varrho_2} = \bar{\varrho}.$$

Let us assume that a profile with a velocity v_0 in direction of the positive axis moves under the interface between two layers. The x axis is located on the undisturbed free surface of the upper layer. The problem of determining the type of flow produced by a moving body is reduced to the following mathematical problem in which it is necessary to find functions $W_1(z)$ and $W_2(z)$, which are analytical in the areas D_1 and D_2, and which satisfy the following conditions:

$$\Phi_{1x} - v_a \psi_1 = 0 \qquad y = 0,$$

$$\psi_2 - \psi_1 = 0 \qquad y = -1,$$

$$\bar{\varrho}(\Phi_{1x} - \nu_a \psi_1) - (\Phi_{2x} - \nu_a \psi_2) = 0 \qquad y = -1,$$

$\Phi_{2n} = \cos(n, x)$ on body contour C, [212

$$\nabla \varphi_k = 0 \quad x \to +\infty$$

$\nabla \varphi_2$ is finite at the given edge of the hydrofoil c.

The area D_1 is an infinite strip $1 \leqslant y \leqslant 0$, while the area D_2 is contained between the contour C_1 and the line $y = -1$.

Using the Fourier method for the complex potentials of the vortex and the source, located at poing L = -ih, the following expressions were obtained:

$$W_1(z) = \frac{\Gamma}{2\pi i}\ln\frac{(z+ih)}{z-ih} + \frac{i\Gamma}{\pi}\int_0^\infty \left(Ae^{-i\lambda(z-ih)} - Be^{i\lambda(z+ih)}\right)\frac{d\lambda}{\lambda} +$$

$$+ \Gamma\left(A_1 e^{-i\nu_d(z-ih)} + A_2 e^{-i\lambda_0(z-ih)} + A_3 e^{i\lambda_0(z+ih)}\right), \qquad \text{(VI.70)}$$

$$W_2(z) = \frac{\Gamma}{2\pi i}\ln\frac{(z+ih)}{(z-i\zeta)} + \frac{i\Gamma}{\pi}\int_0^\infty Ee^{-i\lambda(z-ih)}\frac{d\lambda}{\lambda} +$$

$$+ \Gamma\left(E_1 e^{-i\nu_d(z-ih)} + E_2 e^{-i\lambda_0(z-ih)}\right), \qquad \text{(VI.71)}$$

$$W_1(z) = \frac{Q}{2\pi}\ln(z^2+h^2) + \frac{Q}{\pi}\int_0^\infty \left(Ae^{-i\lambda(z-ih)} + Be^{i\lambda(z+ih)}\right)\frac{d\lambda}{\lambda} -$$

$$- iQ\left(A_1 e^{-i\nu_d(z-ih)} + A_2 e^{-i\lambda_0(z-ih)} - A_3 e^{i\lambda_0(z+ih)}\right), \qquad \text{(VI.72)}$$

$$W_2(z) = \frac{Q}{2\pi}\ln(z^2+h^2) + \frac{Q}{\pi}\int_0^\infty Ee^{-i\lambda(z-ih)}\frac{d\lambda}{\lambda} -$$

$$- iQ\left(E_1 e^{-i\nu_d(z-ih)} + E_2 e^{-i\lambda_0(z-ih)}\right), \qquad \text{(VI.73)}$$

where

$$A = \frac{T_1(\lambda)\left(\lambda(2+\varkappa) - \varkappa\nu_d\right)}{\lambda(\lambda-\nu_d)T_0(\lambda)} - \frac{\varkappa}{2\lambda}e^{\lambda}(\lambda \operatorname{ch}\lambda + \nu_d \operatorname{ch}\lambda);$$

$$B = -\frac{\varkappa(\lambda+\nu)\left(1-e^{-2\lambda}\right)}{2T_0(\lambda)}; \quad E = \frac{2T_1(\lambda)}{(\lambda-\nu_d)T_0(\lambda)};$$

$$T_1(\lambda) = \lambda^2(1+\varkappa \operatorname{ch}^2\lambda) - \varkappa\nu_d^2 \operatorname{sh}^2\lambda;$$

$$T_0(\lambda) = \lambda(2+\varkappa) - \varkappa\nu_d e^{-2\lambda}; \quad \varkappa = \frac{1}{\varrho} - 1;$$

λ_0 is the root of the equation $T_0(\lambda) = 0$;

$$E_1 = -\frac{2T_1(\nu_d)}{\nu_d T_0(\nu_d)}; \quad E_2 = -\frac{2T_1(\lambda_0)}{\lambda_0(\lambda_0-\nu)T_0'(\lambda_0)}; \quad A_1 = E_1;$$ [213

$$A_2 = \frac{\varkappa(\lambda_0+\nu_d)T_1(\lambda_0)}{\lambda_0^2(\lambda_0-\nu_d)T_0'(\lambda_0)}e^{-2\lambda_0}; \quad A_3 = -\frac{\lambda_0(1+\varkappa)}{T_0'(\lambda_0)}.$$

In the same way as above we obtain the expression for the complex velocities of a two-dimensional profile moving under the interface between two fluids as expressed by Kochin's functions:

$$v_1(z) = \frac{1}{2\pi i}\int_C \frac{v_2(\zeta)\,d\zeta}{z-\zeta} + \int_0^\infty \left[\frac{1}{\pi}\overline{H(\lambda)}\left(A - \frac{1}{2}\right)e^{-i\lambda z} + \right.$$

$$+ H(\lambda) Be^{i\lambda z}\Big] d\lambda - i\nu_d A_1 \overline{H}(\nu_d) e^{-i\nu_d z} -$$

$$- i\lambda_0 A_2 \overline{H}(\lambda_0) e^{-i\lambda_0 z} + i\lambda_0 H(\lambda_0) e^{i\lambda_0 z}, \qquad \text{(VI.74)}$$

$$v_2(z) = \frac{1}{2\pi i} \int_C \frac{v_2(\zeta)\, d\zeta}{z - \zeta} + \frac{1}{\pi} \int_0^\infty \overline{H(\lambda)} \left(E - \frac{1}{2}\right) e^{-i\lambda z} d\lambda -$$

$$- i\nu_d E_1 \overline{H(\nu_d)} e^{-i\nu_d z} - i\lambda_0 E_2 \overline{H(\lambda_0)} e^{-i\lambda_0 z}. \qquad \text{(VI.75)}$$

The asymptotic expressions for the function $v_j(z)$ far behind the moving body are in the form

$$\lim_{x \to -\infty} v_1(z) = -2i [\nu_d A_1 \overline{H(\nu_d)} e^{-i\nu_d z} + \lambda_0 A_2 \overline{H(\lambda_0)} e^{-i\lambda_0 z} -$$

$$- \lambda_0 A_3 H(\lambda_0) e^{i\lambda_0 z},$$

$$\lim_{x \to -\infty} v_2(z) = -2i [\nu_d E_1 \overline{H(\nu_d)} e^{-i\nu_d z} + \lambda_0 E_2 \overline{H(\lambda_0)} e^{-i\lambda_0 z}]. \qquad \text{(VI.76)}$$

The equations for the free surface and the interface between fluids are found by the formulas

$$\eta_1(x) = \frac{1}{\nu_d} \operatorname{Re} v_1(z)\Big|_{y=0}, \quad \eta_2(x) = \frac{1}{\nu_d \left(\frac{1}{\varrho} - 1\right)} \operatorname{Re}\left[\frac{1}{\varrho} v_2(z) - v_1(z)\right]. \quad (y = -1) \qquad \text{(VI.77)}$$

Using formulas (VI.76) and (VI.77) let us determine the wave profiles at infinity:

$$\begin{aligned} \eta_1(x) &= 2E_1 \operatorname{Im} [H(\overline{\nu}_d) e^{-i\nu_d x}] - 2\varkappa E_2 e^{-2\lambda_0} \operatorname{Im} [\overline{H}(\lambda_0) e^{-i\lambda_0 x}], \\ \eta_2(x) &= 2E_1 e^{-\nu} \operatorname{Im} [H(\overline{\nu}_d) e^{-i\nu_d x}] + 2E_2 e^{-\lambda_0} \operatorname{Im} [H(\lambda_0) e^{-i\lambda_0 x}]. \end{aligned} \qquad \text{(VI.78)}$$

Thus, there is superposition of two types of waves with lengths $\overline{\lambda}_1 = \frac{2\pi}{\nu_d}$ and $\overline{\lambda}_0 = \frac{2\pi}{\lambda_0}$ and amplitudes a_j and b_j at both interfaces between the fluids far behind the moving body: [214

$$a_1 = 2|E_1||H(\nu_d)|, \quad a_2 = 2|E_1|H(\nu_d)| e^{-\nu_d},$$

$$B_1 = 2\varkappa |E_2||H(\lambda_0)| e^{-2\lambda_0}, \quad B_2 = 2|E_2||H(\lambda_0)| e^{-\lambda_0}.$$

The ratio of the amplitudes

$$\frac{a_2}{a_1}=e^{-\nu_d},\ \frac{B_2}{B_1}=\frac{1}{\varkappa}e^{\lambda_0}$$

shows that the waves of the first type are developing mainly at the free boundary, while the waves of the second type are developing at the interface between two fluids. The expressions for the hydrodynamic lifting force P, the wave drag Q and the moment M are found from S. A. Chaplygin's formulas:

$$P=\frac{\Gamma}{\varrho}+\frac{1}{\varrho 2\pi}\int_0^\infty |H(\lambda)|^2 d\lambda-\frac{1}{\varrho 2\pi}\int_0^\infty E|H(\lambda)|^2 d\lambda, \tag{VI.79}$$

$$Q=-\frac{1}{\varrho}\nu_d E_1|H(\nu_d)|^2-\frac{1}{\varrho}\lambda_0 E_2|H(\lambda_0)|^2, \tag{VI.80}$$

$$M=-\frac{1}{\varrho}\mathrm{Re}\,[iH(0)]+\frac{1}{\varrho}\mathrm{Re}\left[\frac{i}{\pi}\int_0^\infty\left(E-\frac{1}{2}\right)H'(\lambda)\times\right. \tag{VI.81}$$

$$\left.\times\overline{H(\lambda)}\,d\lambda+\nu_d E_1 H'(\nu_d)\overline{H(\nu_d)}+\lambda_0 E_2 H'(\lambda_0)H(\lambda)\right],$$

where

$$\Gamma=\int_C v_2(z)\,dz.$$

The transition to dimensional quantities is made by using the formulas

$$P'=\varrho_1 g d^2\overline{p},\ Q=\varrho_1 g d^2\overline{Q},\ M=\varrho_1 g d^3\overline{M}.$$

V. S. Voytsenya has also suggested an integral equation for determining the unknown density of distribution of sources along the hydrofoil profile. This equation is obtained in the same way as that derived in the Kochin solution (see Ch. II).

This equation has the following form:

$$\gamma(s)=-\int_C K(s,\sigma)\,\gamma(\sigma)\,d\sigma+f(s), \tag{VI.82}$$

$$K(s,\sigma)=\frac{1}{\pi}\mathrm{Re}\left\{\frac{e^{iv}}{t-\zeta}+\frac{e^{iv}}{t-\overline{\zeta}}-2ie^{iv}\int_0^\infty Ee^{-i\lambda(t-\overline{\zeta})}d\lambda-\right.$$ [215

$$\left.-2\pi e^{iv}\left[\nu_d E_1 e^{-i\nu_d(t-\overline{\zeta})}+\lambda_0 E_2 e^{-i\lambda_0(t-\overline{\zeta})}\right]\right\}, \tag{VI.83}$$

$$f(s)=2\cos v+\frac{\Gamma}{\pi}\mathrm{Re}\left\{\frac{ie^{iv}}{t-z_0}-\frac{ie^{iv}}{t-\overline{z}_0}-2e^{iv}\int_0^\infty Ee^{-i\lambda[t-\overline{z}_0]}d\lambda+\right.$$

$$+ 2\pi i e^{i v}[\nu_d E_i e^{-i\nu_d(t-\bar{z}_0)} + \lambda_0 E_2 e^{-i\lambda_0(t-\bar{z}_0)}]\Big\}. \quad \text{(VI.84)}$$

where v is the angle between the outer normal to the contour C and the x axis.

Voytsenya proves that for the contours which satisfy the condition $\int_C |K_0(\xi, \sigma)|\, d\sigma < p < 1$ the solution of the equation may be sought in the form of (II.91) by the iteration method. Here $K_0(s, \sigma) = K(s, \sigma) - \frac{1}{L}$; s is the length of the contour C.

For a special case of a cylinder with circulation Γ, equations for determining forces are:

$$\bar{P} = \frac{1}{\bar{\varrho}}\left[\Gamma + \frac{\Gamma^2}{4\pi h} + \frac{\Gamma R^2}{2h^2} + \frac{\pi R^4}{2h^3} - \int_0^\infty (\Gamma + 2\pi R^2\lambda)\, E e^{-2\lambda h} d\lambda\right], \quad \text{(VI.85)}$$

$$\bar{Q} = + \frac{1}{\bar{\varrho}} \nu_d E_1 (\Gamma + 2\pi R^2 \nu_d)^2 e^{-2\nu_d h} + \frac{1}{\bar{\varrho}} \lambda_0 E_2 (\Gamma + 2\pi R^2 \lambda_0)^2 e^{-2\lambda_0 h}. \quad \text{(VI.86)}$$

$$\bar{M} = hR + \frac{2\pi R^2}{\bar{\varrho}} [\nu_d E_1 (\Gamma + 2\pi R^2 \nu_d) e^{-2\nu_d h} + \lambda_0 E_2 (\Gamma + 2\pi R^2 \lambda_0)\, e^{-2\lambda_0 h}]. \quad \text{(VI.87)}$$

These formulas were obtained in determining function H(λ) from the complex velocity of a cylinder moving in an infinite flow.

6.6. Motion of a System of Hydrofoils Near the Interface Between the Fluids of Different Densities

The problem of interaction of hydrofoils moving in the fluids of different densities is of great interest. This problem can be investigated by the methods discussed in Ch. IV. Given below is a study of the interaction of two hydrofoils, one of which is located above and the other below the interface between two fluids. This case is very convenient for determining the nature of interaction and is of great practical interest. More complex systems may be studied by means of the methods discussed in this section and also in Ch. IV. [216

Let us examine a steady forward motion of a system of hydrofoils of an arbitrary shape which consists of a hydrofoil located above an interface and a hydrofoil located below it. The solution of this problem will be similar to

that of the problem of interaction of hydrofoils below the free boundary.

Let us take a point z_j and draw two contours C_{j1} and C_{j2} in such a manner that point z_j would be outside the C_{j1} contour but inside the C_{j2} contour. Using Cauchy's integral for the doubly-connected area we obtain the expressions for the complex velocities:

$$\left.\begin{aligned} v_j(z) &= -v_0 + v_{1j}(z) + v_{2j}(z) \\ v_{1j}(z) &= \frac{1}{2\pi i}\int\limits_{C_{j_1}} \frac{v_j(\zeta)}{z-\zeta}\,d\zeta \\ v_{2j}(z) &= \frac{1}{2\pi i}\int\limits_{C_{j_2}} \frac{v_j(\zeta)}{z-\zeta}\,d\zeta \end{aligned}\right\}. \tag{VI.88}$$

Let us use a different presentation for function v_{2j}, which is analytical within the C_{j2} contour. From the expressions (VI.42), (VI.43), (VI.48) and (VI.49) we obtain for the complex velocity of two vortex sources:

$$\frac{d\omega_1(z)}{dz} = \frac{B}{2\pi i}\,\frac{1}{z-\zeta_1} + \frac{\bar{B}_1 a}{2\pi}\int\limits_0^\infty \frac{e^{i\lambda(z-\zeta_1)}\lambda - \nu}{\lambda - \bar{\nu}}\,d\lambda +$$

$$+ \bar{B}_1 i\bar{\nu}\,\frac{\bar{\varrho}}{1+\varrho}\,e^{i\bar{\nu}(z-\bar{\zeta})} - \frac{B_2}{\pi}\,\frac{1}{1+\bar{\varrho}}\int\limits_0^\infty e^{i\lambda(z-\zeta_2)}\,\frac{\lambda}{\lambda-\bar{\nu}}\,d\lambda + \frac{B_2}{1+\bar{\varrho}}\,i\bar{\nu}e^{i\bar{\nu}(z-\zeta_2)}, \tag{VI.89}$$

$$\frac{d\omega_2(z)}{dz} = \frac{B_2}{2\pi i}\,\frac{1}{z-\zeta_2} + \frac{\bar{B}_2 a}{2\pi}\int\limits_0^\infty e^{-i\lambda(z-\bar{\zeta}_2)}\,\frac{\lambda+\nu}{\lambda-\bar{\nu}}\,d\lambda +$$

$$+ \bar{B}_2\,\frac{i\bar{\nu}}{1+\bar{\varrho}}\,\bar{e}^{i\nu(z-\bar{\zeta}_2)} + \frac{B_1}{\pi}\,\frac{\bar{\varrho}}{1+\bar{\varrho}}\int\limits_0^\infty \frac{e^{-i\lambda(z-\zeta)}}{\lambda-\nu}\,d\lambda + B_1\frac{\bar{\varrho}}{1+\bar{\varrho}}\,i\bar{\nu}e^{-i\nu(z-\bar{\zeta}_1)}.$$

Introducing Kochin's function $H(\lambda)$ we will find from the expression (VI.89): [217

$$v_{21} = \frac{a}{2\pi}\int\limits_0^\infty e^{i\lambda z}\overline{H_{11}}(\lambda)\,\frac{\lambda-\nu}{\lambda-\bar{\nu}}\,d\lambda + i\,\frac{\bar{\varrho}}{1+\bar{\varrho}}\,\bar{\nu}e^{i\bar{\nu}z}\overline{H_{11}}(-\bar{\nu}) -$$

$$-\frac{1}{\pi(1+\bar{\varrho})}\int_0^\infty e^{i\lambda z}H_{21}(\lambda)\frac{\lambda}{\lambda-\bar{\nu}}d\lambda+\frac{i\bar{\nu}}{1+\bar{\varrho}}e^{i\bar{\nu}z}H_{21}(\bar{\nu}). \qquad \text{(VI.90)}$$

$$v_{23}(z)=\frac{a}{2\pi}\int_0^\infty e^{-i\lambda z}H_{22}(\lambda)\frac{\lambda+\nu}{\lambda-\bar{\nu}}d\lambda+\frac{i\bar{\nu}}{1+\bar{\varrho}}e^{-i\bar{\nu}z}\overline{H_{22}(\bar{\nu})}+$$

$$+\frac{\bar{\varrho}}{\pi(1+\bar{\varrho})}\int_0^\infty e^{-i\lambda z}H_{12}(-\lambda)\frac{\lambda}{\lambda-\bar{\nu}}d\lambda+\frac{i\bar{\nu}\bar{\varrho}}{1+\bar{\varrho}}e^{-i\bar{\nu}z}H_{12}(-\bar{\nu}), \qquad \text{(VI.91)}$$

where $H_{ij}(\lambda)=e^{i\lambda\varepsilon_{ij}}\int_C e^{-i\lambda l(z)}v_j(z)\,dz$ - the Kochin function.

For the lifting force and the wave drag of hydrofoils the following expressions are obtained in the usual way:

$$P_1=\varrho_1 v_0\Gamma_1-\frac{\varrho_1 a}{2\pi}\int_0^\infty|H_{11}(-\lambda)|^2\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda+\frac{\varrho_1}{\pi(1+\bar{\varrho})}\int_0^\infty[H'_{11}(-\lambda)H^{*}_{21}(\lambda)-$$

$$-H^{*}_{11}(-\lambda)H^{*}_{21}(\lambda)]\frac{\lambda}{\lambda-\bar{\nu}}d\lambda+\frac{\varrho_1}{1+\bar{\varrho}}\bar{\nu}[H'_{11}(-\bar{\nu})H^{*}_{21}(\nu)+$$

$$+H^{*}_{11}(-\bar{\nu})H'_{21}(\bar{\nu})], \qquad \text{(VI.92)}$$

$$Q_1=\bar{\varrho}\frac{\varrho_1}{1+\bar{\varrho}}\bar{\nu}|H_{11}(-\bar{\nu})|^2-\frac{\bar{\varrho}_1}{\pi(1+\bar{\varrho})}\int_0^\infty[H'_{11}(-\lambda)H^{*}_{21}(\lambda)+$$

$$+H^{*}_{11}(-\lambda)H'_{22}(\lambda)]\frac{\lambda}{\lambda-\bar{\nu}}d\lambda+\frac{\bar{\nu}\varrho_1}{1+\bar{\varrho}}[H'_{11}(-\bar{\nu})H'_{21}(\bar{\nu})-$$

$$-H^{*}_{11}(-\bar{\nu})H^{*}_{21}(\bar{\nu})], \qquad \text{(VI.93)}$$

$$P_2=\varrho_2 v_0\Gamma_2-\frac{\varrho_2 a}{2\pi}\int_0^\infty|H_{22}(\lambda)|^2\frac{\lambda+\nu}{\lambda-\bar{\nu}}d\lambda-\frac{\varrho_2\bar{\varrho}}{\pi(1+\bar{\varrho})}\int_0^\infty[H'_{22}(\lambda)H'_{12}(-\lambda)-$$

$$-H^{*}_{22}(\lambda)H^{*}_{12}(-\lambda)]\frac{\lambda}{\lambda-\bar{\nu}}d\lambda+$$

$$+\frac{\varrho_2\overline{\nu\varrho}}{1+\bar{\varrho}}[H'_{22}(\bar{\nu})H^{*}_{22}(-\bar{\nu})+H^{*}_{22}(\bar{\nu})H'_{12}(-\bar{\nu})], \qquad \text{(VI.94)}$$ [218

$$Q_2=\frac{\varrho_2}{1+\bar{\varrho}}\bar{\nu}|H_2(\bar{\nu})|^2+\frac{\varrho_2\bar{\varrho}}{\pi(1+\bar{\varrho})}\int_0^\infty[H'_{22}(\lambda)H^{*}_{12}(-\lambda)+H^{*}_{22}(\lambda)H'_{12}(-\lambda)]\times$$

$$\times\frac{\lambda}{\lambda-\bar{\nu}}d\lambda+\bar{\nu}\varrho_2\frac{\bar{\varrho}}{1+\bar{\varrho}}[H'_{22}(\bar{\nu})H'_{12}(-\bar{\nu})-H^{*}_{22}(\bar{\nu})H^{*}_{12}(-\bar{\nu})]. \qquad \text{(VI.95)}$$

With $\bar{\varrho} = 1$ formulas (VI.92) and (VI.94) will give forces acting on the foils of a biplane in an infinite flow.

For identical hydrofoils without a stagger

$$H_{21}(\lambda) = \overline{H}_1(-\lambda), \qquad H_{12}(-\lambda) = \overline{H}_2(\lambda),$$

$$P_1 = \varrho v_0 \Gamma + \frac{\varrho}{2\pi}\int_0^\infty |H_2(\lambda)|^2 d\lambda, \quad P_2 = \varrho v_0 \Gamma_2 - \frac{\varrho}{2\pi}\int_0^\infty |H_2(\lambda)|^2 d\lambda. \tag{VI.96}$$

It follows from formulas (VI.96) that the lifting forces for the upper and lower hydrofoils are different. In the same way as in Chapters II and IV the circulation Γ_j present in formulas (VI.92)-(VI.95) are values which are not known beforehand. To determine these values it is necessary to determine the functions $v_{1j}(z)$ which satisfy the boundary conditions along the C_j contour. Let us map conformally the shape of the C_j contour on the shape of a circle with a radius R_j so that the infinitely distant point z_j would become point u_j and $\left(\frac{du_j}{dz_j}\right)_{z_j\to\infty} = 1$. Then, the boundary condition on the contour may be written in the form of a boundary condition of the problem for determining the function analytical outside the circle in terms of its real part on the circle.

Using the Schwarz formula we obtain after transformations

$$\begin{aligned} v_{11}(z) = \frac{du_1}{dz}\Bigg\{ & -v_0 + \frac{\bar{v}_0 R_1^2}{u_1^2} + \frac{Gi}{u_1} + \frac{a}{4\pi^2 i}\int_0^\infty [G_1(-\lambda, u)\,\overline{H_1(-\lambda)} + \\ & + \frac{R_1^2}{u_1^2} G_1\left(\overline{-\lambda, \frac{R_1^2}{\bar{u}_1}}\right) H_1(-\lambda)\Big]\frac{\lambda - \nu}{\lambda - \bar{\nu}} d\lambda - \frac{1}{2\pi^2 i(1+\bar{\varrho})}\int_0^\infty \Big[G_1(-\lambda, u_1) H_{21}(\lambda) + \\ & + \frac{R_1^2}{u_1^2} G_1\left(\overline{-\lambda, \frac{R^2}{\bar{u}_1}}\right)\overline{H_{21}(\lambda)}\Big]\frac{\lambda}{\lambda - \bar{\nu}} d\lambda + \frac{\nu}{2\pi(1+\bar{\varrho})}\Big\{\bar{\varrho}\Big[G_1(-\bar{\nu}, u_1) H_1\overline{(-\nu)} - \\ & - \frac{R_1^2}{u_1^2} G_1\left(\overline{-\bar{\nu}, \frac{R^2}{\bar{u}_1}}\right) H_1(-\bar{\nu})\Big] + G_1(-\bar{\nu}, u_1) H_{21}(\bar{\nu}) - \\ & - \frac{R^2}{u_1^2} G_1\left(\overline{-\bar{\nu}, \frac{R^2}{\bar{u}_1}}\right)\overline{H_{21}(\bar{\nu})}\Big\}\Bigg\}. \end{aligned} \tag{VI.97}$$ [219

$$v_{12}(z)=\frac{du_2}{dz}\left\{-v_0+\frac{\bar{v}_0R_2^2}{u_2^2}+\frac{C_2i}{u_2}+\frac{a}{4\pi^2 i}\int_0^\infty\left[G_2(\lambda,u_2)\bar{H}_2(\lambda)+\right.\right.$$

$$\left.+\frac{R_2^2}{u_2}\overline{G_2\left(\lambda,\frac{R^2}{\bar{u}_2}\right)}H_2(\lambda)\right]\frac{\lambda+\nu}{\lambda-\bar{\nu}}d\lambda+\frac{1}{2\pi^2}\frac{\bar{\varrho}}{1+\bar{\varrho}}\int_0^\infty\left[G_2(\lambda,u_2)H_{12}(-\lambda)+\right.$$

$$\left.+\frac{R^2}{u_2}\overline{G_2\left(\lambda,\frac{R_2^2}{\bar{u}_2}\right)}\overline{H_{12}(-\lambda)}\right]\frac{\lambda}{\lambda-\bar{\nu}}d\lambda+\frac{\bar{\nu}}{2\pi(1+\bar{\varrho})}\left\{G_2(\bar{\nu},u)\bar{H}_2(\bar{\nu})-\right.$$

$$-\frac{R_2^2}{u_2^2}\overline{G_2\left(\bar{\nu},\frac{R_2^2}{\bar{u}_2}\right)}H_2(\bar{\nu})+\bar{\varrho}\left[G_2(\bar{\nu},u_2)H_{12}(-\bar{\nu})-\right.$$

$$\left.\left.\left.-\frac{R_2^2}{u_2^2}\overline{G_2\left(\bar{\nu},\frac{R_2^2}{\bar{u}_2}\right)}(\overline{H_{12}(-\nu)}\right]\right\}\right\}, \qquad \text{(VI.98)}$$

where $G_l(\lambda,u_l)=\int\limits_{k_l}\frac{e^{-i\lambda z_l}}{\sigma_l-u_l}dz_l.$

Let us examine a special case of hydrofoils obtained with the aid of Zhukovskiy's function. The function $G(\lambda, u)$ in this case will be determined from formula (II.35). From the expressions (VI.97) and (VI.98) it follows:

$$v_{11}(z)=\frac{du_1}{dz_1}\left\{-\left[v_0+\frac{a}{2\pi}\int_0^\infty e^{-\lambda h_1}H_1(-\lambda)I_0(2\lambda R_1)\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda-\right.\right.$$

$$-\frac{1}{\pi(1+\bar{\varrho})}\int_0^\infty e^{-\lambda h_1}\overline{H_{21}(\lambda)}J_0(2\lambda R)\frac{\lambda}{\lambda-\bar{\nu}}d\lambda-\frac{\nu i}{1+\bar{\varrho}}e^{-\bar{\nu}h_1}J_0(2\bar{\nu}R)\times$$

$$\left.\times(H_1(-\bar{\nu})\varrho+\bar{H}_{21}(\bar{\nu}))\right]+\frac{R_1^2}{u_1^2}\left[\bar{v}_0+\frac{a}{2\pi}\int_0^\infty e^{-\lambda h_1}\overline{H_1(-\lambda)}I_0(2\lambda R_1)\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda-\right.$$

$$-\frac{1}{\pi(1+\bar{\varrho})}\int_0^\infty e^{-\lambda h_1}J_0(2\lambda R_1)H_{21}(\lambda)\frac{\lambda}{\lambda-\bar{\nu}}d\lambda+\frac{\bar{\nu}i}{1+\bar{\varrho}}e^{-\bar{\nu}h_1}J_0(2\bar{\nu}R_1)\times \quad [220$$

$$\left.\left.+(\varrho_1H_1(-\bar{\nu})+H_{21}(\bar{\nu}))\right]+\frac{\Gamma}{2\pi i u_1}\right\}+\frac{a}{4\pi^2 i}\int_0^\infty\left[e^{-\lambda h_1}B_1(-\lambda,u)\bar{H}_1(-\lambda)+\right.$$

$$\left.\frac{R^2}{u_1^2}\overline{B_1\left(-\lambda,\frac{R^2}{\bar{u}_1}\right)}H_1(-\lambda)\frac{du_1}{d\frac{R_1^2}{u_1}}\right]\frac{\lambda-\nu}{\lambda-\bar{\nu}}d\lambda+$$

$$+\frac{1}{2\pi^2 i(1+\bar{\varrho})}\int_0^\infty e^{-\lambda h_1}\Big[B_1(-\lambda, u_1)\,H_{21}\omega_0 +$$

$$+\frac{R^2}{u_1^2}B_1(-\lambda)\,\frac{R_1^2}{\bar{u}_1}\,\bar{H}_{21}(\lambda)\,\frac{du}{d\dfrac{R_1^2}{u}}\Big]\frac{\lambda}{\lambda-\nu}\,d\lambda +$$

$$+\frac{\bar{\nu}}{2\pi(1+\bar{\varrho})}\Big\{\bar{\varrho}\Big[B_1(-\bar{\nu}, u)\,\overline{H_1(-\nu)}-\frac{R_1^2}{u_1^2}B_1\Big(\overline{-\bar{\nu}, \frac{R_1^2}{\bar{u}_1}}\Big)\times$$

$$\times H_1(-\bar{\nu})\,\frac{du_1}{d\dfrac{R_1^2}{u_1}}\Big]+B_1(-\bar{\nu}, u_1)\,H_{21}(\bar{\nu})-\frac{R_1^2}{u_1^2}B_1\Big(\overline{-\bar{\nu}, \frac{R_1^2}{\bar{u}_1}}\Big)\overline{H_{21}(\bar{\nu})}\Big\}, \qquad \text{(VI.99)}$$

$$v_{12}(z)=\frac{du_2}{dz}\Big\{-\Big\{v_0+\frac{a}{2\pi}\int_0^\infty e^{-\lambda h_2}J_0(2\lambda R_2)\,H_2(\lambda)\,\frac{\lambda+\nu}{\lambda-\bar{\nu}}\,d\lambda +$$

$$+\frac{\bar{\varrho}}{1+\bar{\varrho}}\,\frac{1}{\pi}\int_0^\infty e^{-\lambda h_2}J_0(2\lambda R_2)\,\overline{H_{12}(-\lambda)}\,\frac{\lambda}{\lambda-\bar{\nu}}\,d\lambda-\frac{\bar{\nu}i}{1+\varrho}\,e^{-\bar{\nu}h_2}J_0(2\bar{\nu}R_2)\times$$

$$\times[\bar{\varrho}\overline{H_{12}(-\bar{\nu})}+H_2(\bar{\nu})]\Big\}+\frac{R_2^2}{u_2^2}\Big\{\bar{v}_0+\frac{a}{2\pi}\int_0^\infty e^{-\lambda h_2}J_0(2\bar{\nu}R_2)\,\overline{H_2(\lambda)}\,\frac{\lambda+\nu}{\lambda-\bar{\nu}}\,d\lambda +$$

$$+\frac{\bar{\varrho}}{(1+\bar{\varrho})\pi}\int_0^\infty e^{-\lambda h_2}J_0(2\lambda R_2)\,H_{12}(-\lambda)\,\frac{\lambda}{\lambda-\bar{\nu}}\,d\lambda +$$

$$+\frac{\nu i}{1+\bar{\varrho}}\,e^{-\nu h_2}J_0(2\bar{\nu}R_2)[\bar{\varrho}H_{12}(-\bar{\nu})+\overline{H_{12}(\bar{\nu})}]\Big\}+\frac{\Gamma_2}{2\pi i u_2}\Big\}+$$ [221

$$+\frac{a}{4\pi^2 i}\int_0^\infty\Big[e^{-\lambda h_2}B_2(\lambda, u_2)\,\overline{H_2(\lambda)}+\frac{R_2^2}{u_2^2}B_2\Big(\overline{\lambda, \frac{R_2^2}{\bar{u}_2}}\Big)H_2(\lambda)\,\frac{du}{d\dfrac{R_2^2}{u_2}}\Big]\frac{\lambda+\nu}{\lambda-\bar{\nu}}\,d\lambda +$$

$$+\frac{1}{2\pi^2 i}\,\frac{\bar{\varrho}}{1+\bar{\varrho}}\int_0^\infty e^{-\lambda h_2}\Big[B_2(\lambda, u_2)\,H_{12}(-\lambda)+\frac{R_2^2}{u_2^2}B_2\Big(\overline{\lambda, \frac{R_2^2}{\bar{u}_2}}\Big)H_{12}(-\lambda)\Big]\times$$

$$\times\frac{\lambda}{\lambda-\bar{\nu}}\,d\lambda+\frac{\bar{\nu}e^{-\bar{\nu}h_2}}{2\pi(1+\bar{\varrho})}\Big[B_2(\bar{\nu})\,\bar{H}_2(\bar{\nu})-\frac{R_2^2}{u_2^2}B_2\Big(\overline{\bar{\nu}, \frac{R_2^2}{\bar{u}_2}}\Big)\Big]H_2(\bar{\nu})\,\frac{du}{d\dfrac{R^2}{u_2}}+$$

$$+\bar{\varrho}\Big(B_2(\bar{\nu})\,H_{12}(-\bar{\nu})-\frac{R_2^2}{u_2^2}B_2\Big(\bar{\nu}, \frac{R_2^2}{u_2}\Big)H_{12}(-\bar{\nu})\,\frac{du}{d\dfrac{R^2}{\bar{u}}}\Big)\Big]. \qquad \text{(VI.100)}$$

Let us assume that the conditions of the Zhukovskiy-Chaplygin postulate are satisfied at the point $u_{0j} = -R_j$. Then, from the expressions (VI.99) and (VI.100) we obtain a system for determining the circulation:

$$\Gamma_1 = 4\pi R_1 J_m \left\{ v_0 + \frac{a}{2\pi} \int_0^\infty e^{-\lambda h_1} H_1(-\lambda) J_0(2\lambda R_1) \frac{\lambda - \nu}{\lambda - \bar{\nu}} d\lambda - \right.$$

$$- \frac{1}{\pi(1+\bar{\varrho})} \int_0^\infty e^{-\lambda h_1} \bar{H}_{21}(\lambda) J_0(2\lambda R_1) \frac{\lambda}{\lambda - \bar{\nu}} d\lambda -$$

$$\left. - \frac{i\bar{\nu} e^{-\bar{\nu} h_1}}{1+\bar{\varrho}} J_0(2\bar{\nu} R_1) [\bar{\varrho} H_1(-\bar{\nu}) + \overline{H_{21}(\bar{\nu})}] \right\}, \qquad \text{(VI.101)}$$

$$\Gamma_2 = 4\pi R_2 J_m \left\{ v_0 + \frac{a}{2\pi} \int_0^\infty e^{-\lambda h_2} H_2(\lambda) J_0(2\lambda R_2) \frac{\lambda + \nu}{\lambda - \bar{\nu}} d\lambda + \frac{\varrho}{\pi(1+\bar{\varrho})} \int_0^\infty e^{-\lambda h_2} \times \right.$$

$$\left. \times J_0(2\lambda R_2) \bar{H}_{12}(-\lambda) \frac{\lambda d\lambda}{\lambda - \bar{\nu}} - \frac{i\bar{\nu}}{1+\varrho} e^{-\bar{\nu} h_2} J_0(2\bar{\nu} R_2) [\varrho \bar{H}_{12}(-\bar{\nu}) + H_2(\bar{\nu})] \right\}. \qquad \text{(VI.102)}$$

To determine the nature of interaction and to obtain the finite results for a simple special case, let us examine the motion of a biplane without a stagger.

We will determine the function $H_{ij}(\lambda)$ from the complex velocity of the hydrofoil motion in an infinite flow: [222

$$H_{11}(-\lambda) = e^{-\lambda h_1} \Gamma_{1\infty} [J_0(2\lambda R_1) + iJ_1(2\lambda R_1)],$$

$$H_{22}(\lambda) = e^{-\lambda h_2} \Gamma_{2\infty} [J_0(2\lambda R_2) - iJ_1(2\lambda R_2)],$$

$$H_{12}(-\lambda) = H_{11}(-\lambda), \quad H_{21}(\lambda) = H_{22}(\lambda).$$

After transforming we obtain

$$P_1 = \varrho_1 v_0 \Gamma_1 - \varrho \frac{\Gamma_{1\infty}^2}{2\pi R_1 v_0} a \left\{ A_{00}(R_1) + A_{11}(R_1) - \frac{\omega_1}{4}\left(1 - \frac{1}{a}\right)\left[B_{00}(R_1) + \right.\right.$$

$$\left.\left. + B_{11}(R_1) \right] \right\} + \frac{\varrho_1 \Gamma_{1\infty} \Gamma_{2\infty}}{\pi(1+\bar{\varrho}) v_0} (D_{00} + D_{11}) + \frac{\varrho_1}{1+\bar{\varrho}} \frac{\Gamma_{1\infty} \Gamma_{2\infty}}{v_0} \bar{\nu} e^{-(h_1+h_2)\bar{\nu}} \times$$

$$\times [-J_0(2\bar{\nu} R_1) J_1(2\nu R_2) + J_1(2\bar{\nu} R_1) J_0(2\bar{\nu} R_2)], \qquad \text{(VI.103)}$$

$$Q_1 = \frac{\bar{\varrho}\varrho_1}{1+\bar{\varrho}} \Gamma_{1\infty}^2 \bar{\nu} e^{-2\bar{\nu} h_1} [J_0^2(2\bar{\nu} R_1) + J_1^2(2\bar{\nu} R_1)] - \frac{\bar{\varrho}}{1+\bar{\varrho}} \Gamma_{1\infty} \Gamma_{2\infty} \{\pi(-D_{01} +$$

$$+ D_{10}) + \bar{\nu} e^{-\bar{\nu}(h_1+h_2)} [J_0(2\bar{\nu} R_1) J_0(2\bar{\nu} R_2) + J_1(2\bar{\nu} R_1) J_1(2\bar{\nu} R_2)] \}, \qquad \text{(VI.104)}$$

$$P_1 = \varrho_2 v_0 \Gamma_2 - \frac{\varrho\Gamma^2_{2\infty} a}{2\pi R v_0}\Big\{A_{00}(R_2) + A_{11}(R_2) - \frac{\omega_2}{4}\left(\frac{1}{a}+1\right)[B_{00}(R_1) +$$

$$+ B_{11}(R_1)]\Big\} - \frac{\varrho_2\bar{\varrho}}{\pi(1+\bar{\varrho})}\frac{\Gamma_{1\infty}\Gamma_{2\infty}}{v_0}(D_{00}+D_{11}) + \frac{\varrho_2\bar{\varrho}}{1+\bar{\varrho}}\frac{\Gamma_{1\infty}\Gamma_{2\infty}}{v_0}\bar{v}e^{-(h_1+h_2)\bar{v}}\times$$

$$\times [J_0(2\bar{v}R_2)J_1(2\bar{v}R_1) - J_1(2\bar{v}R_2)J_0(2\bar{v}R_1)], \qquad \text{(VI.105)}$$

$$Q_2 = \frac{\varrho_2}{1+\bar{\varrho}}\Gamma_2^2\bar{v}e^{-2\bar{v}h_1}[J_0^2(2\bar{v}R_2)+J_1^2(2\bar{v}R_2)] + \frac{\varrho_2\bar{\varrho}}{1+\bar{\varrho}}\Gamma_{1\infty}\Gamma_{2\infty}\{\pi(D_{10} -$$

$$-D_{01}) + \bar{v}e^{-\bar{v}(h_1+h_2)}[J_0(2\bar{v}R_1)J_0(2\bar{v}R_2) + J_1(2\bar{v}R_1)J_1(2\bar{v}R_2)]\}, \qquad \text{(VI.106)}$$

$$\Gamma_1 = 4\pi R_1\Big\{\sin\alpha_1 + \frac{\Gamma_{1\infty}}{R_1 v_0}\Big[\frac{a}{2\pi}\Big(A_{01} - \frac{\omega}{4}\left(1-\frac{1}{a}\right)B_{01}\Big) - \frac{\pi\bar{\omega}_1}{4}J_0^2\left(\frac{\bar{\omega}_1}{2}\right)\times$$

$$\times e^{-2\bar{\omega}_1\bar{h}_{11}}\frac{\bar{\varrho}}{1+\bar{\varrho}}\Big] - \frac{\Gamma_{2\infty}}{(1+\bar{\varrho})v_0}\Big[\frac{1}{\pi}D_{01} + \frac{\bar{\omega}_1}{4R_1}J_0\left(\frac{\bar{\omega}_1}{2}\right)J_0\left(\frac{\bar{\omega}_2}{2}\right)e^{-\bar{\omega}_1(\bar{h}_{11}+\bar{h}_{21})}\Big]\Big\}, \qquad \text{(VI.107)}$$

$$\Gamma_2 = 4\pi R_2\Big\{\sin\alpha_2 - \frac{\Gamma_{2\infty}}{R_2 v_0}\Big[\frac{a}{2\pi}\Big(A_{01} - \frac{\bar{\omega}_2}{4}\left(\frac{1}{a}+1\right)B_{01}\Big] -$$ [223

$$-\frac{1}{4(1+\bar{\varrho})}\bar{\omega}_2 J_0^2\left(\frac{\bar{\omega}_2}{2}\right)e^{-2\bar{\omega}_2\bar{h}_{22}}\Big\}, \quad \bar{h}_{il} = \frac{h_i}{4R_l}, \quad \bar{\omega}_l = 4\bar{v}R_l, \qquad \text{(VI.108)}$$

$$\bar{A}_{nm}(R_l) = R_l\int_0^\infty e^{-2h_l\lambda}J_n(2\lambda R_l)J_m(2\lambda R_l)\,d\lambda,$$

$$B_{nm}(R_l) = \int_{-\infty}^{1} e^{-2\bar{v}(1-\lambda)h_l}\frac{J_n[2\bar{v}(1-\lambda)R_l]J_m[2\bar{v}(1-\lambda)R_l]}{\lambda}\,d\lambda,$$

$$D_{nm} = \int_0^\infty e^{-\lambda(h_1+h_2)}J_n(2\lambda R_1)J_m(2\lambda R_2)\frac{\lambda}{\lambda-\bar{v}}\,d\lambda,$$

$J_n(x)$ is the Bessel function.

The functions A_{nm} and B_{nm} may be presented in the form of a series in powers of the parameter

$$\tau_l = \sqrt{4\bar{h}_l^2+1} - 2\bar{h}_l.$$

The combination $A_{nm} - \frac{\bar{\omega}a_l}{4a}B_{nm}$ will be determined by the formulas

$$\bar{A}_{00} - \frac{\bar{\omega}_l}{4}\frac{a_l}{a}B_{00} = \frac{1}{a}\sum_{n=0}^{\infty}\tau_l^{2n+1}\sum_{s=0}^{n}\frac{(-1)^s!\,2s!\,(s+n)!}{s!\,s!\,s!\,s!\,(n-s)!\,2^{2s+1}}\times$$

$$\times\left[a + a_l\,\mathrm{Re}\,F_{n+s}\left(\frac{\bar{\omega}_l}{2\tau_l}\right)\right]$$

$$
\left.
\begin{aligned}
&\bar{A}_{01}-\frac{\omega_j}{4}\frac{a_j}{a}B_{01}=\frac{1}{a}\sum_{n=0}^{\infty}\tau_j^{2n+2}\sum_{s=0}^{n}\frac{(-1)^s(2s+1)(s+n+1)!}{s!s!(s+1)!(s+1)!(n-s)!2^{2s+g}}\times\\
&\qquad\times\left[a+a_j\operatorname{Re}F_{n+s+1}\left(\frac{\omega_j}{2\tau_j}\right)\right]\\
&\bar{A}_{11}-\frac{\bar{\omega}_j}{4}\frac{a_j}{a}B_{11}=\frac{1}{a}\sum_{n=0}^{\infty}\tau_j^{2n+3}\sum_{s=0}^{n}\frac{(-1)^s(2s+2)!(s+n+2)!}{s!(s+1)!(s+2)!(n-s)!2^{2s+3}}\times\\
&\qquad\times\left[a+a_j\operatorname{Re}F_{n+s+2}\left(\frac{\bar{\omega}_j}{2\tau_j}\right)\right]\quad a_1=1+a,\quad a_2=a-1
\end{aligned}
\right\}\qquad\text{(VI.109)}
$$

When $R_1 \sim R_2$, functions D_{nm} can be determined in the form of a series in powers of parameter τ. [224

With $P_1 \sim P_2\ \Gamma_1 \sim \frac{\Gamma_2}{\bar{\varrho}}$ and then

$$
\begin{aligned}
&P_1=P_{10}+O(\bar{\varrho}),\ Q_1=Q_{10}+O(\bar{\varrho}),\ \Gamma_1=\Gamma_{10}+O(\bar{\varrho}),\\
&P_2=P_{20}+O(1),\ Q_2=Q_{20}+O(1),\ \Gamma_2=\Gamma_{20}+O(1),
\end{aligned}
\qquad\text{(VI.110)}
$$

where P_{j0}, Q_{j0} and Γ_{j0} are the lifting force, wave drag and circulation, respectively, for a single hydrofoil moving near the interface.

Thus, the effect of the lower hydrofoil on the hydromechanical characteristics of the upper hydrofoil is dedetermined by a value of the order of $\bar{\varrho}$, while the effect of the upper hydrofoil on the lower is determined by a value close to unity. For hydrofoils moving near the air/water interface $\bar{\varrho} = \frac{1}{800}$ and the effect of the lower hydrofoil is so negligible that it can be disregarded. The upper hydrofoil, however, changes the characteristics of the lower hydrofoil and this effect should be taken into account.

It is not difficult to show that with an accuracy involving the terms of the first order of smallness for the angle of attack, the lifting force of the hydrofoils will be determined by Zhukovskiy's formula. In such a case the principal effect of the interface will be governed by the change in circulation.

With $\omega_j \to 0$, from formulas (VI.107) and (VI.108) we obtain:

$$\left.\begin{aligned}\Gamma_1 &= 4\pi R_1\left[\sin\alpha_1 + 2a\sin\alpha_1\bar{A}_{01}(R_1) - \frac{\Gamma_{2\infty}}{(1+\bar{\varrho})\pi w_0}A_{01}(R_1R_2)\right]\\ \Gamma_2 &= 4\pi R_2\left[\sin\alpha_2 - 2a\sin\alpha_2\bar{A}_{01}(R_2) - \frac{\Gamma_{1\infty}\bar{\varrho}}{(1+\bar{\varrho})\pi w_0}A_{01}(R_1R_2)\right]\\ A_{nm}(R_1R_2) &= \int_0^\infty e^{-2h_{cp}\lambda}J_n(2\lambda R_2)J_m(2\lambda R_2)\,d\lambda\end{aligned}\right\}. \quad \text{(VI.111)}$$

In conclusion, let us present the formulas for determining $A_{nm}(R_1R_2)$. Using the presentations of functions A_{nm} through the hypergeometrical functions one may prove that the following relationships are valid:

$$\begin{aligned}&\bar{A}_{10}(R_1R_2) + k\bar{A}_{01}(R_1R_2) = [1 - 4h_{cp}\bar{A}_{00}(R_1R_2)],\\ &\bar{A}_{nm}(R_1R_2) = 2R_2A_{nm}(R_1R_2),\ \bar{h}_{cp} = \frac{h_{cp}}{4R_2},\ k = \frac{R_1}{R_2}.\end{aligned} \quad \text{(VI.112)}$$

Function $\bar{A}_{00}(R_1R_2)$ will be determined from the formula [225

$$\bar{A}_{00}(R_1, R_2) = \frac{\sqrt{x}}{\sqrt{2k}}F\left(\frac{1}{4};\ \frac{3}{4};\ 1;\ x^2\right),\ x = \frac{2k}{16\bar{h}^2+1+k}. \quad \text{(VI.113)}$$

Function $\bar{A}_{01}(R_1R_2)$ may be determined by the expansion

$$\begin{aligned}&\bar{A}_{01}(R_1, R_2) = \sum_{p=0}^{\infty}\frac{(-1)^p k^{1+2p}}{p!\,(p+1)!\,2^{1+2p}}J_{2p},\\ &J_{2p} = \int_0^\infty e^{-4\bar{h}_{cp}\lambda}J_0(\lambda)^{1+2p}d\lambda,\ J_0 = \frac{4\bar{h}}{(16\bar{h}^2+1)^{\frac{3}{2}}}.\end{aligned} \quad \text{(VI.114)}$$

With small values of k, function $\bar{A}_{01}(R_1R_2)$ may be determined by the first term only:

$$\bar{A}_{01}(R_1R_2) = k\frac{2\bar{h}}{(16\bar{h}^2+1)^{\frac{3}{2}}} + \theta(k^3). \quad \text{(VI.115)}$$

A_{10} will be determined from the relation (VI.112)

For the motion near air and water, formulas (VI.111) may be rewritten (retaining in the expansions A_{nm} one term only) in the following form:

$$\gamma_1 = \frac{\Gamma_1}{\Gamma_{1\infty}} = 1 + \frac{1}{2}\tau_1^2,$$
$$\gamma_2 = \frac{\Gamma_2}{\Gamma_{2\infty}} = 1 - \frac{1}{2}\tau_2^2 - 2\overline{\Gamma}\overline{A}_{10}(R_1, R_2), \quad \overline{\Gamma} = \frac{\Gamma_{1\infty}}{\Gamma_{2\infty}}\overline{\varrho}. \qquad \text{(VI.116)}$$

With $k = 1$ $\overline{A}_{10} \approx \frac{1}{2}\tau_3^2$, $\tau_3 = \sqrt{4\overline{h}_{cp}^2 + 1} - 2\overline{h}_{cp}$. [226

Then γ_2 will be determined by the formula

$$\gamma_2 = 1 - \frac{1}{2}\tau_2^2 - \overline{\Gamma}\tau_3^2. \qquad \text{(VI.117)}$$

It follows from the expressions (VI.116) and (VI.117) that, under the influence of the upper hydrofoil, the decrease of the hydrofoil lifting force under the interface will be very considerable. The curves showing the dependence of functions γ_2 on H for various values of $\overline{\Gamma}$ are given in Fig. 14.

[225

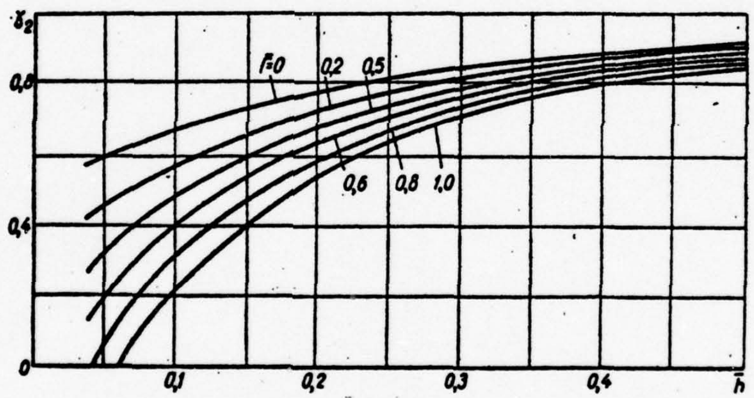

Fig. 14

PART TWO [227

THE THEORY OF THE SUBMERGED HYDROFOIL IN A THREE-DIMENSIONAL FLOW

CHAPTER VII. BASIC CONSIDERATIONS IN THE THEORY OF HYDROFOILS WITH FINITE SPAN [229

7.1. General Considerations. Methods of Studying the Three-Dimensional Motion of Lifting Surfaces

The first part of this monograph considered the theory of the submerged hydrofoil in the plane-parallel flow. It was based on the theory of complex functions. In this part the three-dimensional motion of lifting surfaces will be analyzed. The basic methods for studying it are the methods of the potential flow theory.

For the aerodynamic studies of an airplane wing of finite span moving in an incompressible fluid the most prevalent method used is that which is based on the concept that the lifting surface is formed by a system of entrained and trailing vortices and on the use of the lifting line theory.

However, in the case of a submerged hydrofoil this method leads very seldom to the desired goal. In order to use it a clear and a distinct idea of the physical picture of motion is necessary which will permit the transition from the real lifting surface to a model that represents the combined vortex phenomenon in the flow. A clear physical picture for the submerged hydrofoil may possibly be constructed only for the limiting degenerate cases of motion ($Fr \to 0$ and $Fr \to \infty$). For this case the linear boundary conditions at the free surface are as follows:

$$\frac{\partial \varphi}{\partial y} = 0 \ |Fr \to 0|, \quad \frac{\partial \varphi}{\partial x} = 0 \ |Fr \to \infty|,$$

These conditions can apparently be satisfied by a biplane system consisting of a real and an imaginary wing, with different ($Fr \to 0$) and identical ($Fr \to \infty$) directions of circulation. This method can be used to construct a vortex model for a biplane. However, for the general case of motion it is impossible to find such a model and the lifting line theory becomes inapplicable. Therefore, to study [230
space problems it becomes necessary to use the methods which are based directly on the hydrodynamic equations.

The very convenient methods for investigating such problems are the methods of acceleration potential. Initally these methods were used in the study of the unsteady motion of a finite-span wing by Kussner [92]. They proved to be useful and were subsequently employed in a series of problems dealing with the unsteady motion and the motion in compressible fluid.

T. Nishiyama [213, 214], U. Vu [195, 196], M. D. Khaskind [155] and A. M. Panchenkov [110] use the methods of acceleration potential in the hydrodynamic studies of the hydrofoil. The closest to the acceleration potential method is the method which is based on the representation of the lifting surface and the accompanying zone behind it by a semi-infinite surface with dipoles distributed over it.

The problem based on this concept for the investigation of the hydrofoil has been treated by G. A. Goshev [21, 22]. It will be shown below that this type of physical picture will, from the point of view of the acceleration potential, determine the flow produced by the motion of a lifting line with the span equal to that of the foil.

Before discussing the study of the hydrofoil motion let us clarify the basic aspects of the acceleration potential method by considering a more simple problem of wing motion in an infinite flow.

Let us introduce the concept of velocity potential for an infinite liquid.

The vector form of the hydrodynamic equation in the fixed coordinate system is as follows:

$$\frac{d\bar{v}}{dt} = -\frac{1}{\varrho}\nabla p,$$

i.e., the vector field of the accelerations $\frac{d\vec{v}}{dt}$ contains a potential θ.

The relation between the velocity potential φ and the acceleration potential θ in the linear approximation is established by the following relationship:

$$\Theta = \varphi_t - v_0\varphi_x = \varphi_{1x}. \qquad \text{(VII.1)}$$

For the steady motion the acceleration potential will be determined from the relation

$$\Theta(x, y, z) = -v_0\varphi_x. \qquad \text{(VII.2)}$$

The integral in equation (VII.2), which converges to zero when x → ∞, will be as follows:

$$\varphi = -\frac{1}{v_0}\int_{\infty}^{x} \Theta(\tau, y, z)\, d\tau. \qquad \text{(VII.3)}$$

For the unsteady periodic motion, the acceleration potential may be written [231

$$\Theta(x, y, z, t) = \bar{Q}(x, y, z)e^{i\omega t},$$

and for determining $\bar{Q}(x, y, z)$ the following differential equation is obtained from the expression (VII.1):

$$Q(x, y, z) = v_0(-\varphi_x + ip\varphi);\; p = \frac{\omega}{v_0}. \qquad \text{(VII.4)}$$

The integral form in equation (VII.4), which converges to zero when x = ∞, will be as follows:

$$\varphi = -\frac{e^{ipx}}{v_0}\int_{\infty}^{x} \bar{Q}(\tau, y, z)e^{-ip\tau}\, d\tau. \qquad \text{(VII.5)}$$

If the small values of the second order are disregarded then the pressure in the flow will be determined by the formula (24) as follows:

$$p = -\varrho\,\varphi_{1t}.$$

Equating this equation to that of (VII.1) we obtain the following expression:

$$p = -\varrho\Theta. \qquad \text{(VII.6)}$$

From this it follows that the pressure in the flow is proportional to the acceleration potential and, therefore, the acceleration potential will have a discontinuity at the intersection with the lifting surface S and remain continuous at the intersection with the semi-infinite velocity discontinuity surface Σ formed behind surface S.

Precisely this characteristic determines the essential feature and the advantage of the acceleration potential method, because the conditions that are satisfied at the Σ surface are the general conditions which are satisfied at any other point in the flow. Therefore, in this method the boundary conditions at the surface Σ are trivial because they become automatically satisfied when the velocity potential is determined from formulas (VII.3) and

(VII.5) and the acceleration potential from the discontinuity on the lifting surface.

Since the determination of the function (which is harmonic outside of the surface on which the function values obtained when approaching the surface from above are different from those obtained when approaching it from below) is a classical problem in the theory of potential, the problem of motion of lifting surfaces, therefore, can be effectively solved by the methods used in the theory of potentials.

The advantages that follow from the properties of the acceleration potential are not difficult to appreciate, if we recall that the direct solution of the velocity potential requires a rather intricate selection of the basic solution which satisfies conditions (31) and (35). An example of the solution of the problem dealing with the motion of a circular wing is the Kochin solution [64-67]. [232

Let us examine the steady motion of a lifting surface of an arbitrary shape in an infinite fluid.

The acceleration potential has a discontinuity at the intersection with the surface:

$$\Theta_{-} - \Theta_{+} = -\gamma(\theta) v_0. \tag{VII.7}$$

The acceleration potential will be evaluated as a double-surface potential. By using the known discontinuity it can be expressed by the following formula:

$$\Theta = \frac{v_0}{4\pi} \iint_s \gamma(\theta) \frac{\partial}{\partial \zeta} \frac{1}{r} ds, \tag{VII.8}$$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z-\zeta)^2}$,

ξ, η and ζ are the coordinates of the point Q on surface s.

Using the formula (VII.3) we will obtain the expression for the velocity potential

$$\varphi = -\frac{1}{4\pi} \iint_s \gamma(\theta) \int_{\infty}^{x} \frac{\partial}{\partial \zeta} \frac{1}{r} d\tau ds. \tag{VII.9}$$

By employing the condition (31) the integral equation is obtained. The component of the velocity induced along the z axis will have the following form:

$$\varphi_z = -\frac{1}{4\pi}\iint_s \gamma(\theta)\frac{\partial}{\partial z}\int_\infty^x \frac{\partial}{\partial \xi}\frac{1}{r}\,d\tau ds. \qquad \text{(VII.10)}$$

The expression for φ_2 transforms into the following form:

$$\varphi_z = +\frac{1}{4\pi}\iint_s \frac{\gamma(\theta)}{(y-\eta)^2}\left[1 - \frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\right]ds. \qquad \text{(VII.11)}$$

Let us examine the limiting expression φ_z for $x \to +\infty$ and $x \to -\infty$. From the expression (VII.11) it follows:

$$\text{with } x \to +\infty \quad \varphi_z = 0,$$

$$\text{with } x = \xi \quad \varphi_z = \frac{1}{4\pi}\iint_s \frac{\gamma(\theta)}{(y-\eta)^2}ds, \qquad \text{(VII.12)}$$

$$\text{and with } x \to -\infty \quad \varphi_z = \frac{1}{2\pi}\iint_s \frac{\gamma(\theta)}{(y-\eta)^2}ds. \qquad \text{(VII.13)}$$

The following known aerodynamic result is obtained: [233
the vertical velocity is equal to zero (in front) at infinity, and in the area of the lifting surface it is equal to one-half of its value behind, at infinity.

The integral equation (VII.11) does not lend itself to the analytical treatment. Therefore, additional approximations are introduced into the theory, which hold true for specific foil configurations and, as a result, simpler integral equations can be obtained. For long foils these approximations are given by Prandtl's lifting line theory. The foil span is assumed to be of such large dimensions (in practice $\lambda > 4$), that it can be substituted by a single lifting line, and, in evaluating the hydrodynamic characteristics of the foil, be considered that every cross section of it works independently of the others. Then, substituting the lifting foil and the surface Σ by a single entrained vortex and a system of trailing vortices, we obtain an integro-differential equation which makes the determination of circulation at each cross section possible, while bypassing the need of solving for the potential.

It is interesting to know what type of approximations have to be introduced into the equation (VII.11) in order to arrive at the Prandtl equation. Let us write formula (VII.11) in the following form:

$$\varphi_z = -\frac{1}{4\pi}\frac{\partial}{\partial y}\iint_s \frac{\gamma(\theta)}{y-\eta}\left[1-\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{(x-\xi)}\right]ds. \qquad \text{(VII.12) [sic]}$$

This formula resembles the Reysner formula [124]. In Reysner's formula a φ_x with an opposite sign in the parentheses is used instead of $\varphi_{x-}-\varphi_{x+}=\gamma(\theta)$, due to the different coordinate axis direction.

Let us write the boundary condition (31) in the following form:

$$\varphi_z = -v_0 a. \qquad \text{(VII.13) [sic]}$$

Then, the integral equation can be written as follows:

$$-\frac{1}{4\pi}\frac{\partial}{\partial y}\iint_s \frac{\gamma(\theta)}{y-\eta}\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{(x-\xi))}ds =$$
$$= v_0\left(a-\frac{1}{4\pi v_0}\frac{\partial}{\partial y}\iint_s \frac{\gamma(\theta)}{y-\eta}ds\right). \qquad \text{(VII.14)}$$

Let us consider a wing rectangular in shape in the plane view and introduce an assumption that $\gamma(\theta)=\gamma(\xi)\gamma(\eta)$.

It can be shown by a simple computation that with $\gamma(\eta)$ = const and the span extending to infinity, the integral term in parentheses will be equal to zero and the term in the other part of the equation (VII.14) will describe the two-dimensional parallel flow. For greater lengths of wings we can introduce the following approxima- [234
tion:

$$\sqrt{(x-\xi)^2+(y-\eta)^2}\approx|y-\eta|,$$

then

$$\frac{\partial}{\partial y}\int_{-b}^{b}\gamma(\eta)\frac{|y-\eta|}{(y-\eta)}d\eta = 2\gamma(y)$$

and equation (VII.14) will be as follows:

$$-\frac{\gamma(y)}{2\pi}\int_{-a}^{a}\frac{\gamma(\xi)}{x-\xi}d\xi = v_0\left(a-\frac{1}{4\pi v_0}\int_{-a}^{a}\gamma(\xi)\int_{-b}^{b}\frac{\frac{\partial}{\partial y}\gamma(\eta)}{y-\eta}d\eta\right). \qquad \text{(VII.15)}$$

With $\gamma(-b)=\gamma(b)=0$ the following identity is valid:

Note to final ed.

shouldn't ядро уравнения be nucleus?

See last para. of p. 216. and later.

$$\int_{-b}^{b} \frac{\frac{\partial}{\partial y}\gamma(\eta)}{y-\eta}\,d\eta = \int_{-b}^{b} \frac{\frac{d}{d\eta}\gamma(\eta)}{y-\eta}\,d\eta, \qquad \text{(VII.16)}$$

and then, multiplying expression (VII.15) by $\sqrt{\frac{a-x}{a+x}}$, integrating with respect to x, within the limits from -a to +a, and determining the circulation at the cross section as

$$\Gamma(y) = \int_{-a}^{a} \gamma(\xi)\,\gamma(\eta)\,d\xi,$$

we will obtain the Prandtl equation

$$\Gamma(y) = 2\pi a v_0 \left(\alpha - \frac{1}{4\pi w_0} \int_{-b}^{b} \frac{\Gamma'(\eta)}{y-\eta}\,d\eta\right). \qquad \text{(VII.17)}$$

In exactly the same way one may show that if the expression (VII.10) is written in the form

$$\varphi_z = -\frac{1}{4\pi} \iint_s \gamma(\theta) \frac{\partial}{\partial z} \int_{\xi}^{x} \frac{\partial}{\partial \zeta} \frac{1}{r}\,d\tau\,ds -$$

$$-\frac{1}{4\pi} \iint_s \gamma(\theta) \frac{\partial}{\partial z} \int_{\infty}^{\xi} \frac{\partial}{\partial \zeta} \frac{1}{r}\,d\tau\,ds, \qquad \text{(VII.18)}$$

then the first term in this expression, with span increase, will determine the two-dimensional parallel flow and the second term, which is independent from the variable x, will determine the slope angle of the flow. [235

It follows from the (VII.18) formula that the Prandtl equation can be written as follows:

$$\Gamma(y) = 2\pi a v_0 \left(\alpha - \frac{1}{4\pi w_0} \int_{-b}^{+b} \Gamma(\eta) \frac{\partial}{\partial z} \int_{\infty}^{\xi} \frac{\partial}{\partial \zeta} \frac{1}{r}\,d\tau\,d\eta\right). \quad (z=\zeta) \qquad \text{(VII.19)}$$

The kernel of this equation has a floating characteristic of the $a_i = 2$ order and will diverge, but the integration by parts, under the (VII.16) condition, transforms the kernel into a singular equation and the equation (VII. 19) into the integro-differential equation (VII.17) where, naturally, the integral is determined as the main Cauchy's term. Therefore, in further use of equation (VII.19) we will always consider this operation and treat it as a

singular integro-differential equation.

The approximation, which is opposite to that made in the approximation of the radical, leads to the Jones theory, which is the theory of extremely small wings [222].

Suppose
$$\sqrt{(x-\xi)^2+(y-\eta)^2}\approx|x-\xi|,$$
then
$$\frac{1}{4\pi}\frac{\partial}{\partial y}\int_{-b}^{+b}\frac{\gamma(y)}{y-\eta}\int_{-a}^{a}\gamma(\xi)\left(\frac{|x-\xi|}{x-\xi}-1\right)d\xi=v_0\alpha,$$

from where it is easy to obtain

$$\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{-b}^{+b}\frac{\gamma(\eta)}{y-\eta}\int_{x}^{a}\gamma(\xi)\,d\xi=-v_0\alpha,$$

or

$$\frac{1}{2\pi}\int_{-b}^{b}\frac{\Phi'(\eta,\,x)}{y-\eta}\,d\eta=v_0\alpha,$$

$$\Phi(y,\,x)=\gamma(y)\int_{x}^{a}\gamma(\xi)\,d\xi. \qquad \text{(VII.20)}$$

In is not difficult to show that the equation (VII.17), with the tendency of the relative span to approach zero, gives the value of the flow slope angle different by a factor of two from the results obtained in the small wing span theory.

Clearly, for the finite values of circulation and [236
$a \to \infty$ (which corresponds to the case in which the relative span tends to zero), the expression in parentheses in equation (VII.17) will tend to approach zero and will become as follows:

$$\frac{1}{4\pi}\int_{-b}^{b}\frac{\Gamma'(\eta)}{y-\eta}\,d\eta=v_0\alpha. \qquad \text{(VII.21)}$$

In this case, when the normal velocity is independent of x, $\Phi(y,\,x)=\Gamma(y)$, comparison of equations (VII.20) and (VII.21) leads to the result indicated above.

Later on, the approximations of the lifting line theory in the form of the expression (VII.18) will be utilized in the development of the theory of the submerged hydrofoil of finite span. As it will become evident later

on, this theory happens to be the most simple theory so that the data existing in literature is fully applicable to it. However, this theory has its own drawbacks, the most important of which is its inapplicability to short-span wings noted earlier and the fact that it becomes impossible to obtain more accurate results by introducing a series of lines and solving a system of integral equations.

Besides, there is another point which has a specific meaning only in the problems of the submerged hydrofoil motion. In equation (VII.17) quantity 2π determines the tangent of the angle of inclination of the wing with an infinite span for the relationship $C_y = f(\alpha)$. For the real wings, for which viscosity is taken into account, some average experimental value equal to $a_\infty = 5.45$ is used in calculations instead of 2π. In extending this theory to the submerged hydrofoil, $a_n = a_\infty \psi$ is used instead of a_∞ in equation (VII.18), where $\bar{\Psi}$ is the function obtained from the solution of the two-dimensional problem (see Ch. I-IV). For the general case $\Psi = F(\bar{h}, Fr)$ and the effect of the Froude number is determined by the wave phenomena on the free surface. For the three-dimensional problem the picture of wave formation will be three-dimensional in nature, and, naturally, the question then arises about the possibility of approximating the term, determined by three-dimensional phenomena, with that obtained from the solution of the two-dimensional problem.

For both large and small Froude numbers, when the problem is equivalent to one dealing with the motion of a biplane, limitations imposed on the application of this theory will be governed by those imposed on the lifting line theory application. However, it is impossible to answer this question beforehand for any arbitrary Froude numbers, especially those that are close to unity, i.e., when the wave phenomena are most pronounced. One may only assume that for certain foil spans such an approximation will produce sufficiently accurate results. For determining the values of those spans, however, additional studies are required.

The more rigorous theory is the lifting line theory in which the vortex system, consisting of an entrained vortex and a system of trailing vortices, is analyzed. Here one determines the velocities induced on a certain line (usually on the line located at a distance a from the entrained vortex) for the boundary conditions (31) satisfied [237

on that line. This theory is equally true for long as well as short spans and makes it possible to obtain more accurate results [143, 169], [232, 234].

If we replace the lifting surface by several vortices, the problem will reduce to a system of integral equations, the solutions of which are more accurate. Precisely this approach was used in the development of the theory of wings of short span [169]. This theory is widely used in the study of the more complex wing configurations (for example, swept-back wings [143].

As applied to the submerged hydrofoil, the above approximation makes the determination of Ψ from the two-dimensional problem unnecessary, but there is a danger of inaccurate treatment of the effect of finite dimensions in the direction of the chord. Let us clarify this statement by a simple example. If we increase the span to infinity then the vortex layer intensity of a thin submerged plate in motion with $Fr \to \infty$ will be evaluated from the solution of the integral equation (see Ch. I)

$$\frac{1}{2\pi}\int_{-1}^{+1}\gamma(s)\left(\frac{1}{x-s}+\frac{\bar{x}-\bar{s}}{(x-s)^2+16\bar{h}^2}\right)ds=-1.$$

The solution of this equation was obtained in Ch. I

and the function $\Psi = \dfrac{\int_{-1}^{+1}\gamma(s)\,ds}{\int_{-1}^{1}\gamma_\infty(s)\,ds}$ is given by formula (I.58).

Based on the lifting line theory this equation will be replaced by the relation as follows:

$$\frac{a_n}{2\pi}\left(1+\frac{1}{1+16\bar{h}^2}\right)=1 \qquad \text{(VII.22)}$$

or

$$a_n = 2\pi\psi,$$

where

$$\psi=\frac{1}{2}\left(\frac{1+16\bar{h}^2}{1+8\bar{h}^2}\right).$$

As seen from Fig. 15, formulas (VII.23) (represented by curve 2) and formula (VII.58) (by curve 1) give different results, with the error in determining Ψ by formula (VII.23) amounting to 10% for $\bar{h}$ = 0.2.

In a number of studies the approximate formula [238

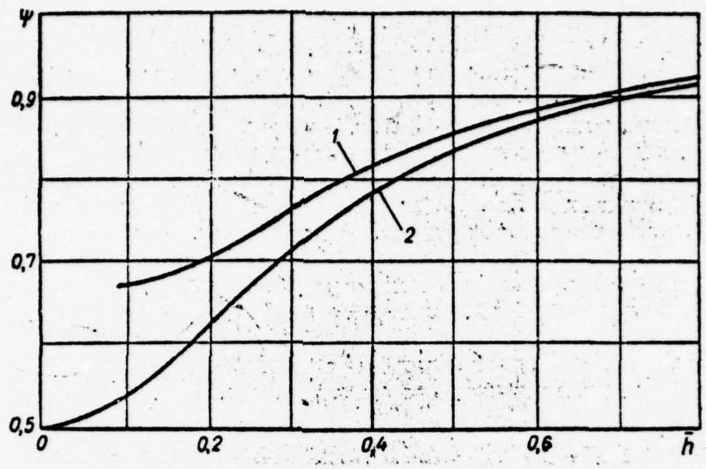

Fig. 15

(VII.23) is used as the basis for the analysis of the submerged hydrofoil problems. The error can be decreased by increasing the number of vortex lines, but this method will inevitably lead to a drastic increase in the computation volume.

By increasing the number of lines we can approach the exact solution of equation (VII.11) as closely as desired. It is not difficult to demonstrate that the integral equation in this theory follows naturally from the integral equation (VII.11) if we pull the chord of the foil together to a point.

For a single line from the expression (VII.11) we obtain the following:

$$\varphi_z = \frac{1}{4\pi}\int_{-b}^{+b}\frac{\Gamma(\eta)}{(y-\eta)^2}\left[1-\frac{x-\xi_1}{\sqrt{(x-\xi_1)^2+(y-\eta)^2}}\right]d\eta. \qquad \text{(VII.24) [sic]}$$

A number of known results can be obtained from this equation. For example, for a rectangular foil the equation derived by P. S. Chushkin [169] and for the sweptback wings the equation given by V. V. Struminskiy and N. K. Lebed' [143] can easily be obtained.

Let us analyze the rectangular foil.

Suppose $x-\xi_1=-a,$

$$\varphi_z = \frac{1}{4\pi}\int_{-b}^{b}\frac{\Gamma(\eta)}{(y-\eta)^2}\left[1+\frac{a}{\sqrt{a^2+(y-\eta)^2}}\right]d\eta$$

and in the dimensionless form [239

$$\bar{\varphi}_2 = \frac{1}{\pi}\int_{-1}^{+1} \frac{\bar{\Gamma}(\bar{\eta})}{(\bar{y}-\bar{\eta})^2}\left[1+\frac{1}{\sqrt{1+\lambda^2(y)(y-\eta)^2}}\right]d\bar{\eta},$$

$$\bar{y} = \frac{y}{b}, \quad \bar{\Gamma} = \frac{\Gamma}{2v_0 b},$$

which can be written as follows

$$\alpha = \frac{1}{\pi}\int_{-1}^{+1} \frac{\bar{\Gamma}'(\eta)}{(\bar{y}-\bar{\eta})}\,d\bar{\eta} + \frac{1}{2\pi}\int_{-1}^{+1} \frac{\bar{\Gamma}(\eta)}{(\bar{y}-\bar{\eta})^2}\left[1 \quad \frac{1}{\sqrt{1+\lambda^2(y)(\bar{y}-\bar{\eta})^2}}\right]d\bar{\eta}. \qquad \text{(VII.25)}$$

For short spans the second integral operator in the equation (VII.25) will be equal to zero and then equation (VII.25) will be transformed into the Jones's equation (VII.20).

The second integral operator in the equation (VII.25) is regular, therefore, the equation can be reduced to the Fredholm's equation with the aid of Cauchy's transformation formulas. This question will be analyzed further during the analysis of the regularization methods of singular integral equations for the submerged hydrofoil.

7.2. <u>The Integral Formulas for the Submerged Hydforoil With a Finite Span</u>

For the fluid with the free surface the hydrodynamic equation is written in the following form:

$$\frac{d\vec{v}}{dt} + g = -\frac{1}{\varrho}\nabla p,$$

where g is the gravitational acceleration. It follows, then, that the acceleration potential Q' will be determined by the following formula:

$$Q' = \varphi_t - v_0\varphi_x + gz \qquad \text{(VII.26)}$$

or

$$Q' = Q + gz. \qquad \text{(VII.27)}$$

Using the Lagrange-Cauchy integral for points on the free surface we obtain Q' = 0 with z = 0.

As is evident, for the potential Q' the boundary conditions on the free surface have a very simple form. The component of the acceleration potential gz for surfaces enclosing a certain volume will produce the Archimedes hydrostatic force and for a thin lifting surface it will [240

be continuous at the intersection with the surface and will not produce any forces. It is convenient to exclude this term from further analysis and use the potential $\theta(x, y, z)$ only. Differentiating the boundary condition expression for the free surface we receive

$$\Theta_{tt} + g\Theta_z = 0, \tag{VII.28}$$

i.e., the φ and θ potentials satisfy the same boundary conditions on the free surface with an accuracy noted by the subscripts. This result can also be obtained from the expression (VII.27).

Now let us derive a formula which will represent the harmonic function in the area between surface s_1, enveloping surface s, and the xy plane. If θ and θ_n are given on the surface, we can select a point P(x, y, z) in the lower half-space and draw two surfaces s_1 and s_2 so that surface s_1 would envelop surface s, but not point p, while surface s_2 would envelop both the surface s and the point p. Using Green's formula we may write as follows:

$$\Theta(x, y, z) = -\frac{1}{4\pi}\iint\limits_{s_1} \frac{1}{r}\frac{\partial\Theta}{\partial n} ds + \frac{1}{4\pi}\iint\limits_{s_1} \Theta \frac{\partial}{\partial n}\frac{1}{r} ds +$$

$$+ \frac{1}{4\pi}\iint\limits_{s_2} \frac{1}{r}\frac{\partial\Theta}{\partial n} ds - \frac{1}{4\pi}\iint\limits_{s_1} \Theta \frac{\partial}{\partial n}\frac{1}{r} ds,$$

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}.$$

The point $\theta(\xi, \eta, \zeta)$ runs over the s_1 and s_2 with the normal to these surfaces assumed to have an outward direction.

The function

$$\Theta_1(x, y, z) = -\frac{1}{4\pi}\iint\limits_{s_1} \frac{1}{r}\frac{\partial\Theta}{\partial n} ds + \frac{1}{4\pi}\iint\limits_{s_1} \Theta \frac{\partial}{\partial n}\frac{1}{r} ds \tag{VII.29}$$

is harmonic outside of surface s_1, while the function

$$\Theta_2(x, y, z) = \frac{1}{4\pi}\iint\limits_{s_2} \frac{1}{r}\frac{\partial\Theta}{\partial n} ds + \frac{1}{4\pi}\iint\limits_{s_2} \Theta \frac{\partial}{\partial n}\frac{1}{r} ds \tag{VII.30}$$

is harmonic inside surface s_2. Surface s_2 may be expanded

infinitely in the lower half-space and the function $\theta(x, y, z)$ may then be represented as follows:

$$\Theta(x, y, z) = \Theta_1(x, y, z) + \Theta_2(x, y, z), \qquad \text{(VII.31)}$$

where $\theta_2(x, y, z)$ is the harmonic function in the lower half-space.

Let us examine the function [241

$$\widetilde{\Theta}(x, y, z) = -\frac{1}{4\pi}\iint_{s_1} \frac{\partial \Theta}{\partial n} G ds + \frac{1}{4\pi}\iint_{s_1} \Theta \frac{\partial G}{\partial n} ds,$$

where $G(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + K(x, y, z, \xi, \eta, \zeta)$ is the function which satisfies the conditions at the free surface (VII.28) and in front at infinity (33).

Function $\widetilde{\theta}(x, y, z)$ will also satisfy the conditions in (VII.28) and (33). Then the function

$$\Delta\Theta(x, y, z) = \Theta(x, y, z) - \widetilde{\Theta}(x, y, z)$$

will be equal to zero, because it is harmonic in the lower half-space and satisfies the boundary conditions (VII.28) and (33).

Function (VII.25), which is harmonic in the lower half-space and which satisfies the boundary conditions on the free surface, determines the free waves on the surface. Since the condition in front at infinity requires absence of these waves, it follows then that function $\Delta\Theta(x, y, z)$ is equal to zero.

Thus, function $\theta(x, y, z)$ is determined by the formula

$$\Theta(x, y, z) = \frac{1}{4\pi}\iint_{s_1} \left(\Theta \frac{\partial G}{\partial n} - \frac{\partial \Theta}{\partial n} G\right) ds. \qquad \text{(VII.32)}$$

For a thin lifting surface, surface s_1 converges toward surface s coming from above and below. Since $\frac{\partial \Theta}{\partial z_-} = \frac{\partial \Theta}{\partial z_+}$, by determining the value of the discontinuity θ by formula (VII.7) we obtain the following:

$$\Theta(x, y, z) = \frac{v_0}{4\pi}\iint_{s} \gamma(\theta) \frac{\partial}{\partial n} G ds. \qquad \text{(VII.33)}$$

In the evaluation of function $G(x, y, z, \xi, \eta, \zeta)$ in

various problems we will mainly employ the separation of variables method. The initial functions will be the Green's integral functions in an infinite three-dimensional space [84] as follows:

$$\frac{1}{r} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\lambda^2 + k^2}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2+k^2}\,|z-\zeta|}\, e^{i[(x-\xi)\lambda+(y-\eta)k]} dk, \tag{VII.34}$$

$$\frac{1}{r} = \frac{1}{2\pi} \int_{0}^{2\pi} d\theta \int_{0}^{\infty} e^{-\lambda(x-x_0)} d\lambda, \tag{VII.35}$$ [242

$$\frac{1}{r} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\theta}{x_0 - x}, \tag{VII.36}$$

where

$$x = z + i(x\cos\theta + y\sin\theta); \; x_0 = \zeta + i(\xi\cos\theta + \eta\sin\theta)$$
$$(z < \zeta)$$

and

$$x = -z + i(x\cos\theta + y\sin\theta); \; x_0 = -\zeta + i(\xi\cos\theta + \eta\sin\theta).$$
$$(z > \zeta)$$

Let us derive now the general integral formulas for the velocity potential, which satisfy the conditions in (33).

Let us first consider the integral

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi(k, \lambda)\, e^{ix\lambda}}{\psi(k, \lambda, i\mu p)} dk d\lambda, \tag{VII.37}$$

and let function $\psi(k, \lambda)$ be simply zero at point λ_0'.

Let us integrate with respect to λ along a path consisting of sections of the real axis $-\infty \div \lambda_0 - \xi$, $\lambda_0 + \xi \div +\infty$ and of the minor semicircle of radius ε_1, which encircles a particular point λ_0; the bypassing of the pole will be determined by the sign of $\mathrm{Im}\lambda_0'$.

Let us analyze the following case:

$$\varphi(k, \lambda) = -\varphi(k, -\lambda), \; \psi(k, \lambda) = \psi(k, -\lambda);$$

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi(k, \lambda)\, e^{ix\lambda}}{\psi(k, \lambda, i\mu p)} d\lambda dk - \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\varphi(k, \lambda)\, e^{-ix\lambda}}{\psi(k, \lambda, -i\mu\lambda)} dk d\lambda.$$

If $\lambda_0' = \lambda_0 - i\mu$, then in the first integral the by-pass of the particular point λ_0 will be from above along L_1, while in the second integral it will be from below along the L_2 path. From this it follows

$$J = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\varphi(k, \lambda) e^{ix\lambda}}{\psi(k, \lambda)} dk d\lambda - \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\varphi(k, \lambda) e^{-ix\lambda}}{\psi(k, \lambda)} dk d\lambda -$$

$$-2\pi i \int_{-\infty}^{\infty} \frac{\varphi(k, \lambda_0)}{\psi'(k, \lambda_0)} \operatorname{Re} e^{ix\lambda_0} dk, \tag{VII.38}$$

where the integrals are in the sense of the main Cauchy value.

Let us now determine the limits of the integral for $x \to \pm\infty$. With $x > 0$ [243

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\varphi(k, \lambda) e^{ix\lambda}}{\psi(k, \lambda)} dk d\lambda = \int_{-\infty}^{\infty}\int_{L_1} \frac{e^{ix\lambda}\varphi(k, \lambda)}{\psi(k, \lambda)} dk d\lambda +$$

$$+ \pi i \int_{-\infty}^{\infty} \frac{\varphi(k, \lambda_0) e^{ix\lambda_0}}{\psi'(k, \lambda_0)} dk,$$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\varphi(k, \lambda) e^{-ix\lambda}}{\psi(k, \lambda)} dk d\lambda = \int_{-\infty}^{\infty}\int_{L_1} \frac{\varphi(k, \lambda) e^{ix\lambda}}{\psi(k, \lambda)} dk d\lambda -$$

$$- \pi i \int_{-\infty}^{\infty} \frac{\varphi(k, \lambda_0) e^{-ix\lambda_0}}{\psi'(k, \lambda_0)} dk. \tag{VII.39}$$

With $x < 0$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\varphi(k, \lambda) e^{ix\lambda}}{\psi(k, \lambda)} dk d\lambda = \int_{-\infty}^{\infty}\int_{L_1} \frac{\varphi(k, \lambda) e^{ix\lambda}}{\psi(k, \lambda)} dk d\lambda -$$

$$- \pi i \int_{-\infty}^{\infty} \frac{\varphi(k, \lambda_0) e^{ix\lambda_0}}{\psi'(k, \lambda_0)} dk, \tag{VII.39) [sic]$$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\varphi(k, \lambda) e^{-ix\lambda}}{\psi(k, \lambda)} dk d\lambda = \int_{-\infty}^{\infty}\int_{L_1} \frac{\varphi(k, \lambda) e^{-ix\lambda}}{\psi(k, \lambda)} dk d\lambda +$$

$$+ \pi i \int_{-\infty}^{\infty} \frac{\varphi(k, \lambda_0) e^{-ix\lambda_0}}{\psi'(k, \lambda_0)} dk.$$

Because $\lim\limits_{k \to \pm\infty} \dfrac{\varphi(k, \lambda)}{\psi(k, \lambda)} = 0,$

then

$$\int\limits_{L_1} \frac{\varphi(k, \lambda)\, e^{-ix\lambda}}{\psi(k, \lambda)} d\lambda = \int\limits_{L_1} \frac{\varphi(k, \lambda)}{\psi(k, \lambda)} d\left(-\frac{e^{-ix\lambda}}{ix}\right) =$$

$$= \frac{1}{ix} \int\limits_{L_1} e^{-ix\lambda} \frac{d}{d\lambda}\left(\frac{\varphi(k, \lambda)}{\psi(k, \lambda)}\right) d\lambda$$

for an unlimited increase x approaches zero. The other integrals also tend to approach zero. [244

Then, for the limiting values of the J integral we will get

$$\underset{x\to\infty}{J} = 0;\ \underset{x\to-\infty}{J} = 4\pi i \int\limits_{-\infty}^{\infty} \frac{\varphi(\lambda_0, k)}{\psi'(\lambda_0, k)} \cos\lambda_0 x dk. \qquad \text{(VII.40)}$$

This characteristic of the integral (VII.37) indicates that we can search for the velocity potential of the submerged hydrofoil in the form of the integral (VII.37).

Let us analyze the potential

$$\varphi = -\frac{1}{4\pi} \iint\limits_S \gamma(\theta) \int\limits_{\infty}^{x} \frac{\partial}{\partial\zeta}\left[\frac{1}{r} + \frac{1}{\pi} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{N(p, g)}{Q(p, g)} e^{ixp} dg dp\right] ds; \qquad \text{(VII.41)}$$

let function Q(p, g) have zero values at all points of λ_i.

Let us transform this expression.

$$\varphi = -\frac{1}{4\pi} \iint\limits_s \gamma(\theta) \left\{\int\limits_{\infty}^{x} \frac{\partial}{\partial\zeta} \frac{1}{r} - \frac{i}{\pi} \frac{\partial}{\partial\zeta} \int\limits_{-\infty}^{\infty} dg \int\limits_0^{\infty} \left[\frac{N(p, g)}{pQ(p, g)} e^{ixp} - \right.\right.$$

$$\left.\left. - \frac{N(-p, g)}{pQ(-p, g)} e^{-ixp}\right] \Bigg|_{\infty}^{x} dp \right\} ds =$$

$$= -\frac{1}{4\pi} \iint\limits_s \gamma(\theta) \left\{\int\limits_{\infty}^{x} \frac{\partial}{\partial\zeta} \frac{1}{r} + \frac{\partial}{\partial\zeta}\left[-\int\limits_{-\infty}^{\infty} \frac{N(0, g)}{Q(0, g)} dg - \right.\right.$$

$$\left.\left. - \frac{i}{\pi} \int\limits_{-\infty}^{\infty} dg \int\limits_0^{\infty} \left(\frac{N(p, g)}{pQ(p, g)} e^{ixp} - \frac{N(-p, g)}{pQ(-p, g)} e^{-ixp}\right) \Bigg|_{\infty}^{x} dp\right]\right\} ds.$$

Let us separate the remainders at points p_i.

$$\varphi = -\frac{1}{4\pi}\iint_s \gamma(\theta)\left\{\int_{\infty}^{x}\frac{\partial}{\partial\zeta}\frac{1}{r}\,d\tau + \frac{\partial}{\partial\zeta}\left[-\int_{-\infty}^{\infty}\frac{N(0,\ g)}{Q(0,\ g)}\,dg + \right.\right.$$

$$+\left(\sum_{i=1}^{k}\mathrm{Sign\,Im}\,p_i'\int_{-\infty}^{\infty}\frac{N(p_i,\ g)}{p_iQ'(p_ig)}\,e^{ixp_i}dg - \right.$$

$$\left.\left. - \sum_{i=1}^{k_1}\mathrm{Sign\,Im}\,\bar{p}_i'\int_{-\infty}^{\infty}\frac{N(-\bar{\lambda}_i,\ g)\,e^{-ix\bar{\lambda}_i}}{p_iQ'(-p_i,\ g)}\,dg\right)\right|_{\infty}^{x} -$$ [245

$$\left.\left. -\frac{i}{\pi}\int_{-\infty}^{\infty}dg\int_0^{\infty}\left(\frac{N(p,\ g)}{pQ(p,\ g)}\,e^{ixp} - \frac{N(-p,\ g)}{pQ(-p,\ g)}\,e^{-ixp}\right)\right|_{\infty}^{x}\right]\right\}ds.$$

With x > 0

$$\int_{-\infty}^{\infty}dg\int_0^{\infty}\frac{N(p,\ g)\,e^{ixp}}{pQ(p,\ g)}\,dp = \int_{-\infty}^{\infty}dg\int_{L_1}\frac{N(p,\ g)\,e^{ixp}}{pQ(p,\ g)}\,dp +$$

$$+\pi i\int_{-\infty}^{\infty}\sum_{i=1}^{k}\frac{N(p_i,\ g_0)\,e^{ixp_i}}{p_iQ'(p_i,\ g)}\,dg;$$

$$\int_{-\infty}^{\infty}dg\int_0^{\infty}\frac{N(-p,\ g)\,e^{-ixp}}{pQ(-p,\ g)}\,dg = \int_{-\infty}^{\infty}dg\int_{L_2}\frac{N(-p,\ g)\,e^{-ixp}}{pQ(-p,\ g)}\,dp -$$

$$-\pi i\sum_{i=1}^{k_1}\int_{-\infty}^{\infty}\frac{N(-\bar{p}_i,\ g)\,e^{-ix\bar{p}_i}}{\bar{p}_iQ'(-\bar{p}_i,\ g)}\,dg.$$

With x < 0

$$\int_{-\infty}^{\infty}dg\int_0^{\infty}\frac{N(p,\ g)\,e^{ixp}}{pQ(p,\ g)}\,dp = \int_{-\infty}^{\infty}dg\int_{L_2}\frac{N(p,\ g)\,e^{ixp}}{pQ(p,\ g)}\,dp -$$

$$-\pi i\sum_{i=1}^{k}\int_{-\infty}^{\infty}\frac{N(\bar{p}_i,\ g)\,e^{ix\bar{p}_i}}{\bar{p}_iQ'(\bar{p}_i,\ g)}\,dg,$$

$$\int_{-\infty}^{\infty}dg\int_0^{\infty}\frac{N(-p,\ g)\,e^{-ixp}}{pQ(-p,\ g)}\,dp = \int_{-\infty}^{\infty}dg\int_{L_1}\frac{N(-p,\ g)\,e^{-ixp}}{pQ(-p,\ g)}\,dp +$$

$$+\pi i\sum_{i=1}^{k}\int_{-\infty}^{\infty}\frac{N(-\bar{p}_i,\ g)\,e^{-ix\bar{p}_i}}{\bar{p}_iQ'(-\bar{p}_i,\ g_0)}\,dg.$$

The case when there are no perturbations in front of the wing is defined by the following expression: [246

$$\text{Sign Im } p_i' = -1.$$

By substituting this expression in the final formula for φ, the latter may be written as follows:

$$\varphi = -\frac{1}{4\pi}\iint_s \gamma(\theta)\left\{\int_\infty^x \frac{\partial}{\partial\zeta}\frac{1}{r}\,d\tau + \frac{\partial}{\partial\zeta}\left[-\int_{-\infty}^{\infty}\frac{N(0,g)}{Q(0,g)}\,dg - \right.\right.$$
$$-\sum_{l=1}^{k}\int_{-\infty}^{\infty}\frac{N(p_l,g)\,e^{ixp_l}}{p_lQ'(p_l,g)}\,dg - \sum_{l=1}^{k_1}\int_{-\infty}^{\infty}\frac{N(-\bar{p}_l,g)\,e^{-ix\bar{p}_l}}{\bar{p}_lQ'(-\bar{p}_l,g)}\,dg -$$
$$\left.\left.-\frac{i}{\pi}\int_{-\infty}^{\infty}dg\int_0^{\infty}\left(\frac{N(p,g)\,e^{ixp}}{pQ(p,g)} - \frac{N(-p,g)\,e^{-ixp}}{pQ(-p,g)}\right)dp\right]\right\}ds. \qquad \text{(VII.43) [sic]}$$

The asymptotic values for the φ potential are as follows:
for $x \to \infty$

$$\varphi = 0,$$

and for $x \to -\infty$

$$\varphi = -\frac{1}{4\pi}\iint_s \gamma(\theta)\left\{-\int_{-\infty}^{\infty}\frac{\partial}{\partial\zeta}\frac{1}{r}\,d\tau + \frac{\partial}{\partial\zeta}\left[-2\int_{-\infty}^{\infty}\frac{N(0,g)}{Q(0,g)}\,dg - \right.\right.$$
$$\left.\left.-2\sum_{l=1}^{k}\int_{-\infty}^{\infty}\frac{N(p_l,g)\,e^{ixp_l}}{p_lQ'(p_l,g)}\,dg - 2\sum_{l=1}^{k_1}\int_{-\infty}^{\infty}\frac{N(-\bar{p}_l,g)\,e^{-ixp_l}}{\bar{p}_lQ'(-p_l,g)}\,dg\right]\right\}ds. \qquad \text{(VII.44)}$$

The formula (VII.43) makes possible the immediate evaluation of the velocity potential of the submerged hydrofoil using the known functions N(p, g) and Q(p, g), which satisfy the conditions on the free surface and possibly some additional conditions. This formula is particularly convenient for using the integral representation (VII.34).

Let us consider the velocity potential formula for the general case of the unsteady motion. For the unsteady periodic motion the velocity potential is related to the acceleration potential by the (VII.5) formula.

When writing the formula for the potential φ, it is necessary to analyze the integral of the following form: [247

$$J=\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} \frac{\varphi(\lambda,\ k)\, e^{ix(\lambda-p)}}{\varphi(\lambda_1 k i\mu)}\, d\lambda.$$

Then, by utilizing the same approach as that used in deriving formula (VII.43), we receive the formula

$$\varphi=-\frac{e^{ipx}}{4\pi}\iint_s \gamma(\theta)\left\{\int_{\infty}^{x} e^{-ip\tau}\frac{\partial}{\partial\zeta}\frac{1}{r}\,d\tau+\right.$$

$$+\frac{\partial}{\partial\zeta}\left[\operatorname{Sign}p\int_{-\infty}^{\infty}\frac{N(\lambda,\ k)}{Q(\lambda,\ k)}\,dk-\sum_{l=1}^{k}\int_{-\infty}^{\infty}\frac{N(\lambda_l,\ k)\, e^{ix(\lambda_l-p)}}{(\lambda_l-p)\, Q'(\lambda_l,\ k)}\,dk-\right.$$

$$-\sum_{l=1}^{k_1}\int_{-\infty}^{\infty}\frac{(N(-\bar{\lambda}_l,\ k)\, e^{-ix(\bar{\lambda}_l+p)}}{(\bar{\lambda}_l+p)\, Q'(-\bar{\lambda}_l,\ k)}\,dk-$$

$$\left.\left.-\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\int_{-\infty}^{\infty}\frac{M(\lambda,\ k)\, e^{ix(k-p)}}{(k-p)\, Q(\lambda,\ k)}\,dk\right]\right\} ds. \qquad \text{(VII.45)}$$

In conclusion, we will present the formulas for determining forces acting during the motion of a lifting surface parallel to the free surface. The resultant vector of the pressure, exerted by the fluid on the body, is determined by the following formula:

$$P=-\iint_{s_1} pnds,$$

where n is the unit vector of the outer normal. Using the projections to the coordinate axes we may write

$$P_x=-\iint_{s_1} p\cos(n,\ x)\, ds,$$

$$P_z=-\iint_{s_1} p\cos(n,\ z)\, ds,$$

$$P_y=0.$$

Let us deform the surface s_1 toward both a surface s passing it above and below and a cylindrical surface δ. Then, for the lifting force we can derive the following expression:

$$P_z=\iint_s (p_- - p_+)\, ds,$$ [248

or

$$P_z = -\varrho \iint_s (\Theta_- - \Theta_+)\, ds.$$

The latter expression together with formula (VII.7) produce

$$P_z = \varrho v_0 \iint_s \gamma(\theta)\, ds. \qquad \text{(VII.46)}$$

Remembering that the circulation along the contour is

$$\Gamma(y) = \int_{-a}^{a} \gamma(y)\, dy,$$

we get

$$P = \varrho v_0 \int_{-b}^{+b} \Gamma(y)\, dy. \qquad \text{(VII.47)}$$

For the drag we have

$$P_x = \iint_s (p_- - p_+) \cos(n,\ x)\, ds - \iint_0 p \cos(n,\ x)\, ds. \qquad \text{(VII.48)}$$

The second term determines the suction force which is produced because of the presence of a high vacuum in the proximity of the leading edge of the foil. At the trailing edge of the foil the condition postulated by Zhukovskiy and Chaplygin is satisfied and the surface envelops the leading edge only.

For determining the suction force let us use the results of the two-dimensional flow problem. In the two-dimensional flow the suction force is determined by the formula

$$Px = \varrho v_z \Gamma,$$

where v_z is the vertical velocity at the leading edge.

For the foil of finite span the suction force per unit length of the foil will be in the form

$$Px = \varrho (v_z - \Delta v_{z\text{инд}}) \Gamma,$$

then the total drag will be determined by the formula of the following type:

$$Px_{\text{инд}} = \varrho v_0 \Gamma_\alpha - \varrho (v_z - \Delta v_{z\text{инд}}) \Gamma,$$

However, since $v_z = v_0\alpha$, we will get the known aerodynamic formula

$$Px_{\text{инд}} = \varrho \Delta v_{z\text{инд}} \Gamma. \qquad \text{(VII.49)}$$

Defining the suction force in formula (VII.48) in this manner, for the total drag of the wing we obtain [249

$$Q = \varrho \int_{-b}^{+b} [v_0 \alpha - v(y)] \Gamma(y)\, dy, \qquad \text{(VII.50)}$$

where v(y) is the vertical velocity at the leading edge of the foil.

Introducing the potential φ in the form of $\varphi = \varphi_1 + \varphi_2$, where φ_1 is the harmonic function outside of the surface s + Σ and φ_2 is the harmonic function in the lower half-space,

$$\varphi_1 = \varphi_{11} + \varphi_{12},$$

where φ_{11} is the harmonic function outside of the surface s, and φ_{12} is the harmonic function outside of the surface Σ. It follows then that the formula for the drag can be expressed in the following form:

$$Q = \varrho \int_{-b}^{b} \varphi_{12z} \Gamma(y)\, dy + \varrho \int_{-b}^{b} \varphi_{2z} \Gamma(y)\, dy. \qquad \text{(VII.51)}$$

Here, the first term will probably define the inductive drag of the wing in an infinite fluid.

CHAPTER VIII. THE THEORY OF THE SUBMERGED HYDROFOIL IN THE THREE-DIMENSIONAL FLUID FLOW OF INFINITE DEPTH [250

8.1. The Velocity Potential of the Submerged Hydrofoil. The Integral Equation for the Hydrofoil of an Arbitrary Shape in the Plan View

For the steady-state motion the boundary conditions (VII.29), considering the dispersion coefficient μ, will have the following form:

$$\Theta_{xx} - \mu\Theta_x + \nu\Theta_z = 0. \qquad \text{(VIII.1)}$$

The determination of the velocity potential by formula (VII.43) will be given later. However, here we will examine the sequential evaluation of the velocity potential by formulas (VII.3) and (VII.33). We will derive the function $G(x, y, z, \xi, \eta, \zeta)$ with the aid of the integral expression (VII. 35), and we will seek function G(x, y, z) in the following form:

$$G = \frac{1}{r} + \frac{1}{r_1} + G_1(x, y, z), \qquad \text{(VIII.2)}$$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$.

The integral expression (VIII.35) [sic] for $\frac{1}{r}$ gives the following for $z + \zeta < 0$:

$$\frac{1}{r_1} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\int_0^\infty e^{\lambda(z+\zeta+i\omega)}d\lambda d\Theta, \qquad \text{(VIII.3)}$$

where

$$\omega = (x-\xi)\cos\Theta + (y-\eta)\sin\Theta.$$

With z = 0

$$\left(\frac{1}{r}\right)_{xx} = \left(\frac{1}{r_1}\right)_{xx}, \quad \left(\frac{1}{r}\right)_z = -\left(\frac{1}{r_1}\right)_z.$$

Hence [251

$$G_{1xx} - \mu G_{1x} + \nu G_{1z} = -2\left(\frac{1}{r_1}\right)_{xx}. \quad (z=0)$$

Utilizing the integral expression (VIII.3) we will obtain the differential equation for evaluating the harmonic function in the lower half-space as follows:

$$G_{1xx} - \mu G_{1x} + \nu G_{1z} = \frac{1}{\pi}\int_{-\pi}^{+\pi}\int_0^\infty \lambda^2 e^{\lambda[z+\zeta+i\omega]}\cos^2\Theta d\Theta d\lambda. \qquad \text{(VIII.4)}$$

Let us seek G_1 in the following form:

$$G_1 = \frac{1}{\pi}\int_{-\pi}^{+\pi}\int_0^{\infty} A(\lambda, \Theta)\, e^{\lambda[z+\zeta+i\omega]}\, d\Theta d\lambda.$$

From the expression (VIII.4) for $A(\lambda, \theta)$ we obtain the following:

$$A[\lambda, \Theta] = -\frac{\lambda\cos^2\Theta}{\lambda\cos^2\Theta + i\mu\cos\Theta - \nu}$$

and

$$G = \frac{1}{r} - \frac{1}{2\pi}\int_{-\pi}^{+\pi}\int_0^{\infty} e^{\lambda[z+\zeta+i\omega]}\,\frac{\lambda\cos^2\Theta + \nu}{\lambda\cos^2\Theta + i\mu\cos\Theta - \nu}\, d\Theta d\lambda. \qquad \text{(VIII.5)}$$

This formula was derived by L. N. Sretenskiy [139]. The prime pole of the integrand function with $\cos\theta > 0$ is located under the abscissa axis and it should be approached from above along the path L_1, while with $\cos\theta < 0$ it is located above this axis and it should be approached from below along the L_2 path.

Taking the remainders from the approach along the L_1 and L_2 (VIII.5), we can write as follows:

$$G = \frac{1}{r} - \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} e^{\lambda(z+\zeta+i\omega)}\,\frac{\lambda\cos^2\Theta + \nu}{\lambda\cos^2\Theta - \nu}\, d\lambda +$$

$$+\operatorname{Re} 2i\nu \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\Theta}(z+\zeta+i\omega)} \sec^2\Theta\, d\Theta. \qquad \text{(VIII.6)}$$

The acceleration potential (VII.33) will be as follows: [252

$$\Theta(x, y, z) = \frac{v_0}{4\pi}\iint_s \gamma(\theta)\Bigg[-\frac{z-\zeta}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} -$$

$$-\frac{1}{2\pi}\int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} e^{\lambda(z+\zeta+i\omega)}\,\frac{\lambda(\lambda\cos^2\Theta + \nu]}{\lambda\cos^2\Theta - \nu}\, d\lambda +$$

$$+\operatorname{Re} 2i\nu^2 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\theta}(z+\zeta+i\omega)} \sec^4\Theta\, d\Theta\Bigg] ds \qquad \text{(VIII.7)}$$

and the velocity potential

$$\varphi=-\frac{1}{4\pi}\iint_s \gamma[\Theta]\left[\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{x-\xi}{r}-1\right)+\right.$$

$$+\operatorname{Re}2\nu\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\Theta}(z+\zeta+i\omega)}\sec^3\Theta\, d\Theta+$$

$$\left.+\frac{i}{2\pi}\int_{-\pi}^{+\pi}\frac{d\Theta}{\cos\Theta}\int_0^\infty e^{\lambda(z+\zeta+i\omega)}\frac{\lambda\cos^2\Theta+\nu}{\lambda\cos^2\Theta-\nu}\,d\lambda\right]ds.$$

The point $\Theta=\left|\frac{\pi}{2}\right|$ is a special point in the double integral expression (VIII.8). Therefore, when integrating with respect to θ the remainder at this point must be taken into account and the path must be selected so as to ensure an appropriate bypass around this point.

Separating the remainders from this integral we have

$$\int_{-\pi}^{+\pi}\frac{d\Theta}{\cos\Theta}\int_0^\infty e^{\lambda(z+\zeta+i\omega)}\frac{\lambda\cos^2\Theta+\nu}{\lambda\cos^2\Theta-\nu}\,d\lambda=$$

$$-2\int_0^\pi\frac{d\Theta}{\cos\Theta}\int_0^\infty e^{\lambda(z+\zeta)}e^{\lambda i(x-\xi)\cos\Theta}\frac{\lambda\cos^2\Theta+\nu}{\lambda\cos^2\Theta-\nu}\cos\lambda(y-\eta)\sin\Theta\, d\lambda=$$

$$=2\int_0^\pi\frac{d\Theta}{\cos\Theta}\int_0^\infty e^{\lambda(z+\zeta)}e^{\lambda i(x-\xi)\cos\Theta}\frac{\lambda\cos^2\Theta+\nu}{\lambda\cos^2\Theta-\nu}\cos\lambda(y-\eta)\sin\Theta\, d\lambda-$$ [253

$$-2\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}\frac{d\Theta}{\cos\Theta}\int_0^\infty e^{\lambda(z+\zeta)}\cos\lambda(y-\eta)\,d\lambda,$$

$$\int_{\frac{\pi}{\varepsilon}-\varepsilon}^{\frac{\pi}{2}+\varepsilon}\frac{d\Theta}{\cos\Theta}=-\int_s^{-s}\frac{du}{u}=-i\int_0^\pi d\varphi=-i\pi.$$

The expression for the velocity potential will now be as follows:

$$\varphi=-\frac{1}{4\pi}\iint_s \gamma(\theta)\left[\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{x-\xi}{r}-1\right)-\right.$$

$$-\int_0^\infty e^{\lambda[z+\zeta]}\cos\lambda(y-\eta)\,d\lambda+\frac{i}{2\pi}\int_{-\pi}^{+\pi}\frac{d\Theta}{\cos\Theta}\int_0^\infty e^{\lambda[z+\zeta+i\omega]}\times$$

$$\times\frac{\lambda\cos^2\Theta+\nu}{\lambda\cos^2\Theta-\nu}\,d\lambda+\mathrm{Re}\,2\nu\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}e^{\frac{\nu}{\cos^2\Theta}[z+\zeta+i\omega]}\sec^3\Theta\,d\Theta\Bigg]ds. \qquad \text{(VIII.9)}$$

M. D. Khaskind [155] determined the double integral as the main Cauchy expression and, as a result, arrived at the wrong conclusion.

At this point, it is not difficult to write a general integral equation, taking the value φ_z with $z = \zeta = -h$ as follows:

$$\frac{1}{4\pi}\iint_s\gamma(\theta)\Bigg[\frac{1}{(y-\eta)^2}\left(\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta)^2}}-1\right)-$$

$$-\int_0^\infty e^{-\lambda h}\lambda\cos\lambda(y-\eta)\,d\lambda+\frac{i}{2\pi}\int_{-\pi}^{+\pi}\frac{d\Theta}{\cos\Theta}\int_0^\infty e^{-\lambda h}e^{i\lambda\omega}\times$$

$$\times\frac{\lambda(\lambda\cos^2\Theta+\nu)}{\lambda\cos^2\Theta-\nu}\,d\lambda+\mathrm{Re}\,2\nu^2\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}e^{\frac{\nu}{\cos^2\Theta}(-h+i\omega)}\sec^5\Theta\,d\Theta\Bigg]ds=v_0\alpha. \qquad \text{(VIII.10)}$$

T. Nishiyama [213] examines this problem by considering the boundary conditions in (VIII.1) for the potential φ. The velocity potential for the hydrofoil motion in an infinite flow he takes as follows: [254

$$\varphi_\infty=\frac{1}{4\pi}\int_{-b}^{+b}\Gamma(\eta)\int_\infty^0\frac{\partial}{\partial z}\frac{d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\eta)^2}}.$$

After the transformations

$$\varphi_\infty=-\frac{i}{8\pi^2}\int_{-b}^{+b}\Gamma(\eta)\,d\eta\int_{-\pi}^{+\pi}\sec\Theta\,d\Theta\int_0^\infty e^{-\lambda(z+h+i\tilde{\omega})}d\lambda+$$

$$+\frac{1}{4\pi}\int_{-b}^{+b}\Gamma(\eta)\,d\eta\int_0^\infty e^{-\lambda(z+h)}\cos\lambda(y-\eta)\,d\lambda$$

$$(\tilde{\omega}=x\cos\Theta+(y-\eta)\sin\Theta. \qquad \text{(VIII.11)}$$

It is clear that from the very outset Nishiyama formulates this problem as the problem of the lifting line

motion. He assumes that the velocity potential of the submerged hydrofoil is in the following form:

$$\varphi = \varphi_\infty + \varphi_1 + \varphi_2,$$

where φ_1 - the potential corresponding to the wing motion near a solid wall;
φ_2 - the so-called wave potential.

The functions φ_1 and φ_2 given below are harmonic in the lower half-space:

$$\varphi_1 = \frac{i}{8\pi^2}\int_{-b}^{+b}\Gamma(\eta)\int_{-\pi}^{+\pi}\sec\Theta d\Theta\int_0^\infty e^{-\lambda(h-z)+i\lambda\widetilde{\omega}}\,d\lambda d\eta +$$

$$+\frac{1}{4\pi}\int_{-b}^{+b}\Gamma(\eta)\,d\eta\int_0^\infty e^{-\lambda(h-z)}\cos\lambda(y-\eta)\,d\lambda,$$

$$\varphi_2 = -\frac{i}{8\pi^2}\nu\int_{-b}^{+b}\Gamma(\eta)\,d\eta\int_{-\pi}^{+\pi}\sec^3\Theta d\Theta\int_0^\infty \frac{e^{-\lambda[(h-z)-i\widetilde{\omega}]}}{\lambda-\nu\sec^2\Theta+i\mu\sec\Theta}\,d\lambda. \quad \text{(VIII.12)}$$

In formulas (VIII.11) and (VIII.12) the signs are opposite of those given in the formulas in [211], because T. Nishiyama uses an opposite direction for the Ox axis.

If we separate the potential (VIII.9) into the corresponding φ_∞, φ_1, φ_2 and use the assumptions for the lifting line, we obtain the expression (VIII.12). G. Goshev [21, 22] obtains the following equation for the wave potential: [255

$$\varphi_2 = \frac{1}{2\pi}\int_{-b}^{+b}\Gamma(\eta)\frac{\partial}{\partial n}\int_0^\infty e^{-\lambda(z+\zeta)}\left(\cos x\sqrt{\nu\lambda}\int_{-\infty}^{x} J_0\cos\xi\sqrt{\nu\lambda}d\xi + \right.$$

$$\left. + \sin x\sqrt{\nu\lambda}\int_{-\infty}^{x} J_0\sin\xi\sqrt{\nu\lambda}d\xi\right)d\lambda d\eta, \quad \text{(VIII.13)}$$

where the argument of the Bessel function is $J_0\,\lambda\sqrt{\xi^2+(y-\eta)^2}$.

8.2. The Integro-Differential Equation for the Lifting Line Theory

The integro-differential equation for the submerged hydrofoil, which generalizes the Prandtl equation, can be

obtained in a similar fashion used for obtaining equations (VIII.17) and (VIII.19). Later on, several forms for writing this equation will be given, which are more convenient in certain cases.

From the expression (VIII.10) we obtain directly the following:

$$\Gamma(y) = 2a(y)\,a_n v_0 \left\{ \alpha(y) - \frac{1}{4\pi v_0} \int_{-B}^{+B} \frac{\Gamma'(\eta)}{y-\eta} d\eta + \right.$$

$$+ \frac{1}{4\pi v_0} \int_{-B}^{+B} \Gamma(\eta) \left[\int_0^\infty e^{-\lambda h} \lambda \cos \lambda (y-\eta)\, d\lambda - 4v^2 \int_0^{\frac{\pi}{2}} \sec^5 \Theta e^{-2vh\sec^2\Theta} \times \right.$$

$$\left. \left. \times \cos v \sec^2 \Theta (y-\eta) \sin \Theta d\Theta \right] d\eta \right\}. \qquad \text{(VIII.14)}$$

The second integral operator can be integrated by parts with the condition that $\Gamma(-B = \Gamma(B) = 0$ and the general nucleus can be constructed which has a singular and regular parts.

In the dimensionless form the equation can be written as follows:

$$\Gamma(\bar{y}) = \frac{a_h}{2\lambda(\bar{y})} \left[\alpha(\bar{y}) - \frac{1}{2\pi} \int_{-1}^{+1} \Gamma'(\bar{\eta}) \left(\frac{1}{\bar{y}-\bar{\eta}} + G(\bar{y}-\bar{\eta}) \right) d\eta \right], \qquad \text{(VIII.15)}$$

where [256

$$\Gamma(\bar{y}) = \frac{\Gamma(y)}{2Bv_0}; \quad \lambda(\bar{y}) = \frac{B}{a(y)}; \quad \bar{y} = \frac{y}{B}; \quad a_h = \frac{dC_{y\bar{h}}}{d\alpha};$$

$G(\bar{y} - \bar{\eta})$ is the regular part of the nucleus of the equation.

In future computations we will utilize the following formula:

$$G(\bar{y}-\bar{\eta}) = \mathrm{Re}\left(2\omega i \int_0^\infty \frac{\lambda+1}{\lambda} e^{-\omega i(\lambda+1)\left[(\bar{y}-\bar{\eta})\sqrt{\frac{\lambda}{\lambda+1}}-4i\bar{H}\right]} \times \right.$$

$$\left. \times d\lambda - \frac{1}{\bar{y}-\bar{\eta}-4i\bar{H}} \right), \qquad \text{(VIII.16)}$$

where $\bar{\omega} = vB = \frac{1}{2\mathrm{Fr}_\beta^2}$; $\bar{H} = \frac{h}{2B}$.

After the examination of the limiting values, function G(y) may be written in a different, more convenient form:

$$G(\bar{y}-\bar{\eta}) = -\int_0^{\infty} e^{-4\lambda\bar{H}} \sin\lambda(\bar{y}-\bar{\eta})\,d\lambda + 2\int_0^{\infty} e^{-4\lambda\bar{H}} \frac{\lambda}{\lambda-\omega} \times$$

$$\times \sin\lambda(\bar{y}-\bar{\eta}) \sqrt{1-\frac{\omega}{\lambda}}\,d\lambda. \qquad \text{(VIII.17)}$$

Formulas (VIII.16) and (VIII.17) use the following integral expression, known from the material discussed previously:

$$\operatorname{Re}\frac{1}{\bar{y}-\bar{\eta}-4i\bar{H}} = \int_0^{\infty} e^{-4\lambda\bar{H}} \sin\lambda(\bar{y}-\bar{\eta})\,d\lambda.$$

From formula (VIII.17) the values of the $G(\bar{y}-\bar{\eta})$ function with $\omega \to 0$ and $\omega \to \infty$ are easily obtained: with $\omega \to 0$

$$G(\bar{y}-\bar{\eta}) = \int_0^{\infty} e^{-4\lambda\bar{H}} \sin\lambda(\bar{y}-\bar{\eta})\,d\lambda,$$

with $\omega \to \infty$

$$G(\bar{y}-\bar{\eta}) = -\int_0^{\infty} e^{-4\lambda\bar{H}} \sin\lambda(\bar{y}-\bar{\eta})\,d\lambda.$$

The case when $\omega \to \infty$ corresponds to the motion of a foil in the proximity of a solid wall, while that in which $\omega \to 0$ corresponds to the motion involving a free surface with the boundary condition of $\theta_x = 0$. It is sometimes convenient to use the following method for obtaining the [257
integro-differential equation. If the function G(x, y, z) is expressed in the form

$$G(x, y, z) = \frac{1}{r} + k_1(x, y, z),$$

then, by considering that $\left(\frac{1}{r}\right)_z = -\left(\frac{1}{r}\right)$ and $k_{1z}(x, y, z) = k_{1\zeta}(x, y, z)$, the integral equation of the problem may be written as follows:

$$v_n = v_{n0} + \frac{1}{4\pi}\int_{-\frac{b}{2}}^{+\frac{b}{2}} \Gamma(\eta)\,d\eta \frac{\partial^2}{\partial z^2}\int_{\infty}^{\varepsilon}\left[\frac{1}{r} - k_1(x, y, z)\right]dz. \text{ with } z=\zeta \qquad \text{(VIII.18)}$$

By means of transformations, the equation (VIII.18) results in a singular integro-differential equation with respect to circulation.

Let us consider that $f(z, \eta) = \int\limits_{\infty}^{c} \left[\frac{1}{r} - k_1(x, y, z) \right] dx$ satisfies the equation

$$\frac{\partial^2 f}{\partial z^2} + a^2 \frac{\partial^2 f}{\partial \eta^2} = f_2(z, \eta).$$

Then

$$v_n = v_{n0} + \frac{1}{4\pi} \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \Gamma(\eta) \left[-a^2 \frac{d^2 f}{d\eta^2} + f_2(\zeta, \eta) \right] d\eta. \quad \text{(VIII.19)}$$

$$v_n = v_{n0} + \frac{1}{4\pi} \left[\int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{d\Gamma}{d\eta} a^2 \frac{df}{d\eta} + \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \Gamma(\eta) f_2(\zeta, \eta) \, d\eta \right]. \quad \text{(VIII.20)}$$

$$v_n = v_{n0} + \frac{1}{4\pi} \left[\int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{d\Gamma}{d\eta} \left[a^2 \frac{df}{d\eta} + \int\limits_{\eta}^{c} f_2(\zeta, \eta) \, d\eta \right] d\eta \right]. \quad \text{(VIII.21)}$$

It will be illustrated in Ch. XI that the problem dealing with a foil moving in a three-dimensional flow can easily be solved by this method. [258

For the case of the submerged hydrofoil moving in the three-dimensional flow, the function k(x, y, z) can be presented as follows:

$$k(x, y, z) = \frac{1}{2\pi} \int\limits_{-\pi}^{+\pi} \int\limits_{0}^{\infty} e^{\lambda\omega} \frac{\lambda + \nu \sec^2 \Theta}{\lambda - \nu \sec^2 \Theta} \, d\lambda d\Theta.$$

After computation we obtain

$$f(z, \eta) = \ln R - \ln R_1 - \mathrm{Re}\, 2 \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\nu \sec^2 \Theta [z + \zeta + i(y - \eta) \sin \Theta]} \sec \Theta d\Theta. \quad \text{(VIII.22)}$$

Let us represent f(z, η) in the form of two functions:

$$f(z, \eta) = f_1(z, \eta) + f_2(z, \eta), \quad \text{(VIII.23)}$$
$$f_1(z, \eta) = \ln R - \ln R_1,$$

$$f_2(z, \eta) = -\operatorname{Re} 2\nu \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\nu \sec^2\Theta[z+\zeta+i(y-\eta)\sin\Theta]} \sec\Theta d\Theta,$$

$$R = \sqrt{(y-\eta)^2 + (z-\zeta)^2}, \quad R_1 = \sqrt{(y-\eta)^2 + (z+\zeta)^2}.$$

The function f(z, η) satisfies the Laplace equation with regard to the variables z and η, while the function f_2(y, η) satisfies the Laplace equation with respect to the variables z, η sin θ.

Then we will have

$$v_n = v_{n0} - \frac{1}{4\pi}\left[\int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{d\Gamma}{d\eta}\left[\frac{1}{y-\eta} - \frac{y-\eta}{(y-\eta)^2+(z-\zeta)^2} - \right.\right.$$

$$\left.\left. - \frac{\partial}{\partial\eta\sin\Theta} \operatorname{Re} 2\nu \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\nu\sec^2\Theta[-2h+i(y-\eta)\sin\Theta]} \frac{1}{\sin\Theta} d\Theta\right] d\eta\right] \qquad \text{(VIII.24)}$$

or

$$\Gamma(y) = a(y)a_n v_0\left[\alpha(y) - \frac{1}{4\pi v_0}\int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{d\Gamma}{d\eta}\left(\frac{1}{y-\eta} - \frac{y-h}{(y-\eta)^2+(2h)^2} + \right.\right.$$

$$\left.\left. + \nu\int\limits_0^{\frac{\pi}{2}} \frac{\sec^3\Theta}{\sin\Theta} \cos\nu\sec^2\Theta(y-\eta)\sin\Theta e^{-\nu\sec^2\Theta 2h} d\Theta\right) d\eta\right]. \qquad \text{(VIII.25)}$$ [259

The lifting force and the drag for the submerged hydrofoil are easily determined in terms of Γ(y) by means of formulas (VII.47) and (VII.51):

$$P = \varrho v_0 \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \Gamma(y)\, dy, \qquad \text{(VIII.26)}$$

$$Q = \frac{\varrho}{4\pi}\int\limits_{-B}^{+B} \Gamma(y) \int\limits_{-B}^{+B} \Gamma'(\eta)\left[\frac{1}{(y-\eta)} + G(y-\eta)\right] d\eta dy. \qquad \text{(VIII.27)}$$

8.3. The Solution of the Integro-Differential Equation Applying the τ-Parameter Method

The equation (VIII.15) represents the generalization of the Prandtl equation for the case of the submerged hydrofoil motion and belongs to the singular integro-differential type of equation. The presence of the regular part in the nucleus makes the solution of this equation more difficult and does not permit the use of the aerodynamic methods directly.

The generalization of the aerodynamic methods here results in the absense of the analytical solution, even for the most simple case of the hydrofoil motion with the optimum distribution of circulation and requires numerical calculations for each submergence of the foil and for each mode of hydrofoil motion.

One may use other approaches to develop the methods for evaluating this problem. This is based on the fact that the corresponding problem for a fluid with the infinite boundaries has already been solved, the deviations from this solution are small, and that the theory of perturbations [84] are applicable to this problem.

It is of interest to obtain an analytical solution at least for the simplest shapes of hydrofoils. We can expect that the effect of the hydrofoil shape in the plan view will produce small values (of the second order) in comparison with the effect of the free surface. Then, this solution would be adequate for a number of applied problems and practical calculations.

An asymptotic method of solution is given below. It is based on the utilization of the τ-parameter. For evaluating the equation (VIII.15), let us introduce a submergence parameter

$$\tau = \sqrt{4\bar{H}^2 + 1} - 2\bar{H}.$$

We will seek the solution in the form of the series

$$\Gamma(\bar{y}) = \Gamma_0(\bar{y}) + \tau^2\Gamma_1(\bar{y}) + \tau^4\Gamma_2(\bar{y}) + \ldots \qquad \text{(VIII.28)}$$

Then

$$\Gamma'(\bar{y}) = \Gamma'_0(\bar{y}) + \tau^2\Gamma'_1(\bar{y}) + \tau^4\Gamma'_2(\bar{y}) + \ldots \qquad \text{(VIII.29)}$$ [260

Taking the following expansion of the regular part of the nucleus in the form

$$G(\bar{y}) = \sum_{n=1}^{\infty} G_n(\bar{y})\,\tau^{2n}, \qquad \text{(VIII.30)}$$

we obtain the recurrent equations for determining $\Gamma_{i}(y)$:

$$\frac{2\Gamma_0(\bar{y})\lambda(\bar{y})}{a_{\bar{h}}} = \alpha(\bar{y}) - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}_0(\bar{y})}{\bar{y}-\bar{\eta}}\,d\bar{\eta}, \qquad (\text{VIII.31})$$

$$\frac{2\Gamma_l(\bar{y})\lambda(\bar{y})}{a_{\bar{h}}} = \frac{1}{2\pi}\int_{-1}^{+1}\sum_{n=1}^{l}\Gamma_{l-n}(\bar{\eta})\,G(\bar{y}-\bar{\eta})\,d\bar{\eta} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}_l(\bar{\eta})}{\bar{y}-\bar{\eta}}\,d\bar{\eta}. \qquad (\text{VIII.32})$$

The first equation represents the equation for the finite-span hydrofoil in an infinite flow and its solution will be $\Gamma_0(\bar{y}) = \Gamma_\infty(\bar{y})$. The remaining equations have the structure of a singular integro-differential equation for the finite-span hydrofoil and for their solutions the aerodynamic methods can be used [19, 39].

Thus, solving several equations, we find the asymptotic solution in the form of the series (VIII.28). This series can produce converging results in the entire lower half-plane, since when $0 < \bar{h} < \infty$, $1 > \tau > 0$.

Later on we will need the expansion of the regular part of the nucleus with respect to powers of the parameter τ. This expansion can be written in the following form:

$$G(\bar{y}) = \sum_{s=2,4,6}^{\infty}\tau^s\sum_{p=0}^{\frac{s}{2}-1}\frac{\bar{y}^{s-1-2p}(s-1-p)\ldots(s+1)(-1)^{\frac{s}{2}-p}}{(s-1-2p)!}\,G_{s,p}\left(\frac{\bar{\omega}}{\tau}\right), \qquad (\text{VIII.33})$$

where $G_{s,p}$ is determined by the formula

$$G_{s,p}\left(\frac{\bar{\omega}}{\tau}\right) = 2e^{-\frac{\bar{\omega}}{\tau}}\frac{\left(\frac{\bar{\omega}}{\tau}\right)^{s-p}}{p(s-1-p)\ldots(s+1)}\int_0^\infty e^{-\frac{\bar{\omega}}{\tau}\lambda}\times$$

$$\times(\lambda+1)^{\frac{s}{2}+\frac{1}{2}}\lambda^{\frac{s}{2}-p-\frac{3}{2}}d\lambda - 1.$$

Let us use this method for the case of the submerged hydrofoil motion with the elliptical distribution of circulation. Then, the solution can be written as follows: [261

$$\Gamma_0(\bar{y}) = \frac{4\alpha\Psi}{\lambda\left[1+\frac{2\Psi}{\lambda}\right]}\sqrt{1-\bar{y}^2},$$

$$\Gamma_1(\bar{y}) = -\frac{4\Psi}{\lambda\left[1+\frac{2\psi}{\lambda}\right]}\left[\frac{1}{2\pi}\int_{-1}^{+1}\sum_{s=2,4}^{l}\sum_{n=1}^{l}\times\right.$$

$$\times\Gamma'_{l-n}\sum_{p=0}^{\frac{s}{2}-1}\frac{(s-1-p)\ldots(p+1)(-1)^{\frac{s}{2}-p}}{(s-1-p)!}\times$$

$$\left.\times(\bar{y}-\bar{\eta})^{s-1-2p}G_{s,p}\left[\frac{\bar{\omega}}{\tau}\right]d\bar{\eta}\right]\sqrt{1-\bar{y}^2}\,d\bar{y}. \qquad \text{(VIII.34)}$$

Making computations, we obtain for the first three functions the following:

$$\left.\begin{aligned}
\Gamma_1(\bar{y}) &= -\frac{4\alpha\Psi^2}{\lambda\left(1+\frac{2\psi}{\lambda}\right)}\cdot\frac{1}{\lambda\left(1+\frac{2\psi}{\lambda}\right)}G_{2,0}\sqrt{1-\bar{y}^2}\\
\Gamma_2(\bar{y}) &= -\frac{4\alpha\Psi^2}{\lambda\left(1+\frac{2\psi}{\lambda}\right)}\cdot\frac{2}{\lambda\left(1+\frac{2\psi}{\lambda}\right)}\left(G_{4,1}-\frac{3}{4}G_{4,0}-\right.\\
&\quad\left.-\frac{\Psi}{2\lambda\left(1+\frac{2\psi}{\lambda}\right)}G_{2,0}^2\right)\sqrt{1-\bar{y}^2}\\
\Gamma_3(\bar{y}) &= -\frac{4\alpha\Psi^2}{\lambda\left(1+\frac{2\psi}{\lambda}\right)}\cdot\frac{2}{\lambda\left(1+\frac{2\psi}{\lambda}\right)}\left[\frac{3}{2}G_{6,2}-3G_{6,1}+\right.\\
&\quad+\frac{25}{16}G_{60}-\frac{2G_{2,0}\psi}{\lambda\left[1+\frac{2\psi}{\lambda}\right]}\left(G_{4,1}-\frac{3}{4}G_{4,0}\right)+\\
&\quad\left.+\frac{\psi'^2}{2\lambda^2\left[1+\frac{2\psi}{\lambda}\right]^2}G_{2,0}^3\right]\sqrt{1-y^2}
\end{aligned}\right\}, \qquad \text{(VIII.35)}$$

where $\Psi = \frac{a_h(\bar{y})}{a_\infty}$ is the function which defines the effect [262

of the free surface on the angular coefficient in the expression $C_y = f(\alpha)$ for an infinite-span hydrofoil.

Designating the angle of the flow downwash by the series

$$\Delta\alpha_i = \Delta\alpha_{i0} + \tau^2\Delta\alpha_{i1} + \tau^4\Delta\alpha_{i2} + \ldots, \qquad \text{(VIII.36)}$$

we obtain

$$\left.\begin{aligned}
\Delta\alpha_{i0} &= \frac{2\alpha\Psi}{\lambda\left(1+\frac{2}{\lambda}\Psi\right)}; \qquad \Delta\alpha_{i1} = \frac{2\alpha\Psi}{\lambda\left(1+\frac{2}{\lambda}\Psi\right)}G_{2,0}\\
\Delta\alpha_{i2} &= \frac{2\alpha\Psi}{\lambda\left(1+\frac{2}{\lambda}\Psi\right)}\left(G_{4,1} - \frac{3}{4}G_{4,0} - \frac{\Psi}{2\lambda\left(1+\frac{2}{\lambda}\Psi\right)}G_{20}^2\right)\\
\alpha\Delta_{i3} &= \frac{2\alpha\Psi}{\lambda\left(1+\frac{2}{\lambda}\Psi\right)}\left(\frac{3}{2}G_{6,2} - 3G_{6,1} + \frac{25}{16}G_{6.0} - \right.\\
&\left. - \frac{2G_{6,0}\psi}{\lambda\left(1+\Psi\frac{2}{\lambda}\right)}\left(G_{4,1} - \frac{3}{8}G_{4,0}\right) + \frac{\Psi^2}{2\lambda^2\left(1+\Psi\frac{2}{\lambda}\right)^2}G_{2,0}^2\right)
\end{aligned}\right\}. \qquad \text{(VIII.37)}$$

Let us write the basic formulas for calculating the lift and the inductive drag in the following form:

$$C_{y\bar{h}} = \frac{\Psi a_\infty}{1 + \frac{\Psi a_\infty}{\pi\cdot\lambda}\zeta_F}(\alpha_0 + \alpha_k - \Delta\alpha_0), \qquad \text{(VIII.38)}$$

$$C_{xi} = \frac{C_{y\bar{h}}^2}{\pi\lambda}\zeta_F. \qquad \text{(VIII.39)}$$

For the correction in the first approximation we have

$$\zeta_F = \zeta_F(\bar{h}_1 \mathrm{Fr}), \qquad \text{(VIII.40)}$$

$$\begin{aligned}
\zeta = {} & 1 + 0{,}5G_{2,0}\tau^2 + (G_{4,1} - 0{,}75G_{4,0})\tau^4 + (1{,}5G_{6,2} + 3G_{6,1} + \\
& + 1{,}5625G_{6,2})\tau^6 + (2G_{8,3} - 7{,}5G_{8,2} + 9{,}375G_{8,1} - 3{,}8281G_{8,0})\tau^8 + \\
& + (2{,}5G_{10,4} - 15G_{10,3} + 32{,}8125G_{10,2} - 30{,}6247G_{10,1} + 10{,}3359G_{10,0})\tau^{10} + \\
& + (3G_{12,5} - 26{,}25G_{12,4} + 87{,}5G_{12,3} - 137{,}8G_{12,2} + 103{,}359G_{12,1} - \\
& - 29{,}7773G_{12,0})\tau^{12} + \ldots
\end{aligned}$$

When $\omega \to 0$ [263

$$\zeta = 1 + 0{,}5\tau^2 + 0{,}25\tau^4 + 0{,}0625\tau^6 + 0{,}0469\tau^8 + 0{,}0257\tau^{10} + 0{,}0188\tau^{12}. \qquad \text{(VIII.41)}$$

where

$$\tau = \sqrt{4\left(\frac{4\bar{h}}{\pi\lambda}\right)^2 + 1} - 2\left(\frac{4\bar{h}}{\pi\lambda}\right); \qquad \bar{h} = \frac{h}{b_0};$$

b_0 is the central chord of the hydrofoil. The more accurate value for the correction ζ_{0F} is determined by the formula

$$\zeta_{0F}=\frac{1+M\dfrac{1}{\left(1+\dfrac{2\Psi}{\lambda}\right)}}{1-\dfrac{2\Psi}{\lambda\left(1+\dfrac{2\Psi}{\lambda}\right)}M}, \qquad \text{(VIII.42)}$$

where

$$M=0{,}5G_{2,0}\tau^2+\left[G_{4,1}-0{,}75G_{4,0}-\frac{\Psi G_{2,0}^2}{2\lambda\left(1+\dfrac{2\Psi}{\lambda}\right)}\right]\tau^4+$$

$$+\left[\frac{3}{2}G_{6,2}+3G_{6,1}+\frac{25}{16}G_{6,0}-\frac{2\Psi G_{2,0}}{\lambda\left(1+\dfrac{2\Psi}{\lambda}\right)}\times\right.$$

$$\left.\times\left(G_{4,1}-\frac{3}{4}G_{4,0}-\frac{\Psi}{2\lambda\left(1+\dfrac{2\Psi}{\lambda}\right)}\right)\right]\tau^6.$$

For the series of functions $G_{s,p}(\lambda)$ the following approximate formulas were obtained:

$$G_{2,0}=-1+e^{-\lambda}\left(2+3\lambda+\frac{5}{8}\lambda^2\right)+E_i(-\lambda)\left[-\frac{3}{4}\lambda^2+\frac{5}{8}\lambda^3\right]$$

$$G_{4,0}=-1+e^{-\lambda}\left(2+\frac{5}{3}\lambda+\frac{5}{8}\lambda^2+\frac{5}{48}\lambda^3-\frac{7}{768}\lambda^4\right)+$$

$$+E_i(-\lambda)\left(\frac{5}{384}\lambda^4-\frac{7}{768}\lambda^5\right)$$

$$G_{4,1}=-1+e^{-\lambda}\left(2+\frac{5}{2}\lambda+\frac{15}{8}\lambda^2+\frac{35}{128}\lambda^3\right)+$$

$$+E_i(-\lambda)\left(-\frac{5}{16}\lambda^3+\frac{35}{128}\lambda^4\right)$$

$$G_{6,0}=-1+e^{-\lambda}\left(2+\frac{7}{5}\lambda+\frac{7}{16}\lambda^2+\frac{7}{96}\lambda^3+\frac{7}{1536}\lambda^4+\right.$$ [264

$$\left.+\frac{7}{15360}\lambda^5+\frac{3}{40960}\lambda^6\right)+E_i(-\lambda)\left(-\frac{1}{61440}\lambda^6+\frac{3}{40960}\lambda^7\right]$$

$$G_{6,1}=-1+e^{-\lambda}\left(2+\frac{7}{4}\lambda+\frac{35}{48}\lambda^2+\frac{35}{192}\lambda^3+\frac{35}{1536}\lambda^4-\right.$$

$$\left.-\frac{7}{4096}\lambda^5\right)+E_i(-\lambda)\left(\frac{7}{3072}\lambda^5-\frac{7}{4096}\lambda^6\right)$$

$$G_{6,2}=-1+e^{-\lambda}\left(2+\frac{7}{3}\lambda+\frac{35}{24}\lambda^2+\frac{35}{48}\lambda^3+\frac{21}{256}\lambda^4\right)+ \qquad \text{(VIII.43)}$$

$$+E_i(-\lambda)\left(-\frac{35}{384}\lambda^4+\frac{21}{256}\lambda^5\right)$$

$$G_{8,0}=-1+e^{-\lambda}\left(2+\frac{9}{7}\lambda+\frac{3}{8}\lambda^2+\frac{1}{16}\lambda^3+\frac{3}{512}\lambda^4-\right.$$

$$\left.-\frac{1}{5120}\lambda^5-\frac{1}{122880}\lambda^6+\frac{1}{513440}\lambda^7\right)+E_i(-\lambda)\frac{1}{1835008}\lambda^8$$

$$G_{8,1}=-1+e^{-\lambda}\left(2+\frac{3}{2}\lambda+\frac{21}{40}\lambda^2+\frac{7}{64}\lambda^3+\frac{7}{512}\lambda^4+\right.$$

$$\left.+\frac{7}{10240}\lambda^5-\frac{7}{122880}\lambda^6\right)-\frac{1}{81920}\lambda^7E_i(-\lambda)$$

$$G_{8,2}=-1+e^{-\lambda}\left(2+\frac{9}{5}\lambda+\frac{63}{80}\lambda^2+\frac{7}{32}\lambda^3+\frac{21}{512}\lambda^4+\right.$$

$$\left.+\frac{21}{5120}\lambda^5-\frac{11}{40960}\lambda^6\right)+E_i(-\lambda)\left(\frac{7}{20480}\lambda^6-\frac{11}{40960}\lambda^7\right]$$

$$G_{8,3}=-1+e^{-\lambda}\left(2+\frac{9}{4}\lambda+\frac{21}{16}\lambda^2+\frac{35}{64}\lambda^3+\frac{105}{512}\lambda^4+\right.$$

$$\left.+\frac{77}{4096}\lambda^5\right)+E_i(-\lambda)\left(-\frac{21}{1024}\lambda^5+\frac{77}{4096}\lambda^6\right)$$

Using the expressions (VIII.43), the evaluation of the function $G_{s,p}(\lambda)$ for the values of the argument $0 \leqslant \leqslant \lambda \leqslant 15$ has been performed. The results are given in Table 5 and in Figures 16-18.

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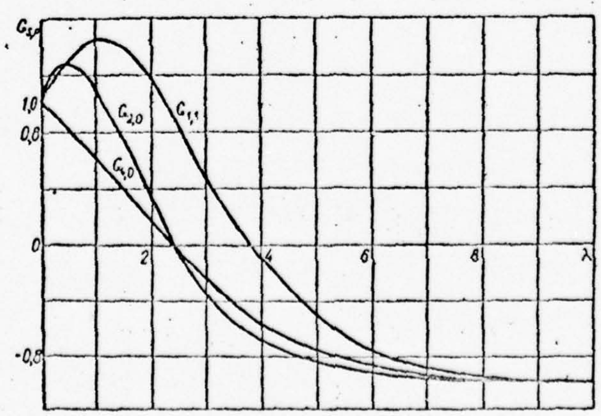

Fig. 16

Table 5

λ	$G_{2,0}$	$G_{4,0}$	$G_{4,1}$	$G_{6,0}$	$G_{6,1}$	$G_{6,2}$	$G_{8,0}$	$G_{8,1}$	$G_{8,2}$	$G_{8,3}$
0	1	1	1	1	1	1	1	1	1	1
0,01	1,0102	0,9965	1,0050	0,9940	0,9975	1,0045	0,9928	0,9949	0,9930	1,0025
0,02	1,0204	0,9933	1,0101	0,9880	0,9950	1,0067	0,9897	0,9904	0,9960	1,0050
0,03	1,0311	0,9899	1,0153	0,9819	0,9925	1,0100	0,9786	0,9850	0,9946	1,0075
0,04	1,0408	0,9866	1,0205	0,9761	0,9900	1,0135	0,9716	0,9800	0,9920	1,0100
0,05	1,0490	0,9832	1,0258	0,9700	0,9874	1,0111	0,9644	0,9750	0,9900	1,0126
0,06	1,0613	0,9894	1,0314	0,9842	0,9849	1,0205	0,9575	0,9701	0,9880	1,0152
0,07	1,0708	0,9765	1,0367	0,9582	0,9824	1,0239	0,9504	0,9651	0,9859	1,0178
0,08	1,0805	0,9730	1,0424	0,9522	0,9800	1,0274	0,9434	0,9601	0,9839	1,0204
0,09	1,0898	0,9696	1,0479	0,9462	0,9773	1,0310	0,9364	0,9552	0,9819	1,0230
0,1	1,0992	0,9605	1,0536	1,9403	0,9748	1,0345	0,9294	0,9503	0,9799	1,0256
0,2	1,1522	0,9314	1,1126	0,8814	0,9491	1,0730	0,8606	0,9010	0,9595	1,0712
0,3	1,2356	0,8956	1,1733	0,8234	0,9229	1,1130	0,7936	0,8522	0,9388	1,0801
0,4	1,2681	0,8587	1,2320	0,7661	0,8963	1,1561	0,7283	0,8041	0,9178	1,1118
0,5	1,2787	0,8205	1,2885	0,7096	0,8680	1,2017	0,6849	0,7564	0,8966	1,1442
0,6	1,2802	0,8360	1,3363	0,6540	0,8410	1,2488	0,6029	0,7093	0,8749	1,1784
0,7	1,2416	0,7375	1,3762	0,5954	0,8092	1,2926	0,5399	0,6534	0,8495	1,2100
0,8	1,2063	0,6990	1,4142	0,5452	0,7831	1,3426	0,4840	0,6166	0,8305	1,2517
0,9	1,1564	0,6565	1,4389	0,4924	0,7531	1,3883	0,4273	0,5711	0,8078	1,2903
1	1,0968	0,6134	1,4504	0,4524	0,7224	1,4512	0,3720	0,5262	0,7845	1,3300
1,2	0,9578	0,5237	1,4601	0,3387	0,6588	1,5065	0,2660	0,4382	0,7364	1,4109
1,4	0,8595	0,4324	1,4250	0,2413	0,5924	1,5658	0,1663	0,3280	0,6591	1,4899

Table 5 (cont.)

1,6	0,6408	0,3293	1,3623	0,1479	0,5821	1,6000	0,0725	0,2697	0,6136	1,5630
1,8	0,4793	0,2482	1,2743	0,0589	0,5525	1,6126	—0,0154	0,1894	0,5791	1,6284
2	0,3232	0,1569	1,1667	0,0260	0,3800	1,6010	—0,0980	0,1120	0,5220	1,6810
2,2	0,1755	0,0684	1,0239	—0,1039	0,3319	1,5677	—0,1748	0,0374	0,4840	1,7081
2,4	—0,0066	—0,0192	0,9132	—0,1816	0,2323	1,5126	—0,2467	—0,1354	0,4014	1,7404
2,6	—0,0872	—0,1098	0,7766	—0,2143	0,1582	1,3659	—0,3132	—0,1518	0,3381	1,7442
2,8	—0,2021	—0,1849	0,5169	—0,3189	0,0849	1,2850	—0,37565	—0,1682	0,2736	1,7306
3	—0,3049	—0,2528	0,5018	—0,3806	0,0128	1,2500	—0,4330	—0,2304	0,2189	1,6998
3,2	—0,3956	—0,3238	0,3687	—0,4378	—0,0577	1,1398	—0,4859	—0,3014	0,1416	1,6522
3,4	—0,4774	—0,3893	0,2410	—0,4912	—0,1261	1,0199	—0,5351	—0,3453	0,0750	1,5896
3,6	—0,5495	—0,4653	0,1199	—0,5412	—0,1921	0,8921	—0,5817	—0,3981	0,0088	1,5137
3,8	—0,6110	—0,5039	0,0048	—0,5855	—0,2553	0,7737	—0,6218	—0,4476	—0,0557	1,4266
4	—0,6668	—0,5564	—0,0992	—0,6278	—0,3156	0,6421	—0,6607	—0,4941	—0,1208	1,3303
4,2	—0,7145	—0,6038	—0,1984	—0,6656	—0,3737	0,5158	—0,6955	—0,5381	—0,1853	1,2230
4,4	—0,7538	—0,6456	—0,2850	—0,6993	—0,4751	0,4077	—0,7266	—0,5781	—0,2447	1,1149
4,6	—0,7892	—0,6837	—0,3674	—0,7327	—0,4773	0,2822	—0,7574	—0,6157	—0,3038	1,0002
4,8	—0,7845	—0,7214	—0,4412	—0,7599	—0,5248	0,1823	—0,7824	—0,6639	—0,3601	0,8846
5	—0,8454	—0,7536	—0,5075	—0,7840	—0,5689	0,0784	—0,8048	—0,7830	—0,3506	0,7670
6	—0,9378	—0,8676	—0,7474	—0,8804	—0,7438	—0,3841	—0,8946	—0,8673	—0,7359	0,2120
7	—0,9[illegible]89	—0,9336	—0,8779	—0,9353	—0,8596	—0,6245	—0,9453	—0,8887	—0,9101	—0,2434
8	—0,9[illegible]79	—0,9693	—0,9439	—0,9659	—0,9211	—0,7242	—0,9732	—0,9389	—0,9475	—0,5446
9	—0,9[illegible]52	—0,9843	—0,9726	—0,9822	—0,9590	—0,8511	—0,9871	—0,9670	—0,9541	—0,7298
10	—0,9[illegible]84	—0,9927	—0,9781	—0,9907	—0,9792	—0,9225	—0,9942	—0,9828	—0,9666	—0,8147
11	—0,9[illegible]37	—0,9971	—0,9895	—0,9951	—0,9904	—0,9596	—0,9974	—0,9919	—0,9773	—0,8609
12	—0,9987	—0,9994	—0,9954	—0,9976	—0,9971	—0,9808	—0,9990	—0,9956	—0,9911	—0,8991
13	—0,9997	—0,9998	—0,9981	—0,9989	—0,9989	—0,9916	—0,9996	—0,9975	—0,9963	—0,9714
14	—1	—0,9999	—0,9999	0,9992	—0,9994	—0,9945	—0,9998	—0,9987	—0,9998	—0,9997
15	—1	—1	—1	—1	—1	—1	—1	—1	—1	—1

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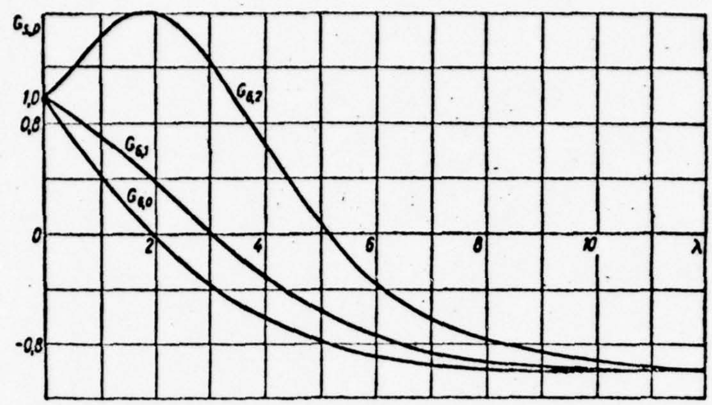

Fig. 17

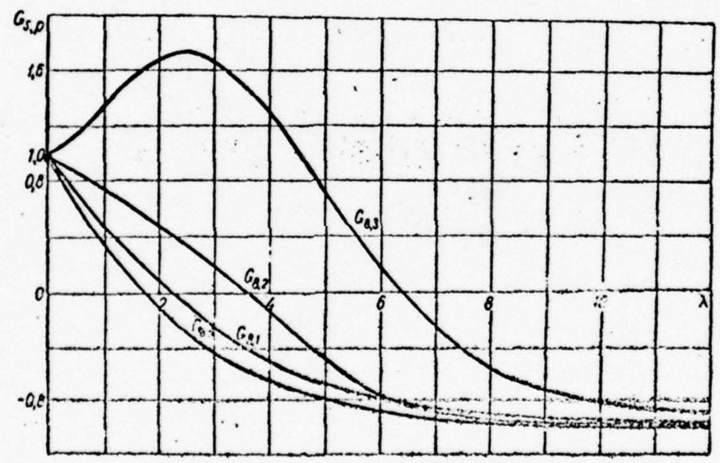

Fig. 18

The function ζ for a number of values of $\frac{\omega}{l}$ and h have [268
been computed by formulas (VIII.40) and (VIII.42); these results are expressed graphically in Fig. 19 and can be used in practical calculations.

In Fig. 20, the values of ζ, obtained by formula (VIII.41), are represented by curve 1 and the aerodynamic results for a biplane with identical wings [39] are shown by curve 2. These results are in good agreement.

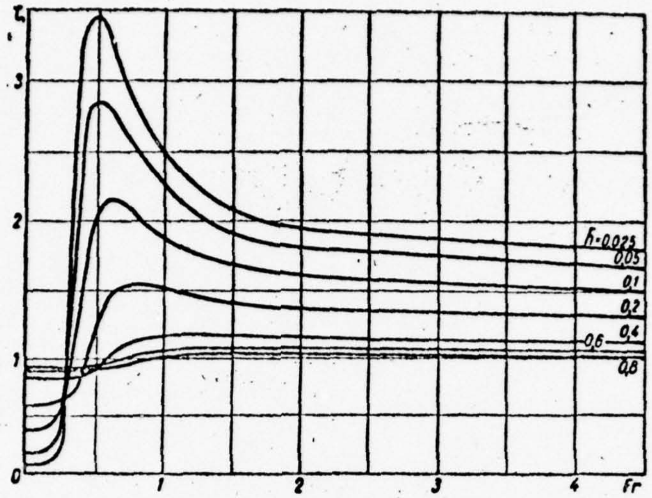

Fig. 19

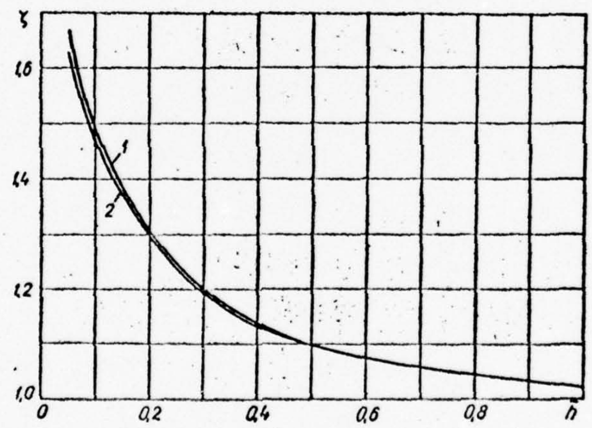

Fig. 20

8.4. Wing Drag with a Constant Circulation Along the Span* [269

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*This part was written together with A. I. Yukhimenko.

The simplest Π-shaped vortex configuration for the wing may also be used to obtain the analytical results for the submerged hydrofoil.

Let us replace a submerged hydrofoil with a horseshoe-shaped vortex. Let us determine the induced drag of such a hydrofoil by formula (VIII.24). When $Fr \to \infty$, $\zeta = 1 + \sigma$,

where σ is determined from the solution of the biplanar motion problem consisting of two Π-shaped vortices.

Function σ may apparently be determined as follows:

$$\sigma = \frac{\Delta Q_h}{Q_\infty},$$

where ΔQ_h - the additional drag on the foil under the free surface;

Q_∞ - the induced drag on the foil in an infinite flow.

Let us determine Q_∞ from the complex potential for the flow in the yz plane as follows:

$$F(t) = \frac{\Gamma}{2\pi i} \ln \frac{t-R}{t+R}, \quad t = y + iz,$$

$$F'(t) = \frac{\Gamma}{2\pi i} \frac{2R}{t^2 - R^2} = v - i\omega$$

or along the span t = y

$$\omega = \frac{\Gamma}{2\pi} \frac{2R}{y^2 - R},$$

from where

$$Q_\infty = \frac{1}{2} \varrho \int_{-\frac{l}{2}}^{+\frac{l}{2}} \Gamma \omega dy = \frac{\varrho \Gamma^2}{4\pi} 2 \ln \frac{l+R}{l-R} \qquad \text{(VIII.44)}$$

and for $\frac{R}{l} = \varkappa_0 = 1{,}04$, according to formula (VIII.44), we obtain $Q_\infty = \frac{2\varrho\Gamma^2}{\pi}$.

For ΔQ_h we have

$$\Delta Q_h = \frac{\varrho \Gamma^2}{4\pi} \int_{-1}^{+1} \overline{W} dy, \qquad \text{(VIII.45)}$$

where $\overline{W} = \frac{2\pi}{\Gamma} W$.

We obtain the expression $\sigma = \frac{1}{8} \int_{-1}^{+1} \overline{W} dy.$ (VIII.46) [270

Because each trailing vortex produces the same induced drag, let us examine one vortex only and multiply the answer by two.

The vertical velocity $\bar{W}$ of the axis of the single vortex will be determined by the expression (VIII.19):

$$\bar{W} = 2\mathrm{Re}\left(2\omega i \int_0^\infty \frac{v+1}{v} e^{-\omega i(v+1)\left[(\bar{y}-\varkappa)\sqrt{\frac{v}{v+1}} - 4i\bar{H}\right]} dv - \frac{1}{\bar{y} - \varkappa_0 - 4i\bar{H}}\right).$$

Then

$$\sigma = \frac{1}{4}\int_{-1}^{+1} \mathrm{Re}\left(2\omega i \int_0^\infty \frac{v+1}{v} e^{-\omega i(v+1)\left[(\bar{y}-\varkappa)\sqrt{\frac{v}{v+1}} - 4i\bar{H}\right]} dv - \right.$$
$$\left. - \frac{1}{\bar{y} - \varkappa_0 - 4i\bar{H}}\right) d\bar{y}.$$

If we determine the integrand function by (VIII.33), then for σ we will get the following:

$$\sigma = \frac{1}{4}\int_{-1}^{+1} \sum_{s=2,4,6\ldots}^{\infty} \tau^s \sum_{p=0,1\ldots}^{\frac{s}{2}-1} \frac{(\bar{y} - \varkappa_0)^{s-1-2p}}{(s-1-2p)!} \times$$
$$\times (s-1-p)\ldots(p+1)(-1)^{\frac{s}{2}-1-p} G_{sp} d\bar{y},$$

or

$$\sigma = \frac{1}{4}\sum_{s=2,4,6\ldots}^{\infty} \tau^s \sum_{p=0,1}^{\frac{s}{2}-1} \frac{[(1-\varkappa_0)^{s-2p} + (-1)^{s-2p+1}(1+\varkappa_0)^{s-2p}]}{(s-2p)!} \times$$
$$\times (s-1-p)\ldots(p+1)(-1)^{\frac{s}{2}-1-p} G_{sp}.$$

Assuming $\varkappa_0 = 1.04$ we obtain

$$\sigma = 0{,}5204\tau^2 G_{2,0} - \tau^4 (1{,}0824 G_{4,0} - 1{,}0408 G_{4,1}) + \tau^6 (3{,}0031 G_{6,0} -$$
$$- 4{,}3297 G_{6,1} + 1{,}5612 G_{6,2}) - \tau^8 (9{,}37327 G_{8,0} - 18{,}0186_{8,1} +$$
$$+ 10{,}8243 G_{8,2} - 2{,}0816 G_{8,3}) + \tau^{10} (31{,}2062 G_{10,0} - 74{,}9862 G_{10,1} +$$
$$+ 63{,}0651 G_{10,2} - 21{,}6486 G_{10,3} + 2{,}602 G_{10,4}) + \ldots \qquad \text{(VIII.47)}$$

For the limiting case of high-velocity motion ($Fr_b \to$ [271
$\to \infty$) $G_{s,p} \to 1$, and then

$$\sigma = 0{,}5204\tau^2 - 0{,}0416\tau^4 + 0{,}2346\tau^6 - 0{,}0974\tau^8 + 0{,}23851\tau^{10} + \ldots \qquad \text{(VIII.48)}$$

The Prandtl data for the evaluation of σ, obtained under the assumption of the elliptical distribution of circulation (curve 1, Fig. 21) [39], and the data received from formula (VIII.48) (curve 2), are nearly the same.

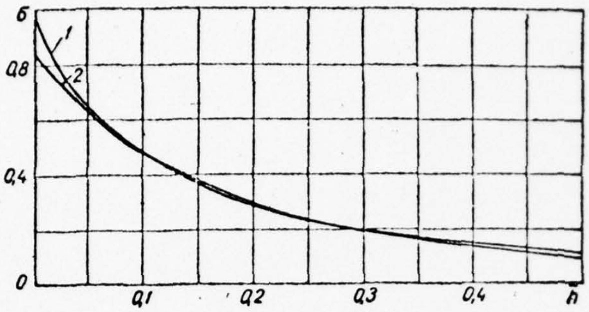

Fig. 21

8.5. The Velocity Potential and the Integral Equation for the Submerged Vertical Hydrofoil

Let us examine the problem of motion of a submerged vertical hydrofoil under a free surface. Using formulas (VII.3), (VII.33) and (VIII.6) the velocity potential of the vertical foil is found to be as follows:

$$\varphi = -\frac{1}{4\pi}\iint_s \gamma(\Theta)\left(\int_\infty^x \frac{\partial}{\partial\eta}\frac{1}{r}\,d\tau - \mathrm{Re}\,2\nu i\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\theta}(z+b+i\omega)}\times\right.$$
$$\times\sec^3\Theta\sin\Theta\,d\Theta + \frac{\mathrm{Re}}{2\pi}\int_{-\pi}^{\pi}\frac{d\Theta}{\cos\Theta}\int_0^\infty e^{\lambda(z+\zeta+i\omega)}\frac{\sin\Theta\,(\lambda+\nu\sec^2\Theta)}{(\lambda-\nu\sec^2\Theta)}\,d\lambda -$$
$$\left.-\int_0^\infty e^{\lambda(z+\zeta)}\sin\lambda\,(y-\eta)\,d\lambda\right)ds. \qquad \text{(VIII.49)}$$

The corresponding components of the induced velocities along the axes are determined in the following manner: [272

$$\varphi_y = -\frac{1}{4\pi}\iint_s \gamma(0)\left(\frac{\partial}{\partial y}\int_\infty^y \frac{\partial}{\partial\eta}\frac{1}{r}\,d\tau + \mathrm{Re}\,2\nu^2\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\theta}(z+\zeta+i\omega)}\times\right.$$
$$\times\sec^5\Theta\sin^2\Theta\,d\Theta + \mathrm{Re}\,\frac{i}{2\pi}\int_{-\pi}^{+\pi}\frac{d\Theta}{\cos\Theta}\int_0^\infty \frac{e^{\lambda[(z+\zeta)+i\omega]}\lambda\sin^2\Theta\,(\lambda+\nu\sec^2\Theta)}{\lambda-\nu\sec^2\Theta}\times$$
$$\left.\times\,d\lambda - \int_0^\infty e^{\lambda(z+\zeta)}\lambda\cos\lambda\,(y-\eta)\,d\lambda\right)ds. \qquad \text{(VIII.50)}$$

$$\varphi_x = -\frac{1}{4\pi}\iint_S \gamma(0)\left(\frac{\partial}{\partial x}\int_{\infty}^{x}\frac{\partial}{\partial \eta}\frac{1}{2}\,d\tau + \frac{\mathrm{Re}}{2\pi} i \times \right.$$

$$\left. \times \int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} \frac{e^{\lambda[(z+\zeta)+i(x-\xi)\cos\Theta+i(y-\eta)\sin\Theta]}\sin\Theta\,(\lambda+\nu)\sec^2\Theta}{\lambda-\nu\sec^2\Theta}\,d\lambda\right). \qquad \text{(VIII.51)}$$

Using the assumptions of the lifting line theory, one obtains the expressions for the induced velocities as follows:

$$\varphi_x = 0,$$

$$\varphi_y = \frac{1}{4\pi}\int_{h_1}^{h_2}\Gamma(\zeta)\left[\frac{1}{(z-\zeta)^2} + \int_0^{\infty} e^{\lambda(z+\zeta)}\lambda\cos\lambda(y-\eta)\,d\lambda - \right.$$

$$\left. - \mathrm{Re}\,4\nu^2\int_0^{\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\Theta}(z+\zeta)+i(y-\eta)\sin\Theta}\sec^5\Theta\sin^2\Theta\,d\Theta\right]d\zeta. \qquad \text{(VIII.52)}$$

Using the expression (VIII.52), the integro-differential equation for the vertical hydrofoil can be written as follows:

$$\Gamma(z) = v_0 a(z)\, a_h\left[\alpha(z) - \frac{1}{4\pi v_0}\int_{h_1}^{h_2}\Gamma'(\zeta)\left(\frac{1}{z-\zeta} - \frac{1}{z+\zeta} + \right.\right.$$

$$\left.\left. + 4\nu\int_0^{\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\Theta}(z+\zeta)}\sec^3\Theta\sin^2\Theta\,d\Theta\right)d\zeta\right]. \qquad \text{(VIII.53)}$$

This equation may be given a different form: [273

$$\Gamma(z) = v_0 a(z)\, a_n\left[\alpha(z) - \frac{1}{4\pi v_0}\int_{h_1}^{h_2}\Gamma'(\zeta)\left(\frac{1}{z-\zeta} - \int_0^{\infty} e^{\lambda(z+\zeta)}d\lambda + \right.\right.$$

$$\left.\left. + 2\int_{\nu}^{\infty} e^{\lambda(z+\zeta)}\sqrt{1-\frac{\nu}{\lambda}}\,d\lambda\right)\right], \qquad \text{(VIII.54)}$$

hence it follows that in the limiting cases of nucleus motion, the equation will be as follows:

$$k(z,\zeta) = \frac{1}{z-\zeta} \pm \frac{1}{z+\zeta},$$

where the "plus" sign corresponds to the case of $\nu \to 0$, and the "minus" sign to $-\nu \to \infty$.

Instead of the integro-differential equation (VIII. 54) for determining circulation $\Gamma(z)$, G. V. Sobolev suggests using an approximate integral equation which he attempts to solve by means of the successive approximations method, taking the circulation obtained for $\nu \to \infty$ as the zero approximation value. As seen from equation (VIII.54), the sign of the real part of the nucleus changes when $0 < \nu < \infty$, so that the zero approximation value for the arbitrary values of ν will not be appropriate. Good results will only be obtained for large values of ν.

8.6. Determination of the Velocity Potential Using the Integral Formulas

Let us examine the velocity potential of the submerged hydrofoil using the formula (VII.43).

Let us write the integral representation for $\frac{1}{r_1}$ using the expression (VII.34):

$$\frac{1}{r_1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{k^2+\lambda^2}} \int_{-\infty}^{+\infty} e^{\sqrt{k^2+\lambda^2}(z+\zeta)} e^{i(x-\xi)} e^{\lambda(y-\eta)k}\, d\lambda. \qquad \text{(VIII.55)}$$

Similarly, from the differential equation (VIII.4) we get the following:

$$G = \frac{1}{r} - \frac{1}{r_1} - \frac{\nu}{\pi} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{k^2+\lambda^2}} \int_{-\infty}^{+\infty} \frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)} e^{i(x-\xi)\lambda} e^{i(y-\eta)k}\, d\lambda}{\left(\frac{\lambda^2}{\sqrt{\lambda^2+k^2}} + \frac{i\lambda}{\sqrt{\lambda^2+k^2}} - \nu\right)}, \qquad \text{(VIII.56)}$$

or

$$G = \frac{1}{r} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{k^2+\lambda^2}} \int_{-\infty}^{\infty} \frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)} e^{i(x-\xi)\lambda} e^{i(y-\eta)k} \left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}} + \nu\right)}{\frac{\lambda^2}{\sqrt{k^2+\lambda^2}} + \frac{i\lambda}{\sqrt{k^2+\lambda^2}} - \nu}\, d\lambda. \qquad \text{(VIII.57)}$$ [274

It follows from expression (VIII.57) that

$$N(\lambda, k) = -\frac{1}{2} \frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)} e^{i(y-\eta)k} \left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}} + \nu\right)}{\sqrt{k^2+\lambda^2}},$$

$$Q(\lambda, k) = \frac{\lambda^2}{\sqrt{k^2+\lambda^2}} - \nu,$$

$$Q'(\lambda, k) = \frac{\lambda(2k^2+\lambda^2)}{(k^2+\lambda^2)^{\frac{3}{2}}},$$

$$N(\theta, k) = -\frac{1}{2}\nu\frac{e^{|k|(z+\zeta)}}{|k|}e^{i(y-\eta)k}.$$

$$Q(\theta, k) = -\nu.$$

The zero values of function $Q(\lambda, k)$ are determined from the equation $Q(\lambda, k) = 0$. Then, using formula (VII. 43),

$$\varphi = -\frac{1}{4\pi}\iint_s \gamma(\theta)\left\{\int_\infty^x \frac{\partial}{\partial\zeta}\frac{1}{r}d\tau + \frac{\partial}{\partial\zeta}\left[-\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{|k|(z+\zeta)}e^{i(y-\eta)k}}{|k|}dk + \right.\right.$$

$$+ 2\nu\int_{-\infty}^{\infty}\frac{e^{\sqrt{\lambda_0^2+k^2}(z+\zeta)}e^{i(y-\eta)k}(k^2+\lambda_0^2)}{\lambda_0^2(2k^2+\lambda_0^2)}\cos\lambda_0(x-\xi)\,dk +$$

$$\left.\left. + \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}\frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)}e^{i(y-\eta)k}e^{i(x-\xi)\lambda}\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}+\nu\right)}{\lambda\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}-\nu\right)}d\lambda\right]\right\}ds. \quad \text{(VIII.58)}$$

When $x \to -\infty$, potential φ has the following form: [275

$$\varphi_{-\infty} = -\frac{1}{4\pi}\iint_s \gamma(\theta)\left\{-\int_{-\infty}^{\infty}\frac{\partial}{\partial\zeta}\frac{1}{r} + \frac{\partial}{\partial\zeta}\left[-\int_{-\infty}^{\infty}\frac{e^{|k|(z+\zeta)}}{|k|}e^{i(y-\eta)k}dk + \right.\right.$$

$$\left.\left. + 4\nu\int_{-\infty}^{+\infty}\frac{e^{\sqrt{\lambda_0^2+k^2}(z+\zeta)}e^{i(y-\eta)}(k^2+\lambda_0^2)}{\lambda_0^2(2k^2+\lambda_0^2)}\cos(x-\xi)\,dk\right]\right\}ds. \quad \text{(VIII.59)}$$

The potential of the foil motion near a solid wall is determined from equation (VIII.58) with $\nu \to \infty$

$$\varphi_{\nu=\infty} = -\frac{1}{4\pi}\iint_s \gamma(\theta)\left\{\int_\infty^x \frac{\partial}{\partial\zeta}\frac{1}{r}d\tau + \frac{\partial}{\partial\zeta}\left[-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{e^{|k|(z+\zeta)}e^{i(y-\eta)}dk}{|k|} - \right.\right.$$

$$\left.\left. - \frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}\frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)}e^{i(y-\eta)k}e^{i(x-\xi)\lambda}}{\lambda}d\lambda\right]\right\}ds. \quad \text{(VIII.60)}$$

With $x \to -\infty$

$$\varphi_{-\infty} = -\frac{1}{4\pi}\iint_s \gamma(\theta)\left\{-\int_{-\infty}^{+\infty}\frac{\partial}{\partial\zeta}\cdot\frac{1}{r}d\tau - \frac{\partial}{\partial\zeta}\int_{-\infty}^{+\infty}\frac{e^{|k|(z+\zeta)}e^{i(y-\eta)k}}{|k|}dk. \quad \text{(VIII.61)}\right.$$

For small values, $\nu\lambda_0^2 \approx \nu|k|$. Then the asymptotic value of the potential for small values of ν is as follows:

$$\varphi = -\frac{1}{4\pi}\iint_{-s}\gamma(\theta)\left\{\int_{\infty}^{x}\frac{\partial}{\partial\zeta}\cdot\frac{1}{r}\,d\tau + \frac{\partial}{\partial\zeta}\left[-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{e^{|k|(z+\zeta)}e^{i(y-\eta)k}}{|k|}\,dk + \right.\right.$$
$$+\int_{-\infty}^{+\infty}\frac{e^{|k|(z+\zeta)}e^{i(y-\eta)k}}{|k|}\cos\sqrt{\nu|k|}\,(x-\xi)\,dk +$$
$$\left.\left.+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}\frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)}e^{i(y-\eta)k}e^{i(x-\xi)\lambda}}{\lambda}\,d\lambda\right]\right\}ds. \qquad \text{(VIII.62)}$$

With $x \to -\infty$

$$\varphi_{-\infty} = -\frac{1}{4\pi}\iint_{s}\gamma(\theta)\left\{-\int_{-\infty}^{+\infty}\frac{\partial}{\partial\zeta}\cdot\frac{1}{r}\,d\tau + \frac{\partial}{\partial\zeta}\left[-\int_{-\infty}^{+\infty}\frac{e^{|k|(z+\zeta)}e^{i(y-\eta)k}}{|k|}\,dk + \right.\right.$$
$$\left.\left.+2\int_{-\infty}^{+\infty}\frac{e^{|k|(z+\zeta)}e^{i(y-\eta)k}}{|k|}\cos\sqrt{\nu|k|}\,(x-\xi)\,dk\right]\right\}ds. \qquad \text{(VIII.63)}$$

Let us determine the components of the induced velocities from the following equations: [276

$$\varphi_z = -\frac{1}{4\pi}\iint_{s}\gamma(\theta)\left(\frac{\partial}{\partial z}\int_{\infty}^{x}\frac{\partial}{\partial\zeta}\cdot\frac{1}{r}\,d\tau - \frac{1}{2}\int_{-\infty}^{+\infty}e^{|k|(z+\zeta)}|k|\,e^{i(y-\eta)k}\,dk + \right.$$
$$+2\nu\int_{-\infty}^{+\infty}\frac{e^{\sqrt{\lambda_0^2+k^2}(z+\zeta)}e^{i(y-\eta)k}(k^2+\lambda_0^2)^2}{\lambda_0^2(2k^2+\lambda_0^2)}\cos\lambda_0(x-\xi)\,dk +$$
$$\left.+\frac{i}{2\pi}\int_{-\infty}^{+\infty}dk\int_{-\infty}^{+\infty}\frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)}e^{i(y-\eta)k}e^{i(x-\xi)\lambda}\times\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}+\nu\right)\sqrt{k^2+\lambda^2}}{\lambda\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}-\nu\right)}\,d\lambda\right)ds. \qquad \text{(VIII.64)}$$

$$\varphi_x = -\frac{1}{4\pi}\iint_{s}\gamma(\theta)\left(\frac{\partial}{\partial x}\int_{\infty}^{x}\frac{\partial}{\partial\zeta}\cdot\frac{1}{r}\,d\tau - 2\nu\int_{-\infty}^{+\infty}\frac{e^{\sqrt{\lambda_0^2+k^2}(z+\zeta)}}{\lambda_0^2(2k^2+\lambda_0^2)}\times\right.$$
$$\times\frac{e^{i(y-\eta)k}(k^2+\lambda_0^2)^{\frac{3}{2}}}{1}\sin\lambda_0(x-\xi)\,dk - \int_{-\infty}^{+\infty}dk\int_{-\infty}^{+\infty}\frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)}}{\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}-\nu\right)}\times$$

$$\times \frac{e^{i(y-\eta)k} e^{i(x-\xi)\lambda}\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}+\nu\right)}{1} d\lambda \Bigg). \qquad \text{(VIII.65)}$$

The integral equation of the problem is obtained, as usual, from the expression (VIII.64).

The configuration of the free surface is determined by the formula

$$\eta(x, y) = \frac{v_0}{g} \varphi_x(x, y, \theta).$$

With $x \to -\infty$, φ_x has the following form:

$$\varphi_{x_{-\infty}} = \frac{1}{\pi} \iint_s \gamma(\theta) \left[\nu \int_{-\infty}^{+\infty} \frac{e^{\sqrt{\lambda_0^2+k^2}(z+\zeta)} e^{i(y-\eta)k} (k^2+\lambda_0^2)^{\frac{3}{2}}}{\lambda_0 (2k^2+\lambda^2)} \sin\lambda_0(x-\xi) dk \right] ds, \qquad \text{(VIII.66)}$$

or

$$\varphi_{x_{-\infty}} = \frac{2\nu}{\pi} \iint_s \gamma(\theta) \int_0^\infty \frac{e^{\sqrt{\lambda_0^2+k^2}(z+\zeta)} (k^2+\lambda_0^2)^{\frac{3}{2}} \sin\lambda \times (x-\xi) \cos k(y-\eta)\, ds dk}{\lambda_0 (2k^2+\lambda_0^2)}. \qquad \text{(VIII.67)}$$ [277

The asymptotic formula for $\varphi_{x_{-\infty}}$ with $\nu \to 0$ will be as follows:

$$\varphi_{x-\infty} = \frac{\sqrt{\nu}}{\pi} \iint_s \gamma(\theta) \int_0^\infty e^{k(z+\zeta)} \sqrt{k} \cos k(y-\eta) \sin\sqrt{\nu k}(x-\xi)\, dk ds. \qquad \text{(VIII.68)}$$

Let us examine the values of φ_y when $x \to -\infty$:

$$\varphi_{y_{-\infty}} = -\frac{1}{4\pi} \iint_s \gamma(\theta) \Bigg\{ -\frac{\partial}{\partial y} \int_{-\infty}^{+\infty} \frac{\partial}{\partial \zeta} \frac{1}{r} d\tau + 2 \int_0^\infty e^{k(z+\zeta)} k \sin k(y-\eta) dk -$$

$$- 8\nu \int_0^\infty \frac{e^{\sqrt{\lambda_0^2+k^2}(z+\zeta)} (k^2+\lambda_0^2)^{\frac{3}{2}} \cos\lambda_0(x-\xi) k \sin k(y-\eta)}{\lambda_0^2 (2k^2+\lambda_0^2)} dk \Bigg\} ds. \qquad \text{(VIII.69)}$$

For the limiting values of $\nu \begin{smallmatrix}\to \infty \\ \to 0\end{smallmatrix}$ we have the following:

$$\varphi_{y_{-\infty}} = -\frac{1}{4\pi} \iint_s \gamma(\theta) \Bigg\{ \Bigg(-\frac{\partial}{\partial y} \int_{-\infty}^{+\infty} \frac{\partial}{\partial \zeta} \frac{1}{r} d\tau \pm 2 \int_0^\infty e^{k(z+\zeta)} k \sin(y-\eta) d\eta) ds, \qquad \text{(VIII.70)}$$

but

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial \zeta} \frac{1}{r} d\tau = -\frac{2(z-\zeta)}{[(y-\eta)^2+(z-\zeta)^2]},$$

$$\frac{\partial}{\partial y} \int_{-\infty}^{+\infty} \frac{\partial}{\partial \zeta} \frac{1}{r} d\tau = -\frac{4(z-\zeta)(y-\eta)}{[(y-\eta)^2+(z-\zeta)^2]^2},$$

then (VIII.70) may be rewritten as follows:

$$\varphi_{y-\infty} = -\frac{1}{\pi} \iint_s \gamma(\theta) \left(\frac{(z-\zeta)(y-\eta)}{[(y-\eta)^2+(z-\zeta)^2]^2} \pm \frac{(z+\zeta)(y-\eta)}{[(y-\eta)^2+(z-\zeta)^2]^2} \right). \quad \text{(VIII.71)}$$

Hence we obtain [278

$$(\varphi_{y-} - \varphi_{y+})_{z=\zeta} = +\frac{2}{\pi} \iint_s \gamma(\theta) \frac{\xi(y-\eta)}{[(y-\eta)^2+\xi^2]^2} ds.$$

$$(\varphi_{y-} - \varphi_{y+})_{z=-\zeta} = \pm\frac{2}{\pi} \iint_s \gamma(\theta) \frac{\xi(y-\eta)}{[(y-\eta)^2+\xi^2]^2} ds. \quad \text{(VIII.72)}$$

In the second relation, the "+" sign refers to the $\nu \to 0$ case, while the "-" sign refers to the $\nu \to \infty$ case.

Thus, when passing through surface Σ and the $\bar{\Sigma}$ surface, symmetrical with respect to the xy plane, the induced velocities ζy undergo sudden changes. In the limiting cases, the value of the change on the $\bar{\Sigma}$ surface is equal to that occuring on the Σ surface. However, for $\nu \to 0$ and $\nu \to \infty$ the signs of the change are opposite.

8.7. Planing of Lifting Surfaces

The methods developed below make it possible to solve the problem of planing of lifting surfaces. Let us examine a three-dimensional problem dealing with the steady-state planing of lifting surfaces. If we apply Cauchy's integral to the points on the free surface, then, in the linear approximation, the boundary conditions on the free surface will be expressed as follows:

$$\varphi_{it} + g\eta = 0 \quad \text{outside s}, \quad \text{(VIII.73)}$$

$$\varphi_{1t} + g\eta = -p \quad \text{on s}, \quad \text{(VIII.74)}$$

where p is the pressure at point x, y on the surface s.

Differentiating the expressions (VIII.73) and (VIII.74)

with respect to time, we obtain the following:

$$\varphi_{xx} = -\mu\varphi_x + \nu\varphi_z = 0 \quad \text{outside s,}$$

$$\varphi_{xx} - \mu\varphi_x + \nu\varphi_z = \frac{p(x, y)}{v_0} \quad \text{on s,}$$

and then, for the acceleration potential we will have

$$\theta_{xx} - \mu\theta_x + \nu\theta_z = 0 \quad \text{outside s,} \tag{VIII.75}$$

$$\theta_{xx} - \mu\theta_x + \nu\theta_z = \bar{p}_{xx} \quad \text{on s,} \tag{VIII.76}$$

$$\bar{p} = \frac{p}{\varrho}.$$

Let us examine the fluid flow in the lower half-space only and let us write function $\bar{p}(x, y, z)$ in the form of a two-layer potential as follows:

$$\bar{p} = \frac{1}{2\pi} \iint_s \bar{p}(\xi, \eta) \frac{\partial}{\partial z} \frac{1}{r} ds. \tag{VIII.77}$$

We can look for the acceleration potential in the following form: [279

$$\theta_1 = \frac{1}{2\pi} \iint_s \bar{p}(\xi, \eta) F(\xi, \eta, x, y, z) ds, \tag{VIII.78}$$

where $F(x, y, z, \xi, \eta, \zeta)$ is the unknown harmonic function in the lower half-space, with the exception of the surface s, which envelops the planing surface from below.

Let us take the integral expression for $\frac{1}{r}$ in the following form:

$$\frac{1}{r} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} e^{\lambda(z + i\omega)} d\lambda;$$

now we can express the unknown function F as follows:

$$F(x, y, z, \xi, \eta, \Theta) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} e^{\lambda(z + i\omega)} A(\lambda, \Theta) d\lambda.$$

Using expression (VIII.76) and employing the Fourier method we obtain the following:

$$A(\lambda, \Theta) = -\frac{\lambda^2 \cos^2 \Theta}{\lambda \cos^2 \Theta + i\mu \cos \Theta - \nu},$$

and then the potentials θ and φ will be expressed as follows:

$$\Theta = -\frac{1}{4\pi^2}\iint_s \bar{p}(\Theta)\int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} \frac{e^{\lambda(z+i\omega)}\lambda^2\cos^2\Theta}{\lambda\cos^2\Theta + i\mu\cos\Theta - \nu}\,d\lambda, \qquad \text{(VIII.79)}$$

$$\varphi = -\frac{i}{4\pi v_0}\iint_s \bar{p}(\Theta)\int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} \frac{e^{\lambda(z+i\omega)}\lambda\cos\Theta}{\lambda\cos^2\Theta + i\mu\cos\Theta - \nu}\,d\lambda. \qquad \text{(VIII.80)}$$

For $\nu \to \infty$ the acceleration potential $\theta = 0$ and for $\nu \to \infty$

$$\Theta_0 = -\frac{1}{4\pi^2}\iint_s \bar{p}(\Theta)\int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} e^{\lambda(z+i\omega)}\lambda\,d\lambda$$

or $\theta_0 = -\bar{p}$, which agrees with the formula (VIII.6).

Let us now calculate the value of the induced vertical velocity as follows:

$$\varphi_z = -\frac{i}{4\pi^2 v_0}\iint_s \bar{p}(\Theta)\int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} \frac{e^{\lambda(z+i\omega)}\lambda^2\cos\Theta}{\lambda\cos^2\Theta + i\mu\cos\Theta - \nu}\,d\lambda.$$

After the separation of the remainders we have [280

$$\varphi_z = -\frac{i}{4\pi^2 v_0}\iint_s \bar{p}(\Theta)\left(\int_{-\pi}^{+\pi} d\Theta \int_0^{\infty} \frac{e^{\lambda(z+i\omega)}\lambda^2\cos\Theta}{\lambda\cos^2\Theta - \nu}\,d\lambda + \right.$$

$$\left. + 2\pi i\nu^2 \,\mathrm{Re}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{\frac{\nu}{\cos^2\Theta}(z+i\omega)}}{\cos^5\Theta}\,d\Theta\right). \qquad \text{(VIII.81)}$$

At large distances behind the hydrofoil

$$\varphi_{z-\infty} = \frac{1}{\pi}\nu^2\iint_s \bar{p}(\Theta)\,\mathrm{Re}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\Theta}(z+i\omega)}\sec^5\Theta\,d\Theta\,ds, \qquad \text{(VIII.82)}$$

or

$$\varphi_{z-\infty} = \frac{1}{\pi}\iint_s \bar{p}(\Theta)\,\mathrm{Re}\int_{\nu}^{\infty} e^{\lambda\left[z+i(x-\xi)\sqrt{\frac{\nu}{\lambda}}+i(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}\right]}\frac{\lambda}{\sqrt{1-\frac{\nu}{\lambda}}}\,d\lambda. \qquad \text{(VIII.83)}$$

It follows from (VIII.83) that behind the planing surface a vortical wake is formed, which produces a vertical velocity directed downward; with $\nu \to 0$ the vertical

velocity will be formed by the wake of vortices only.

$$\varphi_{\substack{z\to-\infty\\ \nu\to 0}} = \frac{1}{\pi}\iint\limits_{s}\bar{p}(\Theta)\int\limits_{0}^{\infty} e^{\lambda z}\cos\lambda(y-\eta)\,\lambda\, d\lambda. \qquad \text{(VIII.84)}$$

For an arbitrary lifting surface, the integral equation is derived from the expression (VIII.81) as follows:

$$\frac{1}{2\pi v_0}\iint\limits_{s}\bar{p}(\Theta)\left(\frac{i}{2\pi}\int\limits_{-\pi}^{+\pi}d\Theta\int\limits_{0}^{\infty}\frac{e^{\lambda(-\xi+i\omega)}\lambda^2\cos\Theta}{\lambda\cos^2\Theta-\nu}\,d\lambda - \nu^2\,\mathrm{Re}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\Theta}(-\xi+i\omega)}\sec^5\Theta\, d\Theta\right)ds = v_0\alpha. \qquad \text{(VIII.85)}$$

In the equation (VIII.85), after the final results are obtained, it is necessary to direct ξ toward zero from the side of positive values.

Using the lifting line theory approximation, one may write φ_z as follows: [281

$$\varphi_z = \frac{1}{2\pi\nu_0}\int\limits_{-b}^{+b} p(\eta)\,\nu^2\,\mathrm{Re}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\Theta}[z+i(y-\eta)\sin\Theta]}\sec^5\Theta\, d\eta, \qquad \text{(VIII.86)}$$

and then the integral equation in the lifting line theory may be written as follows:

$$p(y) = f(y)\left[\alpha(y) + \frac{1}{2\pi v_0^2}\int\limits_{-b}^{+b} p(\eta)\,\nu^2\,\mathrm{Re}\times\right.$$
$$\left.\times\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\Theta}[-\xi+i(y-\eta)\sin\Theta]}\sec^5\Theta\, d\Theta\, d\eta\right], \qquad \text{(VIII.87)}$$

or

$$p(y) = f(y)\left[\alpha(y) + \frac{1}{2\pi v_0^2}\int\limits_{-b}^{+b} p(\eta)\times\right.$$
$$\left.\times\int\limits_{\nu}^{\infty}\frac{e^{-\lambda\xi}\lambda\cos\lambda\sqrt{1-\frac{\nu}{\lambda}}\,(y-\eta)}{\sqrt{1-\frac{\nu}{\lambda}}}\,d\lambda\, d\eta\right]. \qquad \text{(VIII.88)}$$

When $\nu \to \infty$, the nucleus of the equation is equal to zero, and when $\nu \to 0$

$$\frac{(y-\eta)^2-\xi^2}{[(y-\eta)^2+\xi^2]^2}=\int_0^\infty e^{-\lambda\xi}\lambda\cos\lambda(y-\eta)\,d\lambda.$$

Let us write the equation (VIII.88) in the form of the integro-differential equation:

$$p(y)=f(y)\left[\alpha(y)-\frac{1}{2\pi v_0^2}\int_{-b}^{+b}p'(\eta)\times\right.$$
$$\left.\times\int_\nu^\infty e^{-\lambda\xi}\frac{\lambda}{\lambda-\nu}\sin\lambda\sqrt{1-\frac{\nu}{\lambda}}(y-\eta)\,d\lambda d\eta\right]. \qquad \text{(VIII.89)}$$

With $\nu \to 0$ the equation (VIII.89) transforms into the Prandtl singular integro-differential equation as follows: [282

$$p(y)=f(y)\left[\alpha(y)-\frac{1}{2\pi v_0^2}\int_{-b}^{+b}\frac{p'(\eta)}{y-\eta}d\eta\right]. \qquad \text{(VIII.90)}$$

However, the angle of the downwash, determined from this expression, will be double that of the hydrofoil in an infinite fluid.

The function f(y) in the expressions can be taken in the form of $f(y)=v_0^2 a(y)a_r$, where $a_r=\frac{dC_{yr}}{d\alpha}$.

The lifting force and the induced drag are determined through $\bar{P}$ and the vertical velocity as follows:

$$P=\varrho\int_{-b}^{+b}p(y)dy, \qquad \text{(VIII.91)}$$

$$Q=-\frac{\varrho}{2\pi v_0^2}\int_{-b}^{+b}p(y)\int_{-b}^{+b}p(\eta)\int_\nu^\infty\frac{e^{-\lambda\xi}\lambda\cos\lambda\sqrt{1-\frac{\nu}{\lambda}}(y-\eta)}{\sqrt{1-\frac{\nu}{\lambda}}}\times$$
$$\times d\lambda d\eta dy \qquad \text{(VIII.92)}$$

The equation (VIII.90) is solved by the aerodynamic methods, while the approximate solution of equation (VIII.

89) for small values of parameter $\bar{\omega} = \frac{1}{2\mathrm{Fr}^2}$ may be sought in the form of a series:

$$p(y) = p_0(y) + \omega p_1(y) + \omega^2 p_2(y) + \ldots$$

The approximate results can be obtained by employing the method of the parameter τ. However, in this case one should evaluate a great number of approximations and compare the results for correlation, because, in the final result, τ has to converge to unity.

If we determine C_y and C_{xi} for the elliptical lifting surface in the plan view from the formulas

$$C_y = \frac{a_r}{1 + \frac{a_r}{\pi\lambda}\zeta_r}(\alpha_0 + \alpha_\kappa),$$

$$C_{xi} = \frac{C_y^2}{\pi\lambda}\zeta_r, \qquad \text{(VIII.93)}$$

then the value of ζ_r determined in terms of parameter τ will be [283

$$\begin{aligned}\zeta_r = {} & 0{,}5G_{2,0}(\omega) + G_{4,2}(\omega) - 0{,}75G_{4,0}(\omega) + 1{,}5G_{6,2}(\omega) + 3G_{6,1}(\omega) + \\ & + 1{,}5625G_{6,2}(\omega) + 2G_{8,3}(\omega) - 7{,}5G_{8,2}(\omega) + 9{,}375G_{8,1}(\omega) - \\ & - 3{,}8281G_{8,0}(\omega) + 2{,}5G_{10,4}(\omega) - \\ & - 15G_{10,3}(\omega) + 32{,}8125G_{10,2}(\omega) - \\ & - 30{,}6247G_{10,1}(\omega) + 10{,}3359G_{10}(\omega) + \\ & + 3G_{12,5}(\omega) - 26{,}25G_{12,4}(\omega) + \\ & + 87{,}5G_{12,3}(\omega) - 137{,}8G_{12,2}(\omega) + \\ & + 103{,}359G_{12,1}(\omega) - \\ & - 29{,}7773G_{12,0}(\omega) + 0{,}90. \text{ (VIII.94)}\end{aligned} \qquad \text{(VIII.94)}$$

For $\omega = \infty$, the formula (VIII.94) gives $\zeta_r = 0$ and for $\omega = 0$, $\zeta_r = 1.80$ instead of the exact value of 2. The graph of ζ_r as a function of Fr is shown in Fig. 22.

8.8. The Basic Relationships for an Optimum Submerged Hydrofoil

The solution of the problem of distribution of circulation that results in minimum drag and the determination

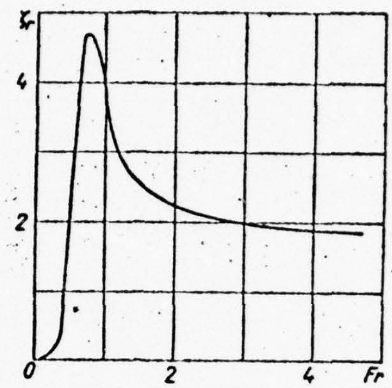

Fig. 22

of the corresponding parameters of the hydrofoil is of great practical interest.

If, for the case of infinite fluid, this problem has a simple solution, with ordinary elliptical distribution being optimal, it is not, however, possible to produce such a simple closed solution for the submerged hydrofoil and the optimal distribution is more complicated, depending on the depth of submergence $\bar{h}$ and the Froude number $F_r = \frac{v_0}{\sqrt{gb}}$. The optimum condition for the submerged hydrofoil is formulated in the similar way as in aerodynamics.

If the lifting force and the drag are determined by formulas (VIII.27)

$$P = \rho v_0 \int_{-b}^{+b} \Gamma(y)\, dy,$$

$$Q = \rho \int_{-b}^{+b} \Gamma(y)\, w\, dy,$$

then the problem for determining the minimum drag with a constant lifting force leads to the condition

$$\delta Q - \lambda \delta P = 0$$

or [284

$$\int_{-b}^{+b} \delta \Gamma w\, dy - \lambda v_0 \int_{-b}^{+b} \delta \Gamma\, dy = 0,$$

$$\int_{-b}^{+b} \delta \Gamma (w - c)\, dy = 0. \qquad \text{(VIII.95)}$$

Since the latter condition must be satisfied for any $\delta\Gamma$, then

$$w = \text{const.} \qquad \text{(VIII.96)}$$

This condition determines constancy of the induced velocity along the span of the foil with an optimum distribution of circulation.

If the integro-differential equation for the hydrofoil is taken in the form (VIII.15), then the condition (VIII.96) leads to the equation

$$\int_{-1}^{+1} \bar{\Gamma}'(\bar{\eta}) \left[\frac{1}{\bar{y}-\bar{\eta}} + G(\bar{y}-\bar{\eta}) \right] \overline{d\eta} = \Phi, \qquad \text{(VIII.97)}$$

where Φ is a constant.

The equation (VIII.97) can be reduced to a quasi-regular equation. Let us write it in the form

$$\int_{-1}^{+1} \frac{\bar{\Gamma}'(\bar{\eta})}{\bar{y}-\bar{\eta}} \overline{d\eta} = f(\bar{y}).$$

The solution of this equation, unlimited at the ends of -1 and +1, is defined by the formula similar to Cauchy's conversion integral [15]

$$\bar{\Gamma}'(\bar{\eta}) = \frac{1}{\pi^2 \sqrt{1-\bar{\eta}^2}} \int_{-1}^{+1} \frac{f(\bar{y}) \sqrt{1-\bar{y}^2}}{\bar{y}-\bar{\eta}} d\bar{y}.$$

Here, the constant a_0 is assumed to equal zero and

$$f(\overline{y}) = \Phi - \int_{-1}^{+1} \bar{\Gamma}'(\eta) G(\bar{y}-\bar{\eta}).$$

Then, after calculations, we obtain

$$\bar{\Gamma}'(\bar{\eta}) = \bar{\Gamma}'(\bar{\eta}) - \frac{1}{\pi^2 \sqrt{1-\bar{\eta}^2}} \int_{-1}^{+1} \bar{\Gamma}'(\bar{s}) \overline{ds} \times$$

$$\times \int_{-1}^{+1} \frac{\sqrt{1-\bar{y}^2}}{\bar{y}-\bar{\eta}} G(\bar{y}-\bar{s}) \overline{dy}, \qquad \text{(VIII.98)}$$

where [285

$$\bar{\Gamma}'(\bar{\eta}) = -\frac{\Phi\bar{\eta}}{\pi \sqrt{1-\bar{\eta}^2}}.$$

The equation (VIII.98) can be written as follows:

$$\varphi(s) = f(s) - \frac{1}{\pi^2}\int_{-1}^{+1} \varphi(\overline{\eta}) K(\overline{s}, \overline{\eta})\, d\overline{s}, \qquad \text{(VIII.99)}$$

where

$$K(\overline{s}, \eta) = \frac{1}{\sqrt{1-\overline{\eta}^2}} \int_{-1}^{+1} \frac{\sqrt{1-\overline{y}^2}}{\overline{y}-\overline{s}} G(\overline{y}-\overline{\eta})\, dy;$$

$$\varphi(\overline{s}) = \overline{\Gamma}'(\overline{s})\sqrt{1-\overline{y}^2}\; f(\overline{s}) = -\frac{\Phi \overline{s}}{\pi}.$$

The circulation $\Gamma(y)$ is determined through $\varphi(\eta)$ by the formula

$$\Gamma(\overline{y}) = \int_{-1}^{y} \frac{\varphi(\overline{s})}{\sqrt{1-s^2}}\, d\overline{s}.$$

If the circulation $\Gamma(\bar{y})$, which has been derived from equation (VIII.97), is written in the following form:

$$\Gamma(\overline{y}) = \frac{\Phi}{\pi}\sqrt{1-\overline{y}^2} f(\overline{y}),$$

then the constant Φ can easily be found from the basic integro-differential equation

$$\Gamma(\overline{y}) = \frac{a_n}{2\lambda(\overline{y})}\left[\alpha(\overline{y}) - \frac{\Phi}{2\pi}\right]. \qquad \text{(VIII.100)}$$

Then

$$\Phi = \frac{\alpha(\overline{y})}{\left[\frac{1}{2\pi} + \frac{2}{\pi}\,\frac{\sqrt{1-\overline{y}^2} f(\overline{y})\lambda(\overline{y})}{a_n}\right]}. \qquad \text{(VIII.101)}$$

Thus, in order to ensure optimum distribution of circulation along the submerged hydrofoil span, the correlation between the distribution of angles $\alpha(y)$ along the span and the relative span $\lambda(y)$ has to conform to the relation in (VIII.101). Since this relation is satisfied along the entire span, it is convenient to determine Φ along the central cross section ($y = 0$) as follows:

$$\Phi = \frac{\alpha(0)}{\left[\frac{1}{2\pi} + \frac{2}{\pi}\,\frac{f(0)\lambda(0)}{ah}\right]}. \qquad \text{(VIII.102)}$$

For the constant angle of attack along the span, the relation (VIII.101) produces [286

$$\sqrt{1-\overline{y}^2} f(\overline{y})\lambda(\overline{y}) = \text{const}.$$

We will define the circulation using a series

$$\Gamma(\bar{y}) = \frac{\Phi}{\pi}\sqrt{1-\bar{y}^2}\,(A_1 + \bar{y}^2 A_2 + \bar{y}^4 A_3 + \ldots). \qquad \text{(VIII.103)}$$

Then for the lifting force we get the following:

$$C_y = \lambda\int_{-1}^{+1}\Gamma(\bar{y})\,d\bar{y} = \lambda\bar{\Phi}2\pi\alpha\left(\frac{A_1}{2} + \frac{A_2}{8} + \frac{A_3}{16} + \frac{5}{128}A_4 + \frac{7}{256}A_5 + \ldots\right),$$

$$\bar{\Phi} = \frac{\Phi}{2\pi\alpha}.$$

The relative span and the span at the central cross section of the foil are related by the expression $\lambda_0 = \frac{\pi}{4}\lambda k_1$, where k_1 is the coefficient taking into account the shape of the foil in plan view:

$$k_1 = \frac{2}{\pi}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,f_1(\bar{\eta})\,d\bar{\eta}.$$

Here $f_1(\eta)$ is the function which is introduced by the relationship

$$\frac{a_h}{\lambda(\bar{y})} = \frac{a_h}{\lambda(0)}\sqrt{1-\bar{y}^2}\,f_1(\bar{y}), \quad f_1(0) = 1. \qquad \text{(VIII.104)}$$

Now the formulas for $\bar{\Phi}$ and C_y may be written as follows:

$$\bar{\Phi} = \frac{a_h}{\pi\lambda A_1 k_1\left(1 + \frac{a_h}{\pi\lambda A_1 k_1}\right)},$$

$$\text{(VIII.105)}$$

$$C_y = \frac{a_h\alpha}{\left(1 + \frac{a_h}{\pi\lambda}\frac{1}{A_1 k_1}\right)}\cdot\frac{A_1 + \frac{1}{4}A_2 + \frac{1}{8}A_3 + \frac{5}{64}A_4 + \frac{7}{128}A_5}{A_1 k_1}.$$

For an infinite fluid, $A_1 = 1$, $k_1 = 1$, and then we again obtain the known results [287

$$\Gamma_\infty = \frac{2a_\infty\alpha}{\pi\lambda\left(1 + \frac{a_\infty}{\pi\lambda}\right)}\sqrt{1-\bar{y}^2}, \qquad C_y = \frac{a_\infty\alpha}{1 + \frac{a_\infty}{\pi\lambda}}.$$

With α = const

$$A_1 k = A_1 + \frac{A_3}{4} + \frac{A_5}{8} + \ldots$$

For the coefficient of the lifting force we will obtain

$$C_y = \frac{a_h \alpha}{1 + \frac{a_h}{\pi\lambda}\zeta}. \qquad \text{(VIII.106)}$$

The drag coefficient will be determined as follows:

$$C_{x\text{в}} = \overline{\Phi}\alpha C_y,$$

from where

$$C_{x\text{в}} = \frac{C_y^2}{\pi\lambda}\zeta, \qquad \text{(VIII.107)}$$

where

$$\zeta = \frac{1}{A_1 + \frac{A_3}{4} + \frac{A_5}{8} + \ldots} \qquad \text{(VIII.108)}$$

is the correction for the influence of the free surface on the drag of the submerged hydrofoil.

Formulas (VIII.106) and (VIII.107) are also obtained for an arbitrary distribution of angles of attack along the span without any difficulty.

Formulas (VIII.106) and (VIII.107) are also used when determining the hydromechanical coefficients of the hydrofoil with the elliptical distribution of circulation. For that case the formulas produce the approximate values of C_y and C_x, while for the case of an optimal submerged hydrofoil they produce the exact values of hydrodynamic coefficients determined in the lifting line theory. Similarly to aerodynamics, the formulas (VIII.106) and (VIII. 107) may be used for determining coefficients C_y and C_x for the hydrofoil of an arbitrary shape, provided we introduce small corrections which depend on the shape of the hydrofoil in the plan view. In aerodynamics these corrections are determined from the solution of the Prandtl integro-differential equation [19, 38].

For the submerged hydrofoil this type of approach is inconvenient, since one may not be certain that the corrections, which characterize the shape of the wing in the plan view, are the same for various modes of motion.

With this approach, instead of ζ in formulas (VIII. 106) and (VIII.107), one should use $(1 + \tau)\zeta$ and $(1 + \delta)\zeta$, respectively, where τ and δ can themselves depend on $\bar{h}$ and Fr.

It is more convenient to account for the effect of the free surface in each formula by a single function which can be obtained from the solution of the equation (VIII. 15) and to write the general formulas as follows: [288

$$C_y = \frac{a_\infty \psi}{1 + \frac{a_\infty \psi}{\pi\lambda}\zeta_1} a_y, \tag{VIII.109}$$

$$C_x = \frac{C_y^2}{\pi\lambda}\zeta_2, \tag{VIII.110}$$

where functions ζ_1 and ζ_2 differ very little from each other.

8.9. Determination of the Optimal Distribution of Circulation Along the Span of the Submerged Hydrofoil*

*This part has been written together with P. I. Zinchuk. He also suggested a solution for the semi-regular equation.

For the submerged hydrofoil moving near the surface of a fluid of infinite depth, the equations (VIII.97) and (VIII.99) are solved by using the expansion (VIII.30).

Let us obtain an approximate solution of the equation (VIII.97). Let us find it in the form of the series

$$\bar{\Gamma}'(\bar{\eta}) = \bar{\Gamma}_0'(\bar{\eta}) + \tau^2\bar{\Gamma}_2'(\bar{\eta}) + \tau^4\bar{\Gamma}_4'(\bar{\eta}) + \dots$$

The nucleus $G_n(y)$ of eqation (VIII.97) can be presented as follows:

$$G(y) = \sum_{n=2,4}^{\infty}{}' G_n(y)\tau^n.$$

Then

$$\int_{-1}^{+1} \frac{\bar{\Gamma}_0'(\bar{\eta})}{\bar{y} - \bar{\eta}} d\bar{\eta} = \Phi,$$

$$\int_{-1}^{+1} \frac{\bar{\Gamma}_2'(\bar{\eta})}{\bar{y} - \bar{\eta}} d\bar{\eta} = -\int_{-1}^{+1} \bar{\Gamma}_0'(\bar{\eta})\, G_2(\bar{y} - \bar{\eta})\, d\bar{\eta},$$

$$\int_{-1}^{+1} \frac{\bar{\Gamma}'_4(\bar{\eta})}{\bar{y}-\eta} d\bar{\eta} = -\int_{-1}^{+1} \bar{\Gamma}'_2(\bar{\eta}) G_2(\bar{y}-\bar{\eta}) + \bar{\Gamma}'_0(\bar{\eta}) G_n(\bar{y}-\bar{\eta})) d\bar{\eta},$$

. . . .

$$\int_{-1}^{+1} \frac{\bar{\Gamma}'_n(\bar{\eta})}{\bar{y}-\eta} d\bar{\eta} = -\int_{-1}^{+1} \sum_{m=2}^{n} \bar{\Gamma}'_{m-2}(\eta) G_{n-m}(\bar{y}-\bar{\eta}) d\eta. \qquad \text{(VIII.111)}$$

The solution of equation (VIII.111) will also be determined by the formula of the integral conversion. [289

The function $G_n(y)$ is obtained from the formulas on page 242

$$G_n(y) = \sum_{p=0}^{\frac{n}{2}-1} \frac{y^{n-1-2p}(n-1-p)\ldots(p+1)(-1)^{\frac{n}{2}-p}}{(s-1-2p)!} G_{sp}\left(\frac{\bar{\omega}}{\tau}\right).$$

By solving it, we obtain:

$$\left.\begin{aligned} \bar{\Gamma}'_0(\bar{\eta}) &= -\frac{\Phi\bar{\eta}}{\pi\sqrt{1-\bar{\eta}^2}} \\ \bar{\Gamma}'_2(\bar{\eta}) &= -\frac{\Phi\bar{\eta}}{\pi\sqrt{1-\bar{\eta}^2}} \cdot \frac{G_{2,0}}{2} \\ \bar{\Gamma}'_4(\bar{\eta}) &= -\frac{\Phi}{\pi\sqrt{1-\bar{\eta}^2}}\left[\bar{\eta}\left(\frac{G_{20}^2}{L} + G_{4,1} + \frac{3}{8}G_{4,0}\right) - \frac{3}{2}G_{4,0}\bar{\eta}^3\right] \end{aligned}\right\}. \qquad \text{(VIII.112)}$$

Then

$$\Gamma(\bar{y}) = \frac{\Phi}{\pi}\sqrt{1-\bar{y}^2}\left\{1 - \frac{G_{2,0}}{2}\tau^2 - \left(-\frac{G_{2,0}^2}{4} + G_{4,1} - \frac{5}{8}G_{4,0}\right)\tau^4 - \right.$$

$$-\left(\frac{G_{2,0}^3}{8} - G_{2,0}G_{4,1} + \frac{11}{16}G_{4,0}G_{2,0} + \frac{9}{8}G_{6,0} - \frac{5}{2}G_{6,1} + \frac{3}{2}G_{6,2}\right)\tau^6 + \ldots +$$

$$\left. + \left[\frac{1}{2}G_{4,0}\tau^4 - \left(\frac{3}{2}G_{6,0} - 2G_{6,1} + \frac{1}{4}G_{2,0}G_{4,0}\right)\tau^6\right]\bar{y}^2 - \frac{1}{2}G_{6,0}\bar{y}^4\right\}. \qquad \text{(VIII.113)}$$

When $Fr \to \infty$, formula (VIII.113) becomes

$$\Gamma(\bar{y}) = \frac{\Phi}{\pi}\sqrt{1-\bar{y}^2}\left[1 - 0{,}5\tau^2 - \frac{1}{8}\tau^4 + \frac{1}{16}\tau^6 + \ldots + \right.$$

$$\left. + \left(\frac{1}{2}\tau^4 + \frac{1}{4}\tau^6\right)\bar{y}^2 - \frac{1}{2}\tau^6\bar{y}^4\right]. \qquad \text{(VIII.114)}$$

The solution of the integral equation (VIII.99) can be obtained by the iteration method.

Introducing the resolvent of the equation we can write the solution as follows:

$$\varphi(\bar{\eta}) = f(\bar{s}) - \frac{1}{\pi^2}\int_{-1}^{+1} f(\bar{\eta})\, R\left(\bar{s}, \bar{\eta}, -\frac{1}{\pi^2}\right) d\bar{s}, \qquad \text{(VIII.115)}$$

where $R\left(\bar{s}, \bar{\eta}, -\frac{1}{\pi^2}\right) = \sum_{m=1}^{\infty}\left(-\frac{1}{\pi^2}\right)^{m-1} K_m(\bar{s}, \bar{\eta})$ is the resolvent of the [290
equation, and $K_m(\bar{s}, \eta)$ are the iterated nuclei.

For determining the resolvent let us take the function $G(y - \eta)$ also in the form of (VIII.30).

For determining the iterated nuclei we have the recurrent formula

$$K_{n+p}(\bar{s}, \bar{\eta}) = \int_{-1}^{+1} K_n(\bar{s}, t)\, K_p(t, \bar{\eta})\, dt. \qquad \text{(VIII.116)}$$

By solving it we obtain the following:

$$\begin{aligned}
K_1(\bar{s}, \bar{\eta}) = & \frac{1}{\sqrt{1-\bar{\eta}^2}}\left[\tau^2 G_{2,0}\pi\left(\frac{1}{2} - \bar{s}^2 + \bar{\eta}\bar{s}\right) - \right. \\
& - \tau^4\left\{G_{4,0}\pi\left[\frac{1}{8} + \frac{1}{2}\bar{s}^2 - \bar{s}^4 - 3\bar{\eta}\bar{s}\left(\frac{1}{2} - \bar{s}^2\right) + 3\bar{\eta}^2\left(\frac{1}{2} - \bar{s}^2\right) + \right.\right. \\
& \left. + \bar{\eta}^3\bar{s}\right] - 2G_{4,1}\pi\left(\frac{1}{2} - \bar{s}^2 + \bar{\eta}\bar{s}\right)\Big\} + \tau^6\left\{G_{6,0}\pi\left[\frac{1}{16} + \frac{1}{8}\bar{s}^2 + \right.\right. \\
& + \frac{1}{2}\bar{s}^4 - \bar{s}^6 - 5\bar{\eta}\bar{s}\left(\frac{1}{8} + \frac{1}{2}\bar{s}^2 - \bar{s}^4\right) + 10\bar{\eta}^2\left(\frac{1}{8} + \frac{1}{2}\bar{s}^2 - \bar{s}^4\right) - \\
& - 10\bar{\eta}^3\bar{s}\left(\frac{1}{2} - \bar{s}^2\right) + 5\bar{\eta}^4\left(\frac{1}{2} - \bar{s}^2\right) + \bar{\eta}^5\bar{s}\Big] - 4G_{6,1}\pi\left[\frac{1}{8} + \frac{1}{2}\bar{s}^2 - \right. \\
& \left. - \bar{s}^4 - 3\bar{\eta}\bar{s}\left(\frac{1}{2} - \bar{s}^2\right) + 3\bar{\eta}^2\left(\frac{1}{2} - \bar{s}^2\right) + \bar{\eta}^3\bar{s}\right| + 3G_{6,2}\pi\left[\left(\frac{1}{2} - \right.\right. \\
& \left.\left. - \bar{s}^2 + \bar{\eta}\bar{s}\right)\right]\Big\} - \tau^8\left\{G_{8,0}\pi\left[\frac{5}{128} + \frac{1}{16}s^2 + \frac{1}{8}\bar{s}^4 + \frac{1}{2}\bar{s}^6 - \bar{s}^8 - \right.\right. \\
& - 7\bar{\eta}\bar{s}\left(\frac{1}{16} + \frac{1}{8}\bar{s}^2 + \frac{1}{2}\bar{s}^4 - \bar{s}^6\right) + 21\bar{\eta}^2\left(\frac{1}{16} + \frac{1}{8}\bar{s}^2 + \frac{1}{2}\bar{s}^4 - \bar{s}^6\right) - \\
& - 35\bar{\eta}^3\bar{s}\left(\frac{1}{8} + \frac{1}{2}\bar{s}^2 - \bar{s}^4\right) + 35\bar{\eta}^4\left(\frac{1}{8} + \frac{1}{2}\bar{s}^2 - \bar{s}^4\right) -
\end{aligned}$$

$$-21\bar\eta^5\bar s\left(\frac{1}{2}-\bar s^2\right)+7\bar\eta^6\left(\frac{1}{2}-\bar s^2\right)+\bar\eta^7\bar s\Big]-$$

$$-6G_{8,1}\pi\Big[\frac{1}{16}+\frac{1}{8}\bar s^2+\frac{1}{2}\bar s^4-\bar s^6-5\bar\eta\bar s\left(\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4\right)+$$

$$+10\bar\eta^2\left(\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4\right)-10\bar\eta^3\bar s\left(\frac{1}{2}-\bar s^2\right)+5\bar\eta^4\left(\frac{1}{2}-\bar s^2\right)+$$ [291

$$+\bar\eta^5\bar s\Big]+10G_{8,2}\pi\Big[\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4-3\bar\eta\bar s\left(\frac{1}{2}-\bar s^2\right)+3\bar\eta^2\left(\frac{1}{2}-\bar s^2\right)+$$

$$+\bar\eta^3\bar s\Big]-4G_{8,3}\pi\left(\frac{1}{2}-\bar s^2+\bar\eta\bar s\right)\Big)\Big\}+\tau^{10}\Big\{G_{10,0}\pi\Big[\frac{7}{256}+\frac{5}{128}\bar s^2+$$

$$+\frac{1}{16}\bar s^4+\frac{1}{8}\bar s^6+\frac{1}{8}\bar s^8-\bar s^{10}-9\bar\eta\bar s\left(\frac{5}{128}+\frac{1}{16}\bar s^2+\frac{1}{8}\bar s^4+\frac{1}{2}\bar s^6-\bar s^8\right)+$$

$$+36\bar\eta^2\left(\frac{5}{128}+\frac{1}{16}\bar s^2+\frac{1}{8}\bar s^4+\frac{1}{2}\bar s^6-\bar s^8\right)-84\bar\eta^3\bar s\left(\frac{1}{16}+\frac{1}{8}\bar s^2+\frac{1}{2}\bar s^4-\bar s^6\right)+$$

$$+126\bar\eta^4\left(\frac{1}{16}+\frac{1}{8}\bar s^2+\frac{1}{2}\bar s^4-\bar s^6\right)-126\bar\eta^5\bar s\left(\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4\right)+$$

$$+84\bar\eta^6\left(\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4\right)-36\bar\eta^7\bar s\left(\frac{1}{2}-\bar s^2\right)+9\bar\eta^8\left(\frac{1}{2}-\bar s^2\right)+$$

$$+\bar\eta^9\bar s\Big]-8G_{10,1}\pi\Big[\frac{5}{128}+\frac{1}{16}\bar s^2+\frac{1}{8}\bar s^4+\frac{1}{2}\bar s^6-\bar s^8-7\bar\eta\bar s\left(\frac{1}{16}+\frac{1}{8}\bar s^2+\right.$$

$$\left.+\frac{1}{2}\bar s^4-\bar s^6\right)+21\bar\eta^2\left(\frac{1}{16}+\frac{1}{8}\bar s^2+\frac{1}{2}\bar s^4-\bar s^6\right)-$$

$$-35\bar\eta^3\bar s\left(\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4\right)+35\bar\eta^4\left(\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4\right)+$$

$$+21\bar\eta^5\bar s\left(\frac{1}{2}-\bar s^2\right)+7\bar\eta^6\left(\frac{1}{2}-\bar s^2\right)+\bar\eta^7\bar s\Big]+21G_{10,2}\pi\Big[\frac{1}{16}+\frac{1}{8}\bar s^2+$$

$$+\frac{1}{2}\bar s^4-\bar s^6-5\bar\eta\bar s\left(\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4\right)+10\bar\eta^2\left(\frac{1}{8}+\frac{1}{2}\bar s^2-\bar s^4\right)-$$

$$-10\bar\eta^3\bar s\left(\frac{1}{2}-\bar s^2\right)+5\bar\eta^4\left(\frac{1}{2}-\bar s^2\right)+\bar\eta^5\bar s\Big]-20G_{10,3}\pi\Big[\frac{1}{8}+\frac{1}{2}\bar s^2-$$

$$-\bar s^4-3\bar\eta\bar s\left(\frac{1}{2}-\bar s^2\right)+\bar\eta^3\bar s\Big]+5G_{10,4}\pi\left(\frac{1}{2}-\bar s^2+\bar\eta\bar s\right)\Big]\Big\};$$

$$K_2(\bar\eta,\bar s)=-\frac{\pi^3}{\sqrt{1-\bar\eta^2}}\Big\{-\tau^4G_{2,0}^2\frac{1}{2}\bar\eta\bar s+\tau^6\Big[G_{2,0}G_{4,0}\left(\frac{3}{2}\bar\eta\bar s^3+\frac{1}{2}\bar\eta^3\bar s\right)-$$

$$- 2G_{2,0}G_{4,1}\bar\eta\bar s\Big] - \tau^8\Big[G_{2,0}G_{6,0}\Big(-\frac{25}{16}\bar\eta\bar s + \frac{5}{4}\bar\eta^3\bar s + \frac{1}{2}\bar\eta^5\bar s +$$ [292

$$+ \frac{1}{2}\bar\eta^5\bar s + \frac{5}{2}\bar\eta\bar s^3 + \frac{5}{2}\bar\eta\bar s^5\Big) - 4G_{2,0}G_{6,1}\Big(\frac{1}{2}\bar\eta^3\bar s + \frac{3}{2}\bar\eta\bar s^3\Big) +$$

$$+ 3G_{2,0}G_{6,2}\bar\eta\bar s + G_{4,0}^2\Big(-\frac{3}{16}\bar\eta\bar s - \frac{3}{8}\bar\eta^5\bar s + \frac{7}{8}\bar\eta\bar s^3 + \frac{3}{2}\bar\eta^3\bar s^3\Big) -$$

$$- 3G_{4,0}G_{4,1}\Big(\frac{3}{2}\bar\eta\bar s^3 + \frac{1}{2}\bar\eta^3\bar s\Big) - 2G_{4,1}^2\bar\eta\bar s\Big] + \tau^{10}\Big[G_{2,0}G_{8,0}\Big(-\frac{588}{128}\bar\eta\bar s +$$

$$+ \frac{35}{16}\bar\eta^5\bar s + \frac{21}{8}\bar\eta^5\bar s + \frac{1}{2}\bar\eta^7\bar s - \frac{7}{16}\bar\eta\bar s^3 + \frac{91}{8}\bar\eta\bar s^5 + \frac{7}{2}\bar\eta\bar s^7\Big) -$$

$$- G_{2,0}G_{8,1}\Big(-\frac{75}{8}\bar\eta\bar s + \frac{15}{2}\bar\eta^3\bar s - 3\bar\eta^5\bar s + 15\bar\eta\bar s^3 + 15\bar\eta\bar s^5\Big) +$$

$$+ 10G_{2,0}G_{8,2}\Big(\frac{3}{2}\bar\eta\bar s^3 + \frac{1}{2}\bar\eta^3\bar s\Big) - 4G_{2,0}G_{8,3}\bar\eta\bar s + G_{4,0}G_{6,0}\Big(-\frac{15}{8}\bar\eta\bar s +$$

$$+ \frac{15}{8}\bar\eta\bar s^5 - \frac{5}{2}\bar\eta^3\bar s + \frac{25}{4}\bar\eta^3\bar s^3 + \frac{5}{2}\bar\eta^3\bar s^5 - \frac{3}{8}\bar\eta^5\bar s + \frac{3}{2}\bar\eta^5\bar s^3 +$$

$$+ \frac{60}{16}\bar\eta\bar s^3\Big) + G_{4,0}G_{6,1}\Big(\frac{3}{2}\bar\eta\bar s - 9\bar\eta\bar s^3 - 12\bar\eta^3\bar s^3\Big) + G_{4,0}G_{6,2}\Big(\frac{9}{2}\bar\eta\bar s^3 +$$

$$+ \frac{3}{2}\bar\eta\bar s\Big) - G_{4,1}G_{6,0}\Big(-\frac{25}{8}\bar\eta\bar s + \frac{5}{2}\bar\eta^3\bar s + \bar\eta^5\bar s + 5\bar\eta\bar s^3 + 5\bar\eta\bar s^5\Big) -$$

$$- G_{4,1}G_{6,1}(-12\bar\eta\bar s^3 - 4\bar\eta^3\bar s) - 6G_{4,1}G_{6,2}\bar\eta\bar s\Big]\Big\},$$

$$K_3(\bar s, \bar\eta) = \frac{\pi^5}{\sqrt{1-\bar\eta^2}}\Big\{\tau^6 G_{2,0}^3\frac{1}{4}\bar\eta\bar s - \tau^8\Big[G_{2,0}^2G_{4,0}\Big(\frac{3}{8}\bar\eta\bar s + \frac{1}{4}\bar\eta^3\bar s +$$

$$+ \frac{3}{4}\bar\eta\bar s^3\Big) - \frac{3}{2}G_{2,0}^2G_{4,1}\bar\eta\bar s\Big] + \tau^{10}\Big[G_{2,0}^2G_{6,0}\Big(\frac{5}{8}\bar\eta^3\bar s + \frac{1}{4}\bar\eta^5\bar s +$$

$$+ \frac{5}{4}\bar\eta\bar s^3\Big] + \frac{5}{4}\bar\eta\bar s^5 - 4G_{2,0}^2G_{6,1}\Big(\frac{3}{8}\bar\eta\bar s + \frac{1}{4}\bar\eta^3\bar s + \frac{3}{4}\bar\eta\bar s^3\Big) +$$

$$+ \frac{9}{4}G_{2,0}^2G_{6,2}\bar\eta\bar s + G_{2,0}G_{4,0}^2\Big(-\frac{3}{64}\bar\eta\bar s + \frac{3}{16}\bar\eta^3\bar s + \frac{27}{16}\bar\eta\bar s^3 + \frac{3}{4}\bar\eta^3\bar s^3\Big) -$$

$$- 2G_{2,0}G_{4,0}G_{4,1}\Big(\frac{3}{4}\bar\eta\bar s + \frac{1}{2}\bar\eta^3\bar s + \frac{3}{2}\bar\eta\bar s^3\Big) + 3G_{2,0}G_{4,1}^2\bar\eta\bar s\Big)\Big\},$$

$$K_4(\bar s, \bar\eta) = \frac{-\pi^7}{\sqrt{1-\bar\eta^3}}\Big\{-\tau^8 G_{2,0}^4\frac{1}{8}\bar\eta\bar s + \tau^{10}\Big[G_{2,0}^3G_{4,0}\Big(\frac{12}{32}\bar\eta\bar s +$$ [293

$$+ \frac{3}{8}\bar\eta\bar s^3\Big) - G_{2,0}^3G_{4,1}\bar\eta\bar s\Big]\Big\},$$

$$K_8(\bar{s},\bar{\eta}) = \frac{\pi^8}{\sqrt{1-\bar{\eta}^2}}\tau_{10}G_{2,0}\frac{1}{16}\bar{\eta}\bar{s}. \qquad \text{(VIII.117)}$$

Substituting the value of the resolvent into the equation (VIII.99) and integrating we arrive at the solution

$$\begin{aligned}
S(\bar{s}) = & -\frac{\Phi\bar{s}}{\pi} + \frac{\Phi}{\pi}\Bigg\{\frac{1}{2}\tau^2 G_{2,0}\bar{s} - \tau^4\Bigg[G_{4,0}\left(-\frac{3}{8}\bar{s} + \right.\\
& \left. + \frac{3}{2}\bar{s}^3\right) - G_{4,1}\bar{s} + \frac{1}{4}G_{2,0}^2\bar{s}\Bigg] + \tau^6\Bigg[\frac{1}{8}G_{2,0}^3\bar{s} + G_{2,0}G_{4,0}\left(\frac{3}{4}\bar{s}^3 + \right.\\
& \left. + \frac{3}{16}\bar{s}\right) - G_{2,0}G_{4,1}\bar{s} + G_{6,0}\left(-\frac{15}{8}\bar{s} + \frac{5}{2}\bar{s}^3 + \frac{5}{2}\bar{s}^5\right) - \\
& - 4G_{6,1}\left(-\frac{3}{8}\bar{s} + \frac{3}{2}\bar{s}^3\right) + 3G_{6,2}\frac{1}{2}\bar{s}\Bigg] - \tau^8\Bigg[\frac{1}{16}G_{2,0}^4 + \\
& + G_{2,0}^2G_{4,0}\left(\frac{9}{32}\bar{s} + \frac{3}{8}\bar{s}^3\right) - \frac{3}{4}G_{2,0}^2G_{41}\bar{s} + G_{2,0}G_{6,0}\left(-\frac{5}{32}\bar{s} + \right.\\
& \left. + \frac{5}{4}\bar{s}^3 + \frac{5}{4}\bar{s}^5\right) - 4G_{2,0}G_{6,1}\left(\frac{3}{16}\bar{s} + \frac{3}{4}\bar{s}^3\right) + 3G_{2,0}G_{6,1}\frac{1}{2}\bar{s} + \\
& + G_{4,0}^2\left(-\frac{15}{64}\bar{s} + \frac{9}{8}\bar{s}^3\right) - 2G_{4,0}G_{4,1}\left(\frac{3}{16}\bar{s} + \frac{3}{4}s^3\right) - 4G_{8,3}\frac{1}{2}\bar{s}\Bigg] + \\
& + \tau^{10}\Bigg[\frac{1}{32}G_{2,0}^5 + G_{2,0}^3G_{4,0}\left(\frac{15}{64}\bar{s} + \frac{3}{16}\bar{s}^3\right) - \frac{1}{2}G_{2,0}^3G_{4,1}\bar{s} + \\
& + G_{2,0}^2G_{6,0}\left(\frac{5}{16}\bar{s} + \frac{5}{8}\bar{s}^3 + \frac{5}{8}\bar{s}^5\right) + 4G_{2,0}G_{6,1}\left(\frac{9}{32}\bar{s} + \frac{3}{8}\bar{s}^3\right) + \\
& + \frac{9}{8}G_{2,0}^2G_{6,2}\bar{s} + G_{2,0}G_{4,0}^2\left(\frac{6}{128}\bar{s} + \frac{18}{16}\bar{s}^3\right) - 2G_{2,0}G_{4,0}G_{4,1}\times \\
& \times\left(\frac{9}{16}\bar{s} + \frac{3}{4}\bar{s}^3\right) + \frac{3}{2}G_{2,0}G_{4,1}^2\bar{s} + G_{4,0}G_{6,0}\left(-\frac{255}{128}\bar{s} + \frac{75}{16}\bar{s}^3 + \frac{15}{8}\bar{s}^5\right) + \\
& + G_{4,0}G_{6,1}\left(\frac{3}{4}\bar{s} - 9\bar{s}^3\right) + G_{4,0}G_{6,2}\left(\frac{9}{4}\bar{s}^3 + \frac{9}{16}\bar{s}\right) - G_{4,1}G_{6,0}\left(-\frac{15}{16}\bar{s} + \right.\\
& \left. + \frac{5}{2}\bar{s}^3 + \frac{5}{2}\bar{s}^5\right) - G_{4,1}G_{6,1}\left(-\frac{3}{2}\bar{s} - 6\bar{s}^3\right) - 3G_{4,1}G_{6,2}\bar{s} + \\
& + G_{2,0}G_{8,0}\left(-\frac{133}{256}\bar{s} - \frac{7}{32}\bar{s}^3 + \frac{91}{32}\bar{s}^5 + \frac{7}{4}\bar{s}^7\right) - G_{2,0}G_{8,1}\times \\
& \times\left(-\frac{15}{16}\bar{s} + \frac{15}{2}\bar{s}^3 + \frac{15}{2}\bar{s}^5\right) + 10G_{2,0}G_{8,2}\left(\frac{3}{16}\bar{s} + \frac{3}{4}\bar{s}^3\right) - \\
& - 2G_{2,0}G_{8,3}\bar{s} + G_{10,0}\left(-\frac{1503}{128}\bar{s} - \frac{225}{16}\bar{s}^3 + \frac{369}{16}\bar{s}^5 + \frac{117}{4}\bar{s}^7 + \frac{9}{2}\bar{s}^9\right) -
\end{aligned}$$

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$$
-8G_{10,1}\left(-\frac{623}{128}\bar{s}-\frac{7}{16}\bar{s}^{3}+\frac{91}{8}\bar{s}^{5}+\frac{7}{2}\bar{s}^{7}\right)+21G_{10,2}\left(-\frac{15}{8}\bar{s}+\frac{5}{2}\bar{s}^{3}+\right.
$$

$$
\left.+\frac{5}{2}\bar{s}^{5}\right)-20G_{10,3}\left(-\frac{3}{8}\bar{s}+\frac{3}{2}\bar{s}^{3}\right)+5G_{10,4}\frac{1}{2}\bar{s}\Big]\ldots\Big\}, \qquad \text{(VIII.118)}
$$

from which it is easy to find the optimum distribution of circulation along the span:

$$
\Gamma(\bar{y})=\frac{\Phi}{\pi}\sqrt{1-\bar{y}^{2}}\left\{\left[1-\frac{1}{2}G_{2,0}\bar{\tau}^{2}+\tau^{4}\left(\frac{1}{4}G_{2,0}^{2}+\frac{5}{8}G_{4,0}-G_{4,1}\right)+\right.\right.
$$

$$
+\tau^{6}\left(-\frac{1}{8}G_{2,0}^{3}-\frac{11}{16}G_{2,0}G_{4,0}+G_{2,0}G_{4,1}-\frac{9}{8}G_{6,0}+\frac{5}{2}G_{6,1}-\frac{3}{2}G_{6,2}\right)+
$$

$$
+\tau^{8}\left(\frac{1}{16}G_{2,0}^{4}+\frac{17}{32}G_{2,0}^{2}G_{4,0}-\frac{3}{4}G_{2,0}^{2}G_{4,0}+\frac{43}{32}G_{2,0}G_{6,0}-\frac{11}{4}G_{2,0}G_{6,1}+\right.
$$

$$
+\frac{3}{2}G_{2,0}G_{6,2}+\frac{33}{64}G_{4,0}^{2}-\frac{11}{8}G_{4,0}G_{4,1}+G_{4,1}^{2}+\frac{321}{128}G_{8,0}-\frac{27}{4}G_{8,1}+
$$

$$
\left.+\frac{25}{4}G_{8,2}-2G_{8,3}\right)+\tau^{10}\left(-\frac{1}{32}G_{2,0}^{5}-\frac{23}{64}G_{2,0}^{3}G_{4,0}+\frac{1}{2}G_{2,0}^{3}G_{4,1}-\right.
$$

$$
-\frac{34}{32}G_{2,0}^{2}G_{6,0}+\frac{17}{8}G_{2,0}^{2}G_{6,1}-\frac{9}{8}G_{2,0}^{2}G_{6,2}-\frac{51}{64}G_{2,0}G_{4,0}^{2}+
$$

$$
+\frac{17}{8}G_{2,0}G_{4,0}G_{4,1}-\frac{3}{2}G_{2,0}G_{4,1}^{2}-\frac{273}{118}G_{4,0}G_{6,0}+\frac{33}{8}G_{4,0}G_{6,1}-
$$

$$
-\frac{33}{16}G_{4,0}G_{6,2}+\frac{43}{16}G_{4,1}G_{6,0}-\frac{11}{2}G_{4,1}G_{6,1}+3G_{4,1}G_{6,2}-\frac{811}{256}G_{2,0}G_{8,0}+
$$

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$$
+\frac{129}{16}G_{2,0}G_{8,1}-\frac{55}{8}G_{2,0}G_{8,2}+2G_{2,0}G_{8,3}-\frac{817}{128}G_{10,0}+\frac{321}{16}G_{10,1}-
$$

$$
\left.\left.-\frac{189}{8}G_{10,2}+\frac{25}{2}G_{10,3}-\frac{5}{2}G_{10,4}\right)\right]+\left[\frac{1}{2}G_{4,0}\tau^{4}+\tau^{6}\times\right.
$$

$$
\times\left(-\frac{1}{4}G_{2,0}G_{4,0}-\frac{3}{2}G_{6,0}+2G_{6,1}\right)+\tau^{8}\left(\frac{1}{8}G_{2,0}^{2}G_{4,0}+\frac{3}{4}G_{2,0}G_{6,0}-\right.
$$

$$
\left.-G_{2,0}G_{6,1}+\frac{3}{8}G_{4,0}^{2}-\frac{1}{2}G_{4,0}G_{4,1}+\frac{59}{16}G_{8,0}-9G_{8,1}+5G_{8,2}\right)+
$$

$$
+\tau^{10}\left(-\frac{1}{16}G_{2,0}^{3}G_{4,0}-\frac{3}{8}G_{2,0}^{2}G_{6,0}+\frac{1}{2}G_{2,0}^{2}G_{6,1}-\frac{3}{8}G_{2,0}G_{4,0}^{2}+\right.
$$

$$
+\frac{1}{2}G_{2,0}G_{4,0}G_{4,1}-\frac{33}{16}G_{4,0}G_{6,0}+3G_{4,0}G_{6,1}-\frac{3}{4}G_{4,0}G_{6,2}+\frac{3}{2}G_{4,1}G_{6,0}-
$$

$$
-2G_{4,1}G_{6,1}-\frac{59}{32}G_{2,0}G_{8,0}+\frac{9}{2}G_{2,0}G_{8,1}-\frac{5}{2}G_{2,0}G_{8,2}-\frac{145}{16}G_{10,0}+
$$

$$+\frac{59}{2}G_{10,1}-\frac{63}{2}G_{10,2}+10G_{10,3}\Big]\bar{y}^3+\Big[\tau^6\Big(-\frac{1}{2}G_{6,0}\Big)+$$

$$+\tau^8\Big(\frac{1}{4}G_{2,0}G_{6,0}+\frac{23}{8}G_{8,0}-3G_{8,1}\Big)+\tau^{10}\Big(-\frac{1}{8}G_{2,0}^2G_{6,0}-$$

$$-\frac{3}{8}G_{4,0}G_{6,0}+\frac{1}{2}G_{4,1}G_{6,0}-\frac{23}{16}G_{2,0}G_{8,0}+\frac{3}{2}G_{2,0}G_{8,1}-$$

$$-\frac{165}{16}G_{10,0}+23G_{10,1}-\frac{21}{2}G_{10,2}\Big)\Big]\bar{y}^4+\Big[\tau^8\frac{1}{2}G_{8,0}+$$

$$+\tau^{10}\Big(-\frac{1}{4}G_{2,0}G_{8,0}-\frac{19}{4}G_{10,0}+4G_{10,1}\Big)\Big]\bar{y}^6-\frac{1}{2}\tau^{10}G_{10,0}\bar{y}^8+\ldots\Big\}. \quad \text{(VIII.119)}$$

Equating the expressions (VIII.103) and (VIII.119) gives the following:

$$A_1=1-\frac{1}{2}G_{2,0}\tau^2+\tau^4\Big(\frac{1}{4}G_{2,0}^2+\frac{5}{8}G_{4,0}-G_{4,1}\Big)+$$

$$+\tau^6\Big(-\frac{1}{8}G_{2,0}^3-\frac{11}{16}G_{2,0}G_{4,0}+G_{2,0}G_{4,1}-\frac{9}{8}G_{6,0}+\frac{5}{2}G_{6,1}-$$

$$-\frac{3}{2}G_{6,2}\Big)+\tau^8\Big(\frac{1}{26}G_{2,0}^4+\frac{17}{32}G_{2,0}^2G_{4,0}-\frac{3}{4}G_{2,0}^2G_{4,1}+\frac{43}{32}G_{2,0}G_{6,0}- \quad [296$$

$$-\frac{11}{4}G_{2,0}G_{6,1}+\frac{3}{2}G_{2,0}G_{6,2}+\frac{33}{64}G_{4,0}^2-\frac{11}{8}G_{4,0}G_{4,1}+G_{4,1}^2+$$

$$+\frac{321}{128}G_{8,0}-\frac{27}{4}G_{8,1}+\frac{25}{4}G_{8,2}-2G_{8,3}\Big)+\tau^{10}\Big(-\frac{1}{32}G_{2,0}^5-$$

$$-\frac{23}{64}G_{2,0}^3G_{4,0}+\frac{1}{2}G_{2,0}^3G_{4,1}-\frac{34}{32}G_{2,0}^2G_{6,0}+\frac{17}{8}G_{2,0}^2G_{6,1}-$$

$$-\frac{9}{8}G_{2,0}^2G_{6,2}-\frac{51}{64}G_{2,0}G_{4,0}^2+\frac{17}{8}G_{2,0}G_{4,0}G_{4,1}-\frac{3}{2}G_{2,0}G_{4,1}^2-$$

$$-\frac{273}{128}G_{4,0}G_{6,0}+\frac{33}{8}G_{4,0}G_{6,1}-\frac{33}{16}G_{4,0}G_{6,2}+\frac{43}{16}G_{4,1}G_{6,0}-$$

$$-\frac{11}{2}G_{4,1}G_{6,1}+3G_{4,1}G_{6,2}-\frac{811}{256}G_{2,0}G_{8,0}+\frac{129}{16}G_{2,0}G_{8,1}-$$

$$-\frac{55}{8}G_{2,0}G_{8,2}+2G_{2,0}G_{8,3}-\frac{817}{128}G_{10,0}+\frac{321}{16}G_{10,1}-$$

$$-\frac{189}{8}G_{10,2}-\frac{25}{2}G_{10,3}-\frac{5}{2}G_{10,4}\Big),$$

$$A_2=\frac{1}{2}G_{4,0}\tau^4+\tau^6\Big(-\frac{1}{4}G_{2,0}G_{4,0}-\frac{3}{2}G_{6,0}+2G_{6,1}\Big)+$$

$$+\tau^8\left(\frac{1}{8}G_{2,0}^2G_{4,0}+\frac{3}{4}G_{2,0}G_{6,0}-G_{2,0}G_{6,1}+\frac{3}{8}G_{4,0}^2-\frac{1}{2}G_{4,0}G_{4,1}+\right.$$

$$\left.+\frac{59}{16}G_{8,0}-9G_{8,1}+5G_{8,2}\right)+\tau^{10}\left(-\frac{1}{16}G_{2,0}^3G_{4,0}-\frac{3}{8}G_{2,0}^2G_{6,0}+\right.$$

$$+\frac{1}{2}G_{2,0}^2G_{6,1}-\frac{3}{8}G_{2,0}G_{4,0}^2+\frac{1}{2}G_{2,0}G_{4,0}G_{4,1}-\frac{33}{16}G_{4,0}G_{6,0}+$$

$$+3G_{4,0}G_{6,1}-\frac{3}{4}G_{4,0}G_{6,2}+\frac{3}{2}G_{4,1}G_{6,0}-2G_{4,1}G_{6,1}-$$

$$-\frac{59}{32}G_{2,0}G_{8,0}+\frac{9}{2}G_{2,0}G_{8,1}-\frac{5}{2}G_{2,0}G_{8,2}-\frac{145}{16}G_{10,0}+$$

$$\left.+\frac{59}{2}G_{10,1}-\frac{63}{2}G_{10,2}+10G_{10,3}\right),$$

$$A_3=\tau^6\left(-\frac{1}{2}G_{6,0}\right)+\tau^8\left(\frac{1}{4}G_{2,0}G_{6,0}+\frac{23}{8}G_{8,0}-3G_{8,1}\right)+$$ [297

$$+\tau^{10}\left(-\frac{1}{8}G_{2,0}^2G_{6,0}-\frac{3}{8}G_{4,0}G_{6,0}+\frac{1}{2}G_{4,1}G_{6,0}-\frac{23}{16}G_{2,0}G_{8,0}+\right.$$

$$\left.+\frac{3}{2}G_{2,0}G_{8,1}-\frac{165}{16}G_{10,0}+23G_{10,1}-\frac{21}{2}G_{10,2}\right),$$

$$A_4=\frac{1}{2}\tau^8G_{8,0}+\tau^{10}\left(-\frac{1}{4}G_{2,0}G_{8,0}-\frac{19}{4}G_{10,0}+4G_{10,1}\right), \qquad \text{(VIII.120)}$$

$$A_5=-\frac{1}{2}\tau^{10}G_{10,0}.$$

With Fr $\to\infty$ the coefficients A_i will be determined from the formulas

$$\left.\begin{aligned}
A_1&=1-\frac{1}{2}\tau^2-\frac{1}{8}\tau^4+\frac{1}{16}\tau^6+\frac{11}{128}\tau^8+\frac{17}{256}\tau^{10}\\
A_2&=\frac{1}{2}\tau^4+\frac{1}{4}\tau^6-\frac{9}{16}\tau^8-\frac{33}{32}\tau^{10}\\
A_3&=-\frac{1}{2}\tau^6+\frac{1}{8}\tau^8+\frac{9}{4}\tau^{10}\\
A_4&=\frac{1}{2}\tau^8-\tau^{10}\\
A_5&=-\frac{1}{2}\tau^{10}
\end{aligned}\right\}, \qquad \text{(VIII.121)}$$

and the correction ζ will then be as follows:

$$\zeta=\frac{1}{1-\frac{1}{2}\tau^2+\frac{1}{16}\tau^6-\frac{1}{64}\tau^{10}}. \qquad \text{(VIII.122)}$$

Using formulas (VIII.41) and (VIII.122) the following is obtained for the function $\overline{\Delta\zeta} = \zeta_э - \zeta_{опт}$ with an accuracy up to τ^{12}:

$$\overline{\Delta\zeta} = 0{,}0469\tau^{8} + 0{,}0023\tau^{10} - 0{,}0143\tau^{12}. \qquad \text{(VIII.123)}$$

Formula (VIII.123) determines the decrease in drag for an optimal hydrofoil as compared with a hydrofoil elliptical in shape. It indicates that the optimal hydrofoil can be more advantageous in regard to drag by 4-2% as compared with the wing with an elliptical distribution of circulation. The advantages of various wing shapes can be evaluated by calculating the load distribution over the [300
foil and comparing the results with those obtained from formula (VIII.122).

For hydrofoils of arbitrary shape in the plan view one may obtain optimum distribution of circulation by making the appropriate washout according to the formula [301

$$\alpha(\bar{y}) = \frac{\alpha_0}{1 + 4\frac{A_1 a_h}{\lambda_0}}\left[1 + \frac{4a_h(A_1 + \bar{y}^2A_2 + \bar{y}^4A_3 + \bar{y}^6A_4\bar{y}^8A_5)}{f_1(\bar{y})\lambda_0}\right]. \qquad \text{(VIII.124)}$$

Extensive computations of functions A_i and ζ (Tables 6 and 7) were performed using formulas (VIII.108) and (VIII.120). Figure 23 shows functions ζ for the optimum distribution of circulation for different modes of motion. [300

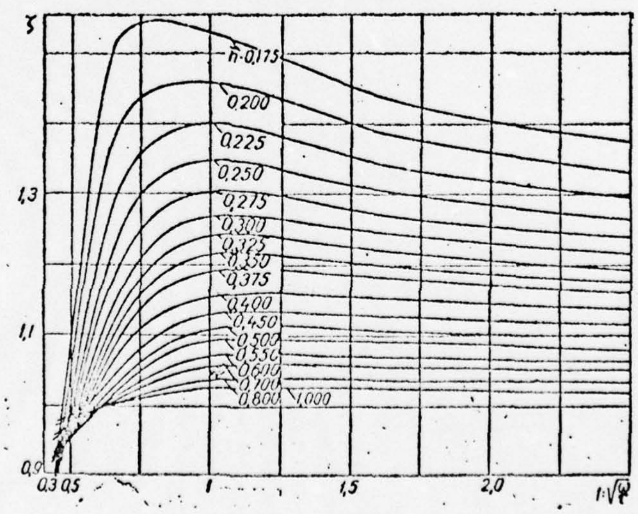

Fig. 23

Table 6

$\frac{\omega}{\tau}$	h	0,025	0,05	0,075	0,1	0,125	0,15	0,175	0,2	0,225	0,25
0,03	A_1	0,6753	0,5910	0,6148	0,6421	0,6703	0,6978	0,7239	0,7480	0,7702	0,7903
	A_2	—0,3953	—0,1524	—0,0197	0,0485	0,0796	0,0901	0,0896	0,0836	0,0753	0,0654
	A_3	1,0508	0,5942	0,3290	0,1765	0,0901	0,0420	0,0159	0,0023	—0,0043	—0,0070
	A_4	—0,2883	—0,1360	—0,0688	—0,0327	—0,0138	—0,0043	0,0002	0,0020	0,0026	0,0026
	A_5	—0,2963	—0,17997	—0,1096	—0,0669	—0,0411	—0,0254	—0,0158	—0,0099	—0,0062	—0,0039
0,06	A_1	0,5614	0,5782	0,6029	0,6312	0,6603	0,6887	0,7156	0,7405	0,7632	0,7840
	A_2	—0,3830	—0,1448	—0,0150	0,0513	0,0813	0,0910	0,0900	0,0838	0,0753	0,0663
	A_3	0,0102	0,5780	0,3194	0,1709	0,0868	0,0401	0,0149	0,0918	—0,0045	—0,0072
	A_4	—0,2453	—0,1286	—0,0646	—0,0303	—0,0125	—0,0036	0,0006	0,0022	0,0027	0,0028
	A_5	—0,2893	—0,1757	—0,1070	—0,0654	—0,0401	—0,0248	—0,0154	0,0096	—0,0061	—0,0086
0,10	A_1	0,5436	0,5616	0,5876	0,6171	0,6474	0,6769	0,7048	0,7307	0,7343	0,7758
	A_2	—0,3873	—0,1350	—0,0090	0,0549	0,0853	0,0922	0,0906	0,8399	0,0733	0,0552
	A_3	0,9876	0,5866	0,3067	0,1634	0,0825	0,0976	0,0135	0,0018	—0,0049	—0,0073
	A_4	—0,2278	—0,1185	—0,0388	0,0271	—0,0187	—0,0029	0,0011	0,0028	0,0028	0,0026
	A_5	0,2801	—0,1701	—0,1036	0,0633	—0,0389	—0,0240	—0,0149	—0,0093	—0,0069	—0,0038
0,40	A_1	1,4469	0,8311	0,5566	0,4565	0,4524	0,4847	0,5321	0,5824	0,6298	0,6723
	A_2	—1,9854	—1,6675	—0,7059	—0,4581	—0,2906	—0,1788	—0,1053	—0,0576	—0,0272	—0,0033
	A_3	0,7199	0,3959	0,2108	0,1066	0,0491	0,0183	0,0025	—0,0051	—0,0081	—0,0038
	A_4	—0,0898	—0,0390	—0,0133	—0,0012	0,0085	0,0056	0,0055	0,0049	0,0041	0,0033
	A_5	—0,2146	—0,1304	—0,0794	—0,0485	—0,0298	—0,0183	—0,0114	—0,0072	—0,0048	—0,0029

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Table 6 (cont.)

$\frac{\omega}{\tau}$	h	0,3	0,35	0,4	0,45	0,5	0,6	0,7	0,8	1,0	1,2
0,03	A_1	0,8247	0,8526	0,8751	0,8933	0,9082	0,9304	0,9458	0,9568	0,9709	0,9792
	A_2	0,0500	0,0370	0,0273	0,0203	0,0152	0,0089	0,0054	0,0035	0,0016	0,0008
	A_3	—0,0075	—0,0059	—0,0043	—0,0030	—0,0021	—0,0010	—0,0005	—0,00036	—0,0001	—0,00003
	A_4	0,0016	0,0012	0,0008	0,0005	0,0003	0,0001	0,00004	0,00002	0,000004	0,000001
	A_5	—0,0016	—0,0007	—0,0003	—0,0001	—0,0001	—0,00002	—0,00001	—0,000002	—0,000003	—0,00000
0,06	A_1	0,8195	0,8482	0,8713	0,8901	0,9054	0,9283	0,9441	0,9554	0,9700	0,9785
	A_2	0,0498	0,0368	0,0272	0,0202	0,0191	0,0088	0,0068	0,0034	0,0019	0,0008
	A_3	—0,0075	—0,0059	—0,0042	—0,0029	—0,0020	—0,0009	—0,0005	—0,0003	—0,00008	—0,00003
	A_4	0,0019	0,0012	0,0007	0,0004	0,0003	0,0001	—0,00004	0,0002	0,000004	0,000001
	A_5	0,0016	—0,0007	—0,0003	—0,0001	—0,00007	—0,00002	—0,00001	—0,000002	—0,0000003	0,00000
0,10	A_1	0,8126	0,8425	0,8665	0,8860	0,9018	0,9256	0,9421	0,9638	0,9689	0,9778
	A_2	0,0496	0,0366	0,0269	0,0260	0,0149	0,0087	0,0063	0,0033	0,0018	0,0008
	A_3	—0,0075	—0,0038	—0,0042	—0,0027	—0,0020	—0,0009	—0,0004	—0,0003	—0,00008	—0,00003
	A_4	0,0019	0,0012	0,0007	0,0004	0,00003	0,0001	0,00004	0,00002	0,000004	0,000001
	A_5	—0,0016	—0,0006	—0,00003	—0,0001	—0,00006	—0,00005	—0,000005	—0,000002	—0,000000	—0,00000
0,40	A_1	0,7412	0,7920	0,8296	0,8580	0,8799	0,9109	0,9315	0,9457	0,9637	0,9741
	A_2	0,0096	0,0145	0,0144	0,0125	0,0104	0,0058	0,0044	0,0029	0,0018	0,0007
	A_3	—0,0079	—0,0055	—0,0038	—0,0029	—0,0017	0,0008	—0,0004	—0,0002	—0,0001	—0,00002
	A_4	0,0019	0,0011	0,0007	0,0004	0,0002	0,00008	0,00003	0,00001	0,000003	0,0000009
	A_5	—0,0012	—0,0005	—0,0002	—0,0001	—0,00005	—0,00001	—0,000004	—0,000001	—0,0000001	—0,00000

Table 7 [300

$\bar{h}$	Values of ζ with ω/τ and $1/\sqrt{\omega/\tau}$									
	5,0	4,0	3,0	2,4	1,8	1,4	1,0	0,8	0,4	0,1
	0,4472	0,5	0,5773	0,6455	0,7453	0,8451	1,00	1,118	1,5812	3,162
0,175	0,9156	1,157	1,535	1,696	1,697	1,648	1,612	1,598	1,502	1,372
0,200	0,9003	1,078	1,346	1,496	1,542	1,539	1,527	1,516	1,439	1,330
0,225	0,8932	1,030	1,230	1,366	1,433	1,452	1,455	1,448	1,387	1,294
0,250	0,8912	1,001	1,167	1,275	1,351	1,382	1,395	1,391	1,342	1,263
0,275	0,8922	0,9810	1,116	1,216	1,290	1,326	1,344	1,343	1,304	1,236
0,3	0,8952	0,9698	1,084	1,167	1,245	1,280	1,302	1,303	1,272	1,213
0,325	0,8993	0,9621	1,060	1,135	1,205	1,2431	1,266	1,269	1,244	1,192
0,350	0,9039	0,9578	1,042	1,106	1,175	1,212	1,236	1,240	1,220	1,175
0,375	0,9089	0,9552	1,031	1,090	1,150	1,186	1,210	1,215	1,199	1,159
0,4	0,9139	0,9547	1,021	1,077	1,131	1,164	1,188	1,193	1,181	1,145
0,45	0,9236	0,9554	1,006	1,058	1,101	1,131	1,153	1,158	1,150	1,122
0,5	0,9323	0,9578	1,001	1,036	1,076	1,106	1,126	1,132	1,127	1,104
0,55	0,9401	0,9608	0,9975	1,026	1,067	1,088	1,106	1,111	1,108	1,089
0,6	0,9468	0,9640	0,9951	1,021	1,052	1,073	1,090	1,095	1,093	1,077
0,7	0,9577	0,9702	0,9931	1,014	1,034	1,053	1,067	1,071	1,071	1,010
0,8	0,9658	0,9751	0,9926	1,006	1,025	1,041	1,052	1,056	1,056	1,047
1,0	0,9766	0,98259	0,9936	1,002	1,012	1,025	1,033	1,036	1,037	1,031
1,2	0,9831	0,9872	0,9951	1,000	1,011	1,017	1,023	1,025	1,026	1,022

The analysis of the results shows that the optimum submerged hydrofoil is fuller in shape (and, therefore, fuller in the plan view with α = const and Fr $\to \infty$) in comparison with an elliptical hydrofoil. However, from the essence of the problem it follows that in the limiting case ($h \to 0$, Fr $\to \infty$) the elliptical form will again be optimal and that for a certain shallow submergence depth the hydrofoil will have the fullest form. This fact has a definite practical value, because by taking the technological strength factors into consideration it becomes apparent that the manufacture of rectangular hydrofoils is advantageous, and the motion at various modes, corresponding to the maximum fullness, can produce a definite saving. [301

8.10. Regularization Methods of Integro-Differential Equations

As was noted earlier, in the regularization of the singular integral equations it is important for the theories being developed to obtain regular integral equations, because they offer a new approach for the study, based on the well-developed methods for solving integral equations.

Singular integral and integro-differential equations for the submerged hydrofoil differ from the equations for an airplane wing because of the existence of a regular integral operator. Therefore, all the regularization methods used in aerodynamics as well as certain exact closed solutions can be used in our study.

The Prandtl equation can easily be transformed into a quasi-Fredholm equation [15, 78]. Let us use this approach for obtaining the quasi-Fredholm equation for the submerged hydrofoil.

In the general case, the integro-differential equation for the submerged hydrofoil may be written as follows

$$\Gamma(y)=\frac{a_h}{2\lambda(y)}\left[\alpha(y)-\frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'(\eta)}{y-\eta}d\eta+\frac{1}{2\pi}\int_{-1}^{+1}\Gamma(\eta)G_1(y-\eta)\,d\eta\right] \quad \text{(VIII.125)}$$

or [302

$$\frac{1}{\pi}\int_{-1}^{+1}\frac{\Gamma'(\eta)}{y-\eta}d\eta=\varphi(y),$$

$$\varphi(y)=2\left[\frac{2\Gamma(y)\lambda(y)}{a_h}-\alpha(y)-\frac{1}{2\pi}\int_{-1}^{+1}\Gamma(\eta)G_1(y-\eta)\,d\eta\right]. \quad \text{(VIII.126)}$$

The solution of equation (VIII.126) in the class of functions unlimited at both ends of the line of cut is determined by the inversion formula of Cauchy's integral [15]:

$$\Gamma'(y)=\frac{1}{\sqrt{1-y^2}}\left[\frac{1}{\pi}\int_{-1}^{+1}\frac{\sqrt{1-\eta^2}}{\eta-y}\varphi(\eta)\,d\eta+a_0\right],$$

where a_0 is an arbitrary constant.

Integrating this equation with respect to y, we arrive at the quasi-Fredholm equation

$$\Gamma(y)+\int_{-1}^{+1}K_1(y,\eta)\Gamma(\eta)\,d\eta=F(y)+C_1+C_2\arcsin y, \quad \text{(VIII.127)}$$

where

$$K_1(y,\eta)=K(y,\eta)-\frac{4}{\pi a_h}\int_{-1}^{+1}\lambda(t)K(y,t)G(\eta-t)\,dt;$$

$$K(y,\eta)=\frac{a_h}{8\pi\lambda(\eta)}\ln\left(\frac{1-y\eta+\sqrt{(1-y^2)(1-\eta^2)}}{1-y\eta-\sqrt{(1-y^2)(1-\eta^2)}}\right);$$

$$F(y)=\frac{8}{a_h}\int_{-1}^{+1}\lambda(\eta)K(y,\eta)\alpha(\eta)\,d\eta.$$

The arbitrary constants C_1 and C_2 will be determined by the conditions at the ends of the interval.

This method may be applied to equation (VII.25) and to its modification which includes the case of submerged hydrofoils.

For the general case, the equation (VII.25) may be written as follows:

$$\alpha(y) = \frac{1}{\pi}\int_{-1}^{+1}\frac{\Gamma'(\eta)}{y-\eta}d\eta +$$

$$+\frac{1}{2\pi}\int_{-1}^{+1}\frac{\overline{\Gamma}(\eta)}{(y-\eta)^2}\left(1-\frac{1}{\sqrt{1+\lambda^2(y)(y-\eta)^2}}\right)d\eta +$$ [303

$$+\frac{1}{2\pi}\int_{-1}^{+1}\overline{\Gamma}(\eta)G_2(y,\eta)\,d\eta, \qquad \text{(VIII.128)}$$

where $G_2(y, \eta)$ is the regular nucleus which takes into consideration the free surface effect.

In the similar manner we obtain the equation (VIII. 127), where

$$K_1(y,\eta) = \frac{1}{4\pi^2}\int_{-1}^{+1}\ln\left(\frac{1-yt+\sqrt{(1-y^2)(1-t)^2}}{1-yt-\sqrt{(1-y^2)(1-t)^2}}\right)G_3(t,\eta)\,dt;$$

$$G_3(t,\eta) = \frac{1}{(t-\eta)^2}\left(1-\frac{1}{\sqrt{1+\lambda^2(t)(t-\eta)^2}}\right)+G_2(t,\eta);$$

$$F(y) = \frac{1}{2\pi}\int_{-1}^{+1}\ln\left(\frac{1-y\eta+\sqrt{(1-y^2)(1-\eta^2)}}{1-y\eta-\sqrt{(1-y^2)(1-\eta^2)}}\right)\alpha(\eta)\,d\eta.$$

The equation (VIII.128) can be reduced to a regular equation using a different approach.

Let us transform the equation (VIII.128) to the following form:

$$\alpha(y) = \frac{1}{\pi}\int_{-1}^{+1}\frac{\Gamma'(\eta)}{(y-\eta)}d\eta - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'(\eta)}{(y-\eta)}[1-\sqrt{1+\lambda^2(y)(y-\eta)^2}]d\eta -$$

$$-\frac{1}{2\pi}\int_{-1}^{+1}\Gamma'(\eta)\,G_4(y,\eta)\,d\eta. \qquad \text{(VIII.130)}$$

Then, using the inversion formula, we obtain the quasi-regular equation directly:

$$\varphi(y)+\frac{1}{2\pi^2}\int_{-1}^{+1}\varphi(\eta)\,K(y,\eta)\,d\eta=F(y), \qquad \text{(VIII.131)}$$

where

$$\varphi(y)=\frac{\Gamma'(\eta)}{\sqrt{1-\eta^2}};$$

$$K(y,\eta)=+\frac{1}{\sqrt{1-\eta)^2}}\int_{-1}^{+1}\frac{\sqrt{1-t^2}}{(t-y)}\left[\frac{1-\sqrt{1+\lambda^2(t)(t-\eta)^2}}{t-\eta}+\right.$$ [304

$$\left.+G_4(y,\eta)\right]d\eta;$$

$$F(y)=\frac{1}{\pi}\int_{-1}^{+1}\frac{\sqrt{1-\eta^2}}{\eta-y}\,\alpha(y)\,d\eta+a_0.$$

The circulation $\Gamma(y)$ will be found from the formula

$$\Gamma(\eta)=\int^{y}\frac{\varphi(\eta)}{\sqrt{1-\eta^2}}\,d\eta+c.$$

Let us proceed now to a more complicated problem of obtaining the regular equation using equation (VIII.125).

An effective solution of this problem for an airplane wing is given by I. N. Vekua [9]. The solution has also been obtained by L. G. Magnaradze [78, 91] using a different approach. Similar results were obtained by G. F. Burago [7]. Although he obtained his results before I. N. Vekua and L. G. Magnaradze, Burago did not publish them in time.

I. N. Vekua obtained the new integral equation by solving Rieman's boundary value problem to which the Prandtl equation is reduced.

If we denote

$$\beta(y)=\frac{1}{2}\left[\alpha(y)+\frac{1}{\pi}\int_{-1}^{+1}\Gamma(\eta)\,G_1(y-\eta)\,d\eta\right.$$

and

$$B(y)=\frac{2a_h}{\lambda(y)}, \qquad \text{(VIII.132)}$$

then the equation (VIII.125) may be written as follows:

$$\frac{8\pi}{B(y)}\Gamma(y)-\int_{-1}^{+1}\frac{\Gamma'(\eta)}{\eta-y}d\eta=4\pi\beta(y). \qquad \text{(VIII.133)}$$

Let us examine the case

$$\begin{gathered}\Gamma(y)=\Gamma(-y);\quad B(y)=B(-y);\\ \alpha(y)=\alpha(-y);\ y\in(-1,1).\end{gathered} \qquad \text{(VIII.134)}$$

Now we can apply Vekua's method to the regularization of equation (VIII.133).

This problem can be analyzed in greater detail [9]. Here, we will confine ourselves to the case of [305

$$B(y)=\frac{\sqrt{1-y^2}}{p(y)}\quad y\in(-1,+1),$$

where p(y) is the analytical function at the segment (-1, +1), which satisfies the conditions

$$p(y)>0,\quad p(y)=p(-y),\quad y\in(-1;+1).$$

There apparently exists a single-link area T, which contains the segment (-1, +1) and which is limited by a smooth closed curve L. In addition, the function $p(\zeta)$ is holomorphic within T + L. Eliminating points in the segment (-1, +1) from the area T, we obtain a doubly-connected area T*, in which function $b(\zeta)=\frac{\sqrt{1-\zeta^2}}{p(\zeta)}$ is holomorphic.

Later on we will analyze a branch of this function that satisfies the condition

$$B_+(y)=-B_-(y)=B(y)>0.$$

Let us assume that functions $\Gamma(y)$, $\alpha(y)$ and $\Gamma'(y)$, continuous according to Gel'der within the segment (-1, +1), are

$$\Gamma'(y)=\frac{\Gamma^*(y)}{(1-y^2)^\lambda}\qquad \lambda<1.$$

Let us analyze Cauchy's integral:

$$\Phi(\zeta)=\frac{1}{2\pi i}\int_{-1}^{+1}\frac{\Gamma'(\eta)\,d\eta}{\eta-\zeta}. \tag{VIII.135}$$

Differentiating with respect to ζ and integrating by parts using the condition (VIII.134), we obtain

$$\Phi'(\zeta)=\frac{1}{2\pi i}\int_{-1}^{+1}\frac{\Gamma'(\eta)}{\eta-\zeta}d\eta-\frac{\Gamma(1)}{\pi i(1-\zeta^2)}. \tag{VIII.136}$$

For practical purposes, of interest is the case when $\Gamma(-1)=\Gamma(1)=0$. However, this problem can also be analyzed by disregarding the above restriction and using it in the final results only.

The functions $\Phi(\zeta)$ and $\Phi'(\zeta)$ are continuous to the points within the segment (-1, +1) from both the upper and lower half-planes. At the ends of the segment (-1, +1) the function $\Phi(\zeta)$ can have only the logarithmic-type singularities, while the integral term (VIII.136), at these points, can have singularities of the order of less than unity.

Using the Yu. A. Sokhotskiy formulas we obtain [306

$$\begin{gathered}\Gamma(y)=\Phi_+(y)-\Phi_-(y),\\ \int_{-1}^{+1}\frac{\Gamma'(\eta)}{\eta-y}d\eta=\pi i\,[\Phi'_+(y)+\Phi'_-(y)]+\frac{2\Gamma(1)}{1-y^2}.\end{gathered} \tag{VIII.137}$$

Then the equation (VIII.133) will be in the form

$$\Phi'_+(y)+8i\frac{\Phi_+(y)}{B_+(y)}+\Phi'_-(y)+8i\frac{\Phi_-(y)}{B_-(y)}=4iv\beta(y)+\frac{2i\Gamma(1)}{\pi(1-y^2)}. \tag{VIII.138}$$

Let us introduce a new function

$$F(\zeta)=\left[\Phi'(\zeta)+8i\frac{\Phi(\zeta)}{B(\zeta)}\right]\sqrt{1-\zeta^2}, \tag{VIII.139}$$

which is apparently holomorphic in T* and continuous up to the points within the segment (-1, +1) from both the upper and lower half-planes.

The equation (VIII.138) will then be as follows:

$$F_+(y) - F_-(y) = 4iv\,\beta(x)\sqrt{1-y^2} + \frac{2i\Gamma(1)}{\pi\sqrt{1-y^2}}. \qquad \text{(VIII.140)}$$

According to Cauchy's formula

$$F(\zeta) = \frac{1}{2\pi i}\int_{-a}^{+a}\frac{F_+(\eta) - F_-(\eta)}{\eta - \zeta}\,d\eta + \frac{1}{2\pi i}\int_L \frac{F(\eta)}{\eta - \zeta}\,d\eta, \qquad \text{(VIII.141)}$$

where ζ is the point which belongs to the area T*.

However, considering (VIII.139) and (VIII.140) it follows from formula (VIII.141):

$$F(\zeta) = \frac{2}{\pi}\int_{-1}^{+1}\frac{\beta(\eta)\sqrt{1-\eta^2}}{\eta-\zeta}\,d\eta + \frac{\Gamma(1)}{\pi^2}\int_{-1}^{+1}\frac{1}{\sqrt{1-\eta^2}}\frac{d\eta}{\eta-\zeta} +$$

$$+ \frac{1}{2\pi i}\int_L \frac{\sqrt{1-\eta^2}\,\Phi'(\eta)}{\eta-\zeta}\,d\eta + \frac{4}{\pi}\int_L \frac{\sqrt{1-\eta^2}\,\Phi(\eta)}{B(\eta)(\eta-\zeta)}\,d\eta. \qquad \text{(VIII.142)}$$

Using the formula and Cauchy's theorem we obtain

$$\int_{-1}^{+1}\frac{1}{\sqrt{1-\eta^2}}\frac{d\eta}{\eta-\zeta} = \frac{\pi i}{\sqrt{1-\eta^2}}; \quad \frac{1}{2\pi i}\int_L \frac{\sqrt{1-\eta^2}\,\Phi'(\eta)\,d\eta}{\eta-\zeta} = 0.$$

Then, due to the fact that $B(y) = \frac{\sqrt{1-y^2}}{p(y)}$ and based on (VIII.138), [307

$$\int_L \frac{\sqrt{1-\eta^2}}{B(\eta)}\cdot\frac{\Phi(\eta)}{\eta-\zeta}\,d\eta = \frac{1}{r\pi i}\int_{-1}^{+1}\Gamma(\sigma)\,d\sigma\int_L \frac{p(\eta)\,d\eta}{(\eta-\zeta)(\sigma-\eta)} =$$

$$= \int_{-1}^{+1}\frac{\Gamma(\sigma)}{\sigma-\zeta}\,d\sigma\left(\frac{1}{2\pi i}\int_L \frac{p(\eta)\,d\eta}{\eta-\zeta} + \frac{1}{2\pi i}\int_L \frac{p(\eta)\,d\eta}{\sigma-\eta}\right) =$$

$$= -\int_{-1}^{+1}\frac{p(\sigma)-p(\zeta)}{\sigma-\zeta}\,\Gamma(\sigma)\,d\sigma. \qquad \text{(VIII.143)}$$

Then

$$F(\zeta) = \frac{2v}{\pi}\int_{-1}^{+1}\frac{\beta(\eta)\sqrt{1-\eta^2}}{\eta-\zeta}\,d\eta - \frac{4}{\pi}\int_{-1}^{+1}\frac{p(\sigma)-p(\zeta)}{\sigma-\zeta}\,\Gamma(\sigma)\,d\sigma +$$

$$+ \frac{i\Gamma(1)}{\pi\sqrt{1-\zeta^2}}, \qquad \text{(VIII.144)}$$

and by passing to the limit we obtain

$$F_+(y)+F_-(y)=\frac{4}{\pi}\int_{-1}^{+1}\frac{\beta(\eta)\sqrt{1-\eta^2}}{\eta-y}d\eta-\frac{8}{\pi}\int_{-1}^{+1}\frac{\rho(\sigma)-\rho(y)}{\sigma-y}\Gamma(\sigma)\,d\sigma. \quad \text{(VIII.145)}$$

From the expression (VIII.139) we obtain the following:

$$\begin{aligned}\Phi'_+(y)+\frac{8i}{B(y)}\Phi_+(y)&=\frac{F_+(y)}{\sqrt{1-y^2}},\\ \Phi'_-(y)-\frac{8i}{B(y)}\Phi_-(y)&=-\frac{F_-(y)}{\sqrt{1-y^2}}.\end{aligned} \quad \text{(VIII.146)}$$

Integrating these equations, we obtain

$$\begin{aligned}\Phi_+(y)&=\Phi_+(0)\,e^{-i\Theta(y)}+\int_0^y e^{i[\Theta(\eta)-\Theta(y)]}\frac{F_+(\eta)\,d\eta}{\sqrt{1-\eta^2}},\\ \Phi_-(y)&=\Phi_-(0)\,e^{i\Theta(y)}-\int_0^y e^{-i[\Theta(\eta)-\Theta(y)]}\frac{F_-(\eta)\,d\eta}{\sqrt{1-\eta^2}},\end{aligned} \quad \text{(VIII.147)}$$

$$\Theta(y)=8\int_0^y\frac{d\eta}{B(\eta)}. \quad \text{(VIII.148)}$$ [308

Because function $\Gamma(y)$ is even, from the expression (VIII. 135) we obtain the following:

$$\Phi_+(0)=-\Phi_-(0)=\frac{1}{2}\Gamma(0).$$

It follows, therefore, from formulas (VIII.137) and (VIII. 147) that

$$\Gamma(y)=\Gamma(0)\cos\theta(y)+\int_0^y\frac{\cos[\theta(\eta)-\theta(y)]}{\sqrt{1-\eta^2}}[F_+(\eta)+F_-(\eta)]\,d\eta+$$
$$+\int_0^y\frac{\sin[\theta(\eta)-\theta(y)]}{\sqrt{1-\eta^2}}i[F_+(\eta)-F_-(\eta)]\,d\eta. \quad \text{(VIII.149)}$$

Taking into account expressions (VIII.109) and (VIII. 114) we can write the equation as follows:

$$\Gamma(y)=\Gamma(0)\cos\theta(y)+\Gamma(1)\,\omega_0(y)+\int_{-1}^{+1}K_0(y,\sigma)\Gamma(\sigma)\,d\sigma+g_0(y), \quad \text{(VIII.150)}$$

where

$$\omega_0(y)=-\frac{2}{\pi}\int_0^y\frac{\sin[\theta(\eta)-\theta(y)]}{1-\eta^2}\,d\eta; \quad \text{(VIII.151)}$$

$$g_0(y) = -4\int_0^y \sin[\theta(\eta) - \theta(y)]\beta(\eta)\,d\eta +$$

$$+\frac{4}{\pi}\int_0^y \frac{\cos[\theta(\eta) - \theta(y)]}{\sqrt{1-\eta^2}}\,d\eta\int_{-1}^{+1}\frac{\beta(\sigma)\sqrt{1-\sigma^2}}{\sigma-\eta}\,d\sigma;$$

$$K_0(y, \sigma) = -\frac{8}{\pi}\int_0^y \frac{\cos[\theta(\eta) - \theta(y)]}{\sqrt{1-\eta^2}}\cdot\frac{p(\sigma - p(\eta)}{\sigma-\eta}\,d\eta. \quad \text{(VIII.152)}$$

Introducing into equation (VIII.150) the condition $\Gamma(1) = \Gamma(-1) = 0$, we can rewrite the equation as follows:

$$\Gamma(y) = \Gamma(0)\cos\theta(y) + \int_{-1}^{+1} K_0(y, \sigma)\Gamma(\sigma)\,d\sigma + g_0(y). \quad \text{(VIII.153)}$$

At this point we can substitute $\beta(y)$ given in (VIII.132) into the expression for $g_0(y)$ (VIII.151): [309

$$g_0(y) = g_1(y) - \frac{1}{\pi}\int_0^y \sin[\theta(\eta) - \theta(y)]\,d\eta\int_{-1}^{+} G_1(\eta - \sigma)\Gamma(\sigma)\,d\sigma +$$

$$+\frac{1}{\pi^2}\int_0^y \frac{\cos[\theta(\eta) - \Theta(y)]}{\sqrt{1-\eta^2}}\,d\eta\,\frac{\sqrt{1-\sigma^2}}{\sigma-\eta}\int_{-1}^{+1} G_1(\sigma - t)\Gamma(t)\,dt, \quad \text{(VIII.154)}$$

$$g_1(y) = -2\int_0^y \sin[\theta(y) - \theta(\eta)]\alpha(\eta)\,d\eta +$$

$$+\frac{2}{\pi}\int_0^y \frac{\cos[\theta(\eta) - \theta(y)]}{\sqrt{1-\eta^2}}\,d\eta\int_{-1}^{+1}\frac{\alpha(\sigma)\sqrt{1-\sigma^2}}{\sigma-\eta}\,d\sigma. \quad \text{(VIII.155)}$$

If α = const,

$$g_1(y) = -2\alpha\int_0\left\{\sin[\theta(\eta) - \theta(y)] + \frac{\eta\cos[\theta(\eta) - \theta(y)]}{\sqrt{1-\eta^2}}\right\}d\eta.$$

The equation (VIII.153) will now acquire the following form:

$$\Gamma(\eta) = \Gamma(0)\cos\theta(y) + \int_{-1}^{+1} K_1(y, \sigma)\Gamma(\sigma)\,d\sigma + g_1(y). \quad \text{(VIII.157) [sic]}$$

where the new nucleus $K_1(y, \sigma)$ is defined as follows:

$$K_1(y, \sigma) = K_0(y, \sigma) - \frac{1}{\pi}\int_0^y G_1(\eta - \sigma)\sin[\theta(\eta) - \theta(y)]\,d\eta +$$

$$+\frac{1}{\pi^2}\int_0^y \frac{\cos[\theta(\eta)-\theta(y)]}{\sqrt{1-\eta^2}}\,d\eta \int_{-1}^{+1} G_1(t-\sigma)\frac{\sqrt{1-t^2}}{t-\eta}\,dt. \qquad \text{(VIII.158)}$$

In the future it is important to analyze two cases which correspond to the following conditions:

$$\cos\theta(1) \neq 0, \quad \cos\theta(1) = 0.$$

For the case of $\cos\theta(1) \neq 0$ the equation obtained will result in the Fredholm equation we were looking for:

$$\Gamma(y) - \int_{-1}^{+1} K(y,\sigma)\Gamma(\sigma)\,d\sigma = g(y), \qquad \text{(VIII.159)}$$

where [310

$$g(y) = g_1(y) - \frac{g_1(1)\cos\theta(y)}{\cos\theta(1)};$$

$$K(y,\sigma) = K_1(y,\sigma) - \frac{K_1(1,\sigma)\cos\theta(y)}{\cos\theta(1)}.$$

For the second condition of $\cos\theta(1) = 0$ we obtain the following from equation (VIII.157):

$$\int_{-1}^{+1} K_1(1,\sigma)\Gamma(\sigma)\,d\sigma + g_1(1) = 0 \qquad \text{(VIII.160)}$$

and this brings us to the following system of equations:

$$\Gamma(y) = \Gamma(0)\cos\theta(y) + \int_{-1}^{+1} K_1(y,\sigma)\Gamma(\sigma)\,d\sigma + g_1(y),$$

$$\int_{-1}^{+1} K_1(1,\sigma)\Gamma(\sigma)\,d\sigma + g_1(1) = 0. \qquad \text{(VIII.161)}$$

The first of these equations contains the indeterminate constant $\Gamma(0)$, which must be determined from the second equation. In case this constant is not determined from the second equation, then it must be chosen so that the solution of the first equation would give the Prandtl equation.

Thus, instead of the singular integro-differential equation (VIII.125), we will obtain Fredholm's equation (VIII.159) and the system (VIII.161), whose effective solution may be obtained for various wing shapes.

The following section is devoted to the derivation of one such solution.

8.11. The Solution of the Regular Integral Equation*

*This chapter was written with A. V. Miodushevskaya.

For the airplane wing for which the function

$$p(y) = \frac{\sqrt{1-\bar{y}^2}}{B(y)}$$

is rational, the nuclei of the equations resulting from the transformation of (VIII.159) and (VIII.161) degenerate when $h \to \infty$ $G_1(y - \eta)$ and the function $\Gamma(y)$ to be determined is found in the explicit form within the quadratures. Let us present two examples of such solutions obtained by I. N. Vekua [9].

For the elliptical wing

$$B(y) = B_0\sqrt{1-\bar{y}^2};\ p(y) = \text{const},\ K_0(x, \sigma) = 0$$

and

$$\Gamma(y) = \Gamma(0)\cos\theta(y) + g_0(y),$$

$$\Theta(y) = x \arcsin \bar{y};\ \chi = \frac{8}{B_0}. \tag{VIII.162}$$

When α = const, the formula (VIII.161) acquires the form [311

$$\Gamma(y) = \Gamma(0)\cos\theta(y) - \frac{2\alpha}{1+x}\cos\theta(y) + \frac{2\alpha}{1+x}\sqrt{1-\bar{y}^2}.$$

For the condition $\Gamma(\pm 1) = 0$ we obtain the known results

$$\Gamma(\bar{y}) = \frac{2\alpha}{1+x}\sqrt{1-\bar{y}^2}.$$

Let us examine a wing which is described by the function

$$B(x) = B_0\sqrt{1-\bar{y}^2}\,\frac{1+\nu\bar{y}^2}{1+\mu\bar{y}^2}, \tag{VIII.163}$$

where ν and μ are constants and $\mu > -1$, $\nu > -1$. In this case

$$K(y, \eta) = \frac{\chi(\nu-\mu)}{1+\nu\eta^2}\varphi_1(y) + \frac{\chi(\nu-\mu)\eta}{1+\nu\eta^2}\varphi_2(y), \tag{VIII.164}$$

$$\varphi_k(y) = \int_0^y \frac{\cos[\theta(\sigma) - \theta(y)]}{\sqrt{1-\sigma^2}}\,\frac{\sigma^{k-1}}{1+\nu\sigma^2}\,d\sigma -$$

$$-\frac{\cos\theta(y)}{\cos\theta(1)}\int_0^1\frac{\cos[\theta(\sigma)-\theta(y)]}{\sqrt{1-\sigma^2}}\frac{\sigma^{k-1}}{1+\nu\sigma^2}d\sigma,\ (k=1,2\ldots)$$

$$\theta(y)=\frac{\chi\mu}{\nu}\arcsin\bar{y}+\frac{\chi(\nu-\mu)}{\nu\sqrt{1+\nu}}\operatorname{arctg}\frac{\bar{y}\sqrt{1+\nu}}{\sqrt{1-\bar{y}^2}}, \qquad \text{(VIII.165)}$$

with $\nu \neq 0$

$$\theta(y)=\chi\left(1+\frac{\mu}{2}\right)\arcsin\bar{y}-\frac{\chi\mu}{2}\bar{y}\sqrt{1-\bar{y}^2}, \qquad \text{(VIII.166)}$$

with $\nu = 0$.
Then

$$\Gamma(y)=\chi(\nu-\mu)\varphi_1(y)\int_{-1}^{+1}\frac{\Gamma(\sigma)\,d\sigma}{1+\nu\sigma^2}+g(y). \qquad \text{(VIII.167)}$$

From this we find

$$\Gamma(y)=g(y)+\left[\chi(\nu-\mu)\int_{-1}^{+1}\frac{g(\sigma)\,d\sigma}{1+\nu\sigma^2}\right]\times$$

$$\times\left[1-\chi(\nu-\mu)\int_{-1}^{+1}\frac{\varphi_1(\sigma)\,d\sigma}{1+\nu\sigma^2}\right]\varphi_1(x). \qquad \text{(VIII.168)}$$

Formula (VIII.168) gives a solution to the equation for the aircraft wing in the explicit form for any angle of attack and shapes described by the expression (VIII. 163). This formula is of great interest because (with various parameters) ν and μ are applicable to a great number of practically important types of wings. [312

For the submerged hydrofoil the rationality of the function p(y) is not sufficient for obtaining the integral equation with the degenerated nucleus.

Let us examine the foils for which

$$p(y)=\frac{R(y)}{\Theta(y)},$$

$$p(y)=\sum_{k=0}^{r}p_k y^k,\quad \Theta(y)=\sum_{k=0}^{g}g_k y^k,$$

$$\frac{p(\sigma)-p(\eta)}{\sigma-\eta}=\frac{p(\sigma)\Theta(\eta)-p(\eta)\Theta(\sigma)}{\Theta(\sigma)\Theta(\eta)(\sigma-\eta)}=\frac{1}{\Theta(\sigma)\Theta(\eta)}\sum_{k=1}^{n}R_k(\eta)\,\sigma^{k-1},$$

where n - the larger of the numbers r and g;
$R_k(\eta)$ - a polynomial of the order not higher than n - 1.

Now

$$K_0(y, \sigma) = \sum_{k=1}^{n} \frac{\sigma^{k-1}}{\theta(\delta)} U_k(y),$$

$$U_k(y) = -\frac{8}{\pi} \int_{\delta}^{y} \frac{R_k(\eta)}{\theta(\eta)} \frac{\cos[\theta(\eta) - \Theta(y)]}{\sqrt{1-\bar{\sigma}^2}} d\sigma. \qquad \text{(VIII.169)}$$

If we could now represent the nucleus $K_1(y, \sigma)$, which is defined by the $G_1(y - \eta)$ function, in the form of $\sum_{l=1}^{k} x_l(y) x_l(\sigma)$, the nucleus $K(y, \sigma)$ would become degenerate.

From the expansion of (VIII.33) it is easy to obtain the expansion of the function $G_1(y - \eta)$ along the exponents of parameter $\tau = \sqrt{4\bar{h}^2 + 1} - 2\bar{h}$;

$$G_1(y-\eta) = \sum_{n=1}^{\infty} \tau^{2n} \sum_{k=0}^{n-1} \frac{(2n-1-k)\ldots(k+1)}{(2k-1-2k)!} \times$$
$$\times (-1)^{k-1} G_{2,n,k}(\lambda)(y-\eta)^{(2n-1-2k)},$$

but since [313

$$(y-\eta)^{2n-1-2k} = \sum_{l=0}^{2n-1-2k} (-1)^l \frac{(2n-2k)!}{l!\,(2n-1-2k-l)!} y^l \eta^{2n-1-2k-l}$$

then

$$G_1(y-\eta) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \sum_{l=0}^{2n-1-2k} \tau^{2n} \frac{(2n-1-k)!}{k!\,l!\,(2n-1-2k-l)!} \times$$
$$\times (-1)^{n-k+l} G_{2n,k}(\lambda) y^l \eta^{2n+2k-l}. \qquad \text{(VIII.170)}$$

Let us limit the expression (VIII.170) to a certain finite number of terms n = N. Then, the highest power η in the expansion becomes 2N - 1, and therefore we have

$$G_1(y-\eta) = \sum_{m=0}^{2N-1} y^m v_m(\eta, \tau, \lambda), \qquad \text{(VIII.171)}$$

where $v_m(\eta, \tau, \lambda)$ is a polynomial whose power does not exceed 2N - 1 - m; $\lambda = \frac{\omega}{\tau}$.

Now we may write for the nucleus

$$K_1(y,\delta) = K_0(y,\delta) - \frac{1}{\pi}\sum_{m=1}^{2N-1} m v_m(\sigma,\tau,\lambda) \times$$

$$\times \int_0^y \eta^{m-1}\sin[\theta(\eta) - \Theta(y)]\,d\eta + \frac{1}{\pi^2}\sum_{m=0}^{2N-1} m v_m(\sigma,\tau,\lambda) \times$$

$$\times \int_0^y \frac{\cos[\theta(\eta) - \Theta(y)]}{\sqrt{1-\eta^2}}\,d\eta \int_{-1}^{+1} t^{m-1}\frac{\sqrt{1-t^2}}{t-\eta}\,dt,$$

and for the integral

$$\int_{-1}^{+1} t^{m-1}\frac{\sqrt{1-t^2}}{t-\eta}\,dt = -\pi\eta^m + \pi\sum_{k=0,2,4}^{m-2}\eta^{m-2-k}\frac{(k-1)!!}{(k+2)!!},$$

$$K_1(y,\delta) = K_0(y,\delta) - \frac{1}{\pi}\sum_{m=0}^{2N-1} m v_m(\sigma,\tau,\lambda)\int_0^y \Big\{\eta^{m-1}\sin[\Theta(\eta) - \Theta(y)] +$$

$$+ \frac{\cos[\theta(\eta) - \Theta(y)]}{\sqrt{1-\eta^2}}\Big(\eta^m - \sum_{k=0,2,4}^{m-2-\frac{1}{2}[1-(-1)^m]}\eta^{m-2-k}\frac{(k-1)!!}{(k+2)!!}\Big)\Big\}\,d\eta.$$ [314

Let us introduce the designation

$$D_m(y) = \frac{m}{\pi}\int_0^y \Big\{\eta^{m-1}\sin[\theta(\eta) - \Theta(y)] + \frac{\cos[\theta(\eta) - \Theta(y)]}{\sqrt{1-\eta^2}}\Big[\eta^m -$$

$$- \sum_{k=0,2,4}^{m-2-\frac{1}{2}[1-(-1)^m]}\eta^{m-2-k}\frac{(k-1)!!}{(k+2)!!}\Big]\Big\}\,d\eta. \qquad \text{(VIII.172)}$$

Then

$$K_1(y,\delta) = \sum_{k=1}^{n}\frac{\sigma^{k-1}}{Q(\delta)}u_k(y) - \sum_{m=0}^{2N-1} v_m(\sigma,\tau,\lambda)\,D_m(y).$$

Introducing the general designations for both summations

$$A_k(\sigma) = \begin{cases} \dfrac{\sigma^{k-1}}{Q(\sigma)} & 1 < k < n \\ v_{k-n-1}(\sigma) & n+1 \leqslant k < n+1+2N-1, \end{cases}$$

$$B'_k(y) = \begin{cases} u_k(y) & 1 < k < n \\ -D_{k+1}(y) & n+1 < k < n+2N, \end{cases}$$

we obtain

$$K_1(y, \sigma) = \sum_{k=1}^{2N+n} A_k(\sigma) B'_k(y),$$

and the complete nucleus can be written as follows:

$$K_0(y, \delta) = \sum_{k=1}^{2N+n} A_k(\sigma) B_k(y),$$

$$B_k(y) = B'_k(y) - B'_k(1) \frac{\cos\theta(y)}{\cos\theta(1)}. \qquad \text{(VIII.173)}$$

Thus, the approximate evaluation of the function $G_1(y - \eta)$ by means of the expression (VIII.171) leads to the degenerate nucleus of the basic integral equation.

Let us examine the case of $\cos|\theta(y) \neq 0$. Let us designate $\int_{-1}^{+1} A_k(\sigma)\Gamma(\sigma)\,d\sigma = \Gamma_k$. Then by using the regular approach we obtain a system of algebraic equations [315

$$\Gamma_l - \sum_{k=1}^{2N+n} G_{lk}\Gamma_k = g_l, \qquad \text{(VIII.174)}$$

$$l = 1, 2 \ldots 2N+n$$

$$c_{lk} = \int_{-1}^{+1} A_l(y) B_k(y)\,dy, \qquad \text{(VIII.175)}$$

$$g_l = \int_{-1}^{+1} A_l(y) g(y)\,dy. \qquad \text{(VIII.176)}$$

Thus, for determining Γ_i we obtain a system of linear equations with the determinant

$$\Delta = \| \sigma_{lk} - c_{lk} \| \qquad \sigma_{lk} = \begin{cases} 1 & l = k \\ 0 & l \neq k \end{cases}.$$

If $\Delta \neq 0$, the system will have a solution and the circulation is determined from this solution according to the formula

$$\Gamma(y) = g(y) + \sum_{k=1}^{2N+n} B_k(y)\Gamma_k. \qquad \text{(VIII.177)}$$

Let us examine the problem of motion of a submerged hydrofoil, elliptical in shape, in the plan view.

Let us limit the expansion (VIII.170) to a finite number of terms, i.e., to n = 4.

$$\left.\begin{aligned}
&G_1(y-\eta)=A_1+A_2y+A_3y^2+A_4y^3+\ldots+A_8y^7\\
&A_1=A_{11}\eta+A_{13}\eta^3+A_{15}\eta^5+A_{17}\eta^7\\
&A_2=A_{20}+A_{22}\eta^2+A_{24}\eta^4+A_{26}\eta^6\\
&A_3=A_{31}\eta+A_{33}\eta^3+A_{35}\eta^5\\
&A_4=A_{40}+A_{42}\eta^2+A_{44}\eta^4\\
&A_5=A_{51}\eta+A_{53}\eta^3\\
&A_6=A_{60}+A_{62}\eta^2\\
&A_7=A_{71}\eta\\
&A_8=A_{80}\\
&A_{11}=-\tau^2G_{2,0}-2\tau^4G_{4,1}-3\tau^6G_{6,2}-4\tau^8G_{8,3}\\
&A_{13}=\tau^4G_{4,0}+4\tau^6G_{6,1}+10\tau^8G_{8,2}\\
&A_{15}=-\tau^6G_{6,0}-6\tau^8G_{8,1}\\
&A_{17}=\tau^8G_{8,0}\\
&A_{20}=\tau^2G_{2,0}+2\tau^4G_{4,1}+3\tau^6G_{6,2}+4\tau^8G_{8,3}\\
&A_{22}=-3\tau^4G_{4,0}-12\tau^6G_{6,1}-30\tau^8G_{8,2}\\
&A_{24}=5\tau^6G_{6,0}+30\tau^8G_{8,1}\\
&A_{26}=-7\tau^8G_{8,0}\\
&A_{31}=3\tau^4G_{4,0}+12\tau^6G_{6,1}+30\tau^8G_{8,2}\\
&A_{33}=-10\tau^6G_{6,0}-60\tau^8G_{6,1}\\
&A_{35}=21G_{8,0}\\
&A_{40}=-\tau^4G_{4,0}-4\tau^6G_{6,1}-10\tau^8G_{8,2}\\
&A_{42}=10\tau^6G_{6,0}+60\tau^8G_{8,1}\\
&A_{51}=-5\tau^6G_{6,0}-30\tau^8G_{8,1}\\
&A_{53}=35\tau^8G_{8,0}\\
&A_{60}=\tau^6G_{6,0}+6\tau^8G_{8,1}\\
&A_{62}=-21\tau^8G_{8,0}\\
&A_{71}=7\tau^8G_{8,0}\\
&A_{80}=-\tau^8G_{8,0}
\end{aligned}\right\} \qquad \text{(VIII.178)}$$

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Let us evaluate functions $D_m(y)$.

$$\Theta(y) = 8\int_0^y \frac{dt}{b(t)} = \frac{8}{a}\int_0^y \frac{dt}{\sqrt{1-t^2}} = \frac{8}{a}\arcsin y = k\arcsin y, \ \frac{8}{a} = k.$$

Determining the integrals, we obtain

$$J_n = \int_0^y \eta^n \sin\theta(\eta)\,d\eta = \int_0^y \eta^n \sin k\arcsin\eta\,d\eta,$$

$$g_n = \int_0^y \eta^n \cos\theta(\eta)\,d\eta,$$

$$T_n = \int_0^y \frac{\sin\theta(\eta)}{\sqrt{1-\eta^2}}\eta^n d\eta,$$ [317

$$R_n = \int_0^y \frac{\cos\theta(\eta)}{\sqrt{1-\eta^2}}\eta^n d\eta.$$

Substituting, we have

$$\arcsin\eta = \varphi,$$
$$\sin\varphi = \eta,$$
$$d\eta = \cos\varphi,$$
$$\eta^n = \sin^n\varphi,$$
$$\sin k\arcsin\eta = \sin k\varphi.$$

The integrals J_n, g_n, T_n and R_n may be written as follows:

$$J_n = \int_0^y \sin^n\varphi \sin k\varphi \cos k\varphi\,d\varphi,$$

$$g_n = \int_0^y \sin^n\varphi \cos k\varphi \cos\varphi\,d\varphi,$$

$$T_n = \int_0^y \sin k\varphi \sin^n\varphi\,d\varphi,$$

$$R_n = \int_0^y \cos k\varphi \sin^n\varphi\,d\varphi.$$

Expressing $\sin^n\varphi$ through sines and cosines of multiple arcs in each individual case and substituting

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta),$$

$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta),$$

$$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta),$$

we will receive values for these integrals.

If we designate [318

$$\sum_{k=0,2,4}^{m-2-\frac{1}{2}[+1-(1)^{m}]} t^{(m-2-k)} \frac{(k-1)!!}{(k+2)!!} = s_m$$

and

$$m = 1,\ s_1 = 0.$$

$$m = 2,\ k = 0,\ s_2 = \frac{1}{2},$$

$$m = 3,\ k = 0,\ s_3 = \frac{1}{2}t,$$

$$m = 4,\ k = 0, 2,\ s_4 = \frac{1}{2}t^2 + \frac{1}{8},$$

$$m = 5,\ k = 0, 2,\ s_5 = \frac{1}{2}t^3 + \frac{1}{8}t,$$

$$m = 6,\ k = 0,\ 2,\ 4,\ s_6 = \frac{1}{2}t^4 + \frac{1}{8}t^2 + \frac{1}{16},$$

$$m = 7,\ k = 0,\ 2,\ 4,\ s_7 = \frac{1}{2}t^5 + \frac{1}{8}t^3 + \frac{1}{16}t,$$

then we obtain the final expressions

$$D_0 = 0, \qquad \varphi = \arcsin y$$

$$D_1(y) = \left(\frac{\cos k\varphi}{k+1} - \frac{\sqrt{1-y^2}}{k+1}\right)\frac{1}{\pi}$$

$$D_2(y) = \left(-\frac{\sin k\varphi}{2(k+2)} - \frac{y\sqrt{1-y^2}}{2+k}\right)\frac{2}{\pi}$$

$$D_3(y) = \left[\frac{\cos k\varphi}{2(k+1)(k+3)} + \sqrt{1-y^2}\times\right.$$

$$\left.\times\left(-\frac{1}{2}\,\frac{1}{(k+1)(k+3)} - \frac{y^2}{k+3}\right)\right]\frac{3}{\pi}$$

$$D_4(y) = \left[-\frac{k+6}{8(k+2)(k+4)}\sin k\varphi + \right.$$

$$\left.+ y\sqrt{1-y^2}\left(-\frac{1}{(k+2)(k+4)} - \frac{y^2}{k+4}\right)\right]\frac{4}{\pi}$$

$$\left.\begin{aligned}
D_5 &= \left\{\sqrt{1-y^2}\left(-\frac{1}{k+5}y^4 - \right.\right.\\
&\quad\left. -\frac{3y^2}{2(k+3)(k+5)} - \frac{k+9}{8(k+1)(k+3)(k+5)}\right) + \\
&\quad\left. + \frac{(k+9)\cos k\varphi}{8(k+1)(k+3)(k+5)}\right\}\\
D_6 &= \frac{6}{\pi}\left\{-\frac{k^2+12k+40}{16(k+2)(k+4)}\sin k\varphi + \right.\\
&\quad + y\sqrt{1-y^2}\left(-\frac{1}{k+6}y^4 - \right.\\
&\quad\left. - \frac{2}{(k+4)(k+6)}y^2\right) - \\
&\quad\left. - \frac{k+12}{4(k+2)(k+4)(k+6)}\right\}\\
D_7(y) &= \frac{7}{\pi}\left\{\frac{(k^2+16k+75)\cos k\varphi}{16(k+1)(k+3)(k+5)(k+7)} + \right.\\
&\quad + \sqrt{1-y^2}\left(-\frac{y^6}{k+7} - \frac{5y^4}{2(k+5)(k+7)} - \right.\\
&\quad - \frac{3(k+15)y^2}{8(k+3)(k+5)(k+7)} - \\
&\quad\left.\left. - \frac{k^2+16k+75}{16(k+1)(k+3)(k+5)(k+7)}\right)\right\}
\end{aligned}\right\} \quad \text{(VIII.179)}$$ [319

Evaluating g(y):

$$g(y) = g_1(y) - \frac{g_1(1)\cos\theta(y)}{\cos\theta(1)},$$

$$g_1(y) = -2\int_0^y \sin[\theta(\eta) - \theta(y)]\,\alpha(\eta)\,d\eta + \frac{2}{\pi}\int_0^y \frac{\cos[\theta(\eta) - \theta(y)]}{\sqrt{1-\eta^2}}\,d\eta \times$$

$$\times \int_{-1}^{+1} \frac{\partial(\delta)\sqrt{1-\sigma^2}\,d\sigma}{\delta - \eta},$$

$$g(y) = \frac{2\alpha}{k+1}\sqrt{1-y^2},$$

$$\left.\begin{aligned}
B_k(y) &= B'_k(y) - B'_k(1)\frac{\cos\theta(y)}{\cos\theta(1)}\\
B'_k(y) &= -D_{k-1}(y)\\
B_1 &= 0,
\end{aligned}\right|$$ [320

$$
\left.
\begin{aligned}
B_2 &= \frac{1}{\pi}\frac{\sqrt{1-y^2}}{k+1},\\
B_4 &= \frac{3}{\pi}\sqrt{1-y^2}\left(\frac{1}{2(k+1)(k+3)}+\frac{y^2}{k+3}\right)\\
B_6 &= \frac{5}{\pi}\sqrt{1-y^2}\left(\frac{1}{k+5}y^4+\frac{3y^2}{2(k+3)(k+5)}+\right.\\
&\qquad\left.+\frac{k+9}{8(k+1)(k+3)(k+5)}\right)\\
B_8 &= \frac{7}{\pi}\sqrt{1-y^2}\left(\frac{y^6}{k+7}+\frac{5y^4}{2(k+5)(k+7)}+\right.\\
&\qquad+\frac{3(k+15)}{8(k+3)(k+5)(k+7)}+\\
&\qquad\left.+\frac{k^2+16k+75}{16(k+1)(k+3)(k+5)(k+7)}\right)
\end{aligned}
\right|\quad \text{(VIII.180)}
$$

The functions B_3, B_5 and B_7 are of the form $f(y) + y\sqrt{1-y^2}\psi(y)$. Let us proceed to evaluating the coefficients of the system (VIII.174). Taking the integrals in the following form: $P_n=\int\limits_{-1}^{+1} y^n\sqrt{1-y^2}dy$, we will obtain the expressions for c_{ik} and g_i. Because c_{12}, c_{14}, c_{16}, c_{18}, g_1, c_{32}, c_{34}, c_{36}, c_{38}, g_3, c_{52}, c_{54}, c_{56}, c_{58}, g_5, c_{72}, c_{74}, c_{76}, c_{78} and g_7 are equal to zero, the Γ_1, Γ_3, Γ_5 and Γ_7 are also equal to zero. Thus, we arrive at a system of the fourth order, the coefficients of which are determined in the following manner:

$$
\left.
\begin{aligned}
c_{22} &= 1 + aA_{20} + bA_{22} + cA_{24} + dA_{26}\\
c_{42} &= aA_{40} + bA_{42} + cA_{44}\\
c_{62} &= aA_{60} + bA_{62}\\
c_{82} &= aA_{80}
\end{aligned}
\right|
$$

$$
\left.
\begin{aligned}
c_{24} &= 3(a'A_{20} + b'A_{22} + c'A_{24} + d'A_{26})\\
c_{44} &= 3(a'A_{40} + b'A_{42} + c'A_{44} + 1)\\
c_{64} &= 3(a'A_{60} + b'A_{62})\\
c_{84} &= 3(a'A_{80})
\end{aligned}
\right|
$$ [321

$$
\left.
\begin{aligned}
c_{26} &= 5\,(a''A_{20} + b''A_{22} + c''A_{24} + d''A_{26})\\
c_{46} &= 5\,(a''A_{40} + b''A_{42} + c''A_{44})\\
c_{66} &= 5\,(a''A_{60} + b''_{62} + 1)\\
c_{86} &= 5\,(a''A_{80})\\
c_{28} &= 7\,(a'''A_{20} + b'''A_{22} + c'''A_{24} + d'''A_{26})\\
c_{48} &= 7\,(a'''A_{40} + b'''A_{42} + c'''A_{44})\\
c_{68} &= 7\,(a'''A_{60} + b'''A_{62})\\
c_{88} &= 7\,(a'''A_{80} + 1)\\
g_2 &= a^{IV}A_{20} + b^{IV}A_{22} + c^{IV}A_{24} + d^{IV}A_{26}\\
g_4 &= a^{IV}A_{40} + b^{IV}A_{42} + c^{IV}A_{44}\\
g_6 &= a^{IV}A_{60} + b^{IV}A_{62}\\
g_8 &= a^{IV}A_{80}\\
a &= -\frac{1}{2(k+1)},\quad b = -\frac{1}{8(k+1)}\\
c &= -\frac{1}{16(k+1)},\quad d = -\frac{5}{128(k+1)}\\
a' &= -\frac{1}{8(k+1)}\\
b' &= -\frac{k+2}{16(k+1)(k+3)}\\
c' &= -\frac{9+5k}{128(k+1)(k+3)}\\
d' &= -\frac{7k+12}{256(k+1)(k+3)}\\
a'' &= -\frac{k^2+8k+15}{16(k+1)(k+3)(k+5)}\\
b'' &= -\frac{5k^2+34k+45}{128(k+1)(k+3)(k+5)}\\
c'' &= -\frac{7k^2+45k+54}{256(k+1)(k+3)(k+5)}\\
d'' &= -\frac{21k^2+131k+150}{1024(k+1)(k+3)(k+5)}\\
a''' &= -\frac{5(k^3+15k^2+71k+105)}{128(k+1)(k+3)(k+5)(k+7)}
\end{aligned}
\right|
\qquad \text{(VIII.181)}
$$

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$$b''' = -\frac{7k^3 + 96k^2 + 389k + 420}{256(k+1)(k+3)(k+5)(k+7)}$$

$$c''' = -\frac{21k^3+278k^2+1052k+1050}{1024(k+1)(k+3)(k+5)(k+7)}$$

$$d''' = -\frac{33k^3 + 428k^2 + 1595k + 1500}{2048(k+1)(k+3)(k+5)(k+7)}$$

$$a^{IV} = \frac{a\pi}{k+1}, \quad b^{IV} = \frac{a\pi}{4(k+1)}$$

$$c^{IV} = \frac{a\pi}{8(k+1)}, \quad d^{IV} = \frac{5a\pi}{64(k+1)}$$

After solving the system we will obtain Γ_2, Γ_4, Γ_6 and Γ_8; $\Gamma(y)$ will be as follows:

$$\Gamma(y) = \sqrt{1-y^2}\,(A_0 + A_2y^2 + A_4y^4 + A_6y^6). \qquad \text{(VIII.182)}$$

$$A_0 = \frac{1}{k+1}\left\{2 + \frac{\Gamma_1}{\pi} + \frac{1}{2\pi(k+3)}\left[3\Gamma_2 + \frac{1}{4(k+5)}\left(5\Gamma_3(k+9) + \frac{7\Gamma_4(k^2+16k+75)}{2(k+7)}\right)\right]\right\},$$

$$A_2 = \frac{1}{\pi(k+3)}\left[3\Gamma_2 + \frac{3}{2(k+5)}\left(5\Gamma_3 + \frac{7\Gamma_4(k+15)}{4(k+7)}\right)\right],$$

$$A_4 = \frac{1}{\pi(k+5)}\left(5\Gamma_3 + \frac{35\Gamma_4}{2(k+7)}\right), \quad A_6 = \frac{7\Gamma_4}{\pi(k+7)}.$$

The function ζ_1 in this case will be determined by the formula [323

$$\zeta_1 = \frac{2}{a_0 + \frac{a_1}{4} + \frac{a_2}{8} + \frac{5}{64}a_3} - \frac{\pi\lambda}{a_\infty}. \qquad \text{(VIII.183)}$$

The calculations for the elliptical foils with various spans were performed with the aid of a computer. The results of these computations are given in Table 8. They are close to the results obtained by means of formula (VIII.40).

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
							$\lambda=4$,
0,025	A_0	0,4719	0,4606	0,4532	0,4034	0,3548	0,3370
	A_1	0,1305	0,1387	0,1438	0,1777	0,2140	0,2296
	A_2	—0,1396	—0,1384	—0,1376	—0,1306	—0,1218	—0,1180
	A_3	0,1253	0,1176	0,1127	0,0804	0,0484	0,0359
	ξ	1,7191	1,7995	1,8545	2,2624	2,7339	2,9236
0,05	A_0	0,4863	0,4766	0,4702	0,4275	0,3866	0,3721
	A_1	0,1186	0,1244	0,1281	0,1525	0,1791	0,1904
	A_2	—0,1076	—0,1068	—0,1062	—0,1011	—0,0942	—0,0909
	A_3	0,0862	0,0811	0,0779	0,0564	0,0346	0,0258
	ξ	1,6213	1,6885	1,7342	2,0662	2,4300	2,5683
0,075	A_0	0,4996	0,4912	0,4857	0,4493	0,4154	0,4039
	A_1	0,1045	0,1086	0,1111	0,1283	0,1470	0,1550
	A_2	—0,0824	—0,0818	—0,0813	—0,0773	—0,0718	—0,0690
	A_3	0,0592	0,0558	0,0537	0,0393	0,02452	0,0184
	ξ	1,5400	1,5964	1,6347	1,9063	2,1872	2,2870
0,1	A_0	0,5116	0,5044	0,4997	0,4687	0,4409	0,4320
	A_1	0,0901	0,0929	0,0946	1,0639	0,1192	0,1245
	A_2	—0,0628	—0,0622	—0,0619	—0,0587	—0,0541	—0,0518
	A_3	0,0406	0,0384	0,0370	0,0274	0,0173	0,0131
	ξ	1,4720	1,5196	1,5518	1,7750	1,9924	2,0637
0,125	A_0	0,5223	0,5161	0,5120	0,4856	0,4631	0,4565
	A_1	0,0765	0,0783	0,07951	0,0874	0,0958	0,0992
	A_2	—0,0477	—0,8472	—0,0469	—0,0442	—0,0405	—0,0385
	A_3	0,0279	0,0264	0,0255	0,0190	0,0121	0,0092
	ξ	1,4144	—1,4550	1,4822	1,6670	1,8355	1,8856
0,15	A_0	0,5317	0,5263	0,5228	0,5003	0,4822	0,4773
	A_1	0,0643	0,0655	0,0663	0,0714	0,0767	0,0787
	A_2	—0,0361	—0,0357	—0,0355	—0,0333	—0,0302	—0,0286
	A_3	0,0192	0,0182	0,0176	0,0132	0,0085	0,0065
	ξ	1,3657	1,4003	1,4235	1,5773	1,7084	1,7431
0,175	A_0	0,5399	0,5353	0,5322	0,5130	0,4984	0,4951
	A_1	0,0537	0,0545	0,0542	0,0581	0,0613	0,0623
	A_2	—0,0274	—0,0271	—0,0268	—0,0250	—0,0225	—0,0211
	A_3	0,0133	0,0126	0,0122	0,0092	0,0060	0,0046
	ξ	1,3241	1,3539	1,3737	1,5027	1,6050	1,6282
0,2	A_0	0,5171	0,5431	0,5404	0,5239	0,5122	0,5100
	A_1	0,0147	0,0452	0,0455	0,0473	0,0490	0,0494
	A_2	—0,0208	—0,0205	—0,0203	—0,0188	—0,0167	—0,0156
	A_3	0,0092	0,0088	0,0085	0,0064	0,0012	0,0032
	ξ	1,2885	1,3143	1,3313	1,4401	1,5203	1,5352
0,225	A_0	0,5534	0,5498	0,5475	0,5333	0,5239	0,5226
	A_1	0,0372	0,0374	0,0376	0,0386	0,0393	0,0393

[Typist's note: p. 304 mates to p. 305; rest of pages of Table 8 are similarly paired.]

1,4	1,8	3	4	5	8	12
$a=3,469579$						
0,3119	0,2983	0,3093	0,3658	0,4535	0,7372	0,8474
0,2573	0,2811	0,3279	0,3049	0,2620	−0,1259	−0,3588
−0,1114	−0,0106	−0,0838	−0,0538	0,0088	0,2283	0,3303
0,0152	−0,0013	−0,0410	−0,11045	−0,1032	−0,1758	−0,1972
3,1963	3,3234	2,3908	2,3816	1,5998	0,4700	0,2466
0,3526	0,3438	0,3657	0,4252	0,5080	0,4276	0,7983
0,2098	0,2253	0,2449	0,2135	0,1568	−0,1193	−0,2961
−0,0846	−0,0781	−0,0525	−0,0226	0,0282	0,1647	0,2164
0,0111	−0,0010	−0,0305	−0,0729	−0,0737	−0,01161	−0,1269
2,7526	2,8184	−2,4793	1,9487	1,3643	0,5140	0,3507
0,3896	0,3852	0,4152	0,4743	0,5482	0,7160	0,7629
0,1680	0,1774	0,1803	0,1462	0,0910	−0,1053	−0,1898
−0,0633	−0,0570	−0,0325	−0,0058	0,0320	0,1180	0,1459
0,0080	−0,0007	−0,0223	−0,0482	−0,0519	−0,0769	−0,0826
2,4068	2,4296	2,0951	1,6438	1,1976	0,5578	0,4445
0,4224	0,4215	0,4569	0,5131	0,5767	0,7042	0,7361
0,01328	0,1378	0,1310	0,0978	0,0499	−0,0900	−0,1443
−0,0468	0,0413	−0,0198	0,00234	0,0296	0,0846	0,1005
0,0057	−0,0005	−0,0161	−0,0319	−0,0362	−0,0511	−0,0542
2,1370	2,1309	1,8098	1,4301	1,0836	0,5998	0,5177
0,4505	0,4526	0,4910	0,5426	0,5963	0,6932	0,7153
0,1040	0,1062	0,0941	0,0641	0,0246	−0,0757	−0,1116
−0,0343	−0,0296	−0,0118	0,00576	0,0251	0,0609	0,0705
0,0041	−0,0004	−0,0115	−0,0213	−0,0251	−0,0341	−0,0359
1,9281	1,9011	1,6994	1,2805	1,0076	0,6387	0,5786
0,4745	0,4787	0,5180	0,5645	0,6092	0,6832	0,6988
0,0811	0,0813	0,0671	0,0409	0,0092	−0,0632	−0,0875
−0,0251	−0,0212	−0,0068	0,0007	0,0204	0,0442	0,0501
0,0029	−0,0003	−0,0081	−0,0144	−0,0173	−0,0229	−0,0240
1,7605	1,7239	1,4444	1,1761	0,9585	0,6743	0,6296
0,4945	0,5002	0,5391	0,5803	0,6373	0,6743	0,6855
0,0631	0,0621	0,0475	0,0253	0,0002	−0,0525	−0,0693
−0,0183	−0,0151	−0,0038	0,0065	0,0162	0,0328	0,0301
0,0020	−0,0002	−0,0057	−0,0091	−0,0119	−0,0155	−0,0102
1,6299	1,5867	1,3302	1,1035	0,9279	0,7064	0,0728
0,5112	0,5178	0,5553	0,5915	0,6222	0,6665	0,6747
0,0192	0,0475	0,0335	0,0149	−0,0017	0,0436	−0,0554
−0,0133	−0,0108	−0,0019	0,00584	0,0128	0,0238	0,0263
0,0014	−0,0001	−0,0040	−0,0060	−0,0082	−0,0106	[illegible]
1,5261	1,4797	1,2458	1,0532	0,9098	0,7351	[illegible]
0,5250	0,5322	0,5677	0,5903	0,6249	0,6596	0,6658
0,0384	0,0364	0,0236	0,00820	−0,0073	−0,0962	−0,0446

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
	A_2	—0,0158	—0,0156	—0,0154	—0,0141	—0,0125	—0,0116
	A_3	0,0064	0,0061	0,0059	0,0046	0,0030	0,0022
	ξ	1,2578	1,2802	1,2950	1,3875	1,4505	1,4593
0,25	A_0	0,5589	0,5557	0,5536	0,5413	0,6339	0,5332
	A_1	0,0309	0,0310	0,0311	0,0316	0,0316	0,0314
	A_2	—0,0121	—0,0119	—0,0117	—0,0107	—0,0093	—0,0086
	A_3	0,0045	0,0043	0,0042	0,0032	0,0021	0,0016
	ξ	1,2313	1,2509	1,2638	1,3429	1,3926	1,3970
0,3	A_0	0,5678	0,6653	0,5636	0,5543	0,5495	0,5497
	A_1	0,0215	0,0215	0,0215	0,0213	0,0208	0,0203
	A_2	—0,0075	—0,0701	—0,0692	—0,0062	—0,0053	—0,0048
	A_3	0,0023	0,0022	0,0021	0,0016	0,0011	0,0008
	ξ	1,1882	1,2035	1,2134	1,2724	1,3037	1,3027
0,35	A_0	0,5745	0,5725	0,5712	0,5699	0,5609	0,5616
	A_1	0,0151	0,0151	0,0150	0,0146	0,0139	0,1344
	A_2	—0,0043	—0,0042	—0,0042	—0,0037	—0,0031	—0,0028
	A_3	0,0012	0,0011	0,0011	0,0008	0,0005	0,0004
	ξ	1,1552	1,1674	1,1752	1,2203	1,2402	1,2364
0,4	A_0	0,5798	0,5781	0,5770	0,5712	0,5693	0,5702
	A_1	0,0108	0,0107	0,0107	0,0103	0,0096	0,0091
	A_2	—0,0027	—0,0026	—0,0025	—0,0022	—0,0018	—0,0016
	A_3	0,0006	0,0006	0,0006	0,0004	0,0003	0,0002
	ξ	1,1297	1,1394	1,1458	1,1811	1,1937	1,1887
0,45	A_0	0,5838	0,5825	0,5816	0,5769	0,5757	0,5767
	A_1	0,0078	0,0078	0,0077	0,0073	0,0067	0,0063
	A_2	—0,0017	—0,0016	—0,0016	—0,0014	—0,0011	—0,0010
	A_3	0,0003	0,0003	0,0003	0,0002	0,0002	0,0001
	ξ	1,1095	1,1176	1,1228	1,1509	1,1590	1,1535
0,5	A_0	0,5870	0,5859	0,5852	0,5813	0,5806	0,5816
	A_1	0,0058	0,0057	0,0057	—0,0053	0,0048	0,0045
	A_2	—0,0011	—0,0010	—0,0010	—0,0009	—0,0007	0,0006
	A_3	0,0002	—0,0002	0,0002	0,0001	0,00009	0,00007
	ξ	1,0935	1,1002	1,1045	1,1274	1,1325	1,1270
0,6	A_0	0,5917	0,5909	0,5903	0,5876	0,5874	0,5883
	A_1	0,0033	0,0032	0,0032	0,0030	0,0026	0,0024
	A_2	—0,0005	—0,0005	—0,0005	—0,0004	—0,0003	—0,0003
	A_3	0,00007	0,00006	0,00006	0,00005	0,00003	0,00002
	ξ	1,0700	1,0748	0,0779	1,0938	1,0956	1,0905
0,7	A_0	0,5948	0,5942	0,5938	0,5918	0,5918	0,5925
	A_1	0,0020	0,0019	0,0019	0,0018	0,0015	0,0014
	A_2	—0,0002	—0,0002	—0,0002	—0,0002	—0,0001	—0,0001
	A_3	0,00002	0,00002	0,00002	0,00002	0,00001	$0,9 \cdot 10^{-5}$
	ξ	1,0540	1,0577	1,0600	1,0716	1,0719	1,0676

1,4	1,8	3	4	5	8	12
0,0097	—0,0077	—0,0008	0,0050	0,00998	0,0176	0,0193
0,0010	$-10^{-4}\cdot 0,9$	—0,0028	—0,0046	—0,0057	—0,0072	0,0075
1,4432	1,3967	1,1832	1,01848	0,9001	0,7606	0,7408
0,5364	0,5439	0,5771	0,6047	0,6261	0,6537	0,6583
0,0302	0,0280	0,0165	0,00387	—0,0083	—0,0300	—0,0361
—0,0071	—0,0055	0,0002	0,00417	0,0078	0,0132	0,0144
0,0007	$-0,7\cdot 10^{-4}$	—0,0020	—0,00318	—0,0040	—0,0050	—0,0052
1,3764	1,3294	1,1366	0,9948	0,8960	0,7833	0,7678
0,5538	0,5611	0,5895	0,6107	0,6260	0,6440	0,6468
0,0189	0,0169	0,0081	—0,00051	—0,0082	—0,0209	—0,0242
—0,0038	—0,0020	0,0003	0,00281	0,0048	0,0076	0,0082
0,0004	$-0,3\cdot 10^{-4}$	—0,0010	—0,0016	—0,0020	—0,0025	—0,0025
1,2778	1,2341	1,0754	0,9683	0,8976	0,8213	0,8114
0,5659	0,5727	0,5967	0,6132	0,6213	0,6367	0,638
0,0121	0,0105	0,0039	—0,0020	—0,0070	—0,0147	—0,0166
—0,0021	—0,0015	0,0004	0,00185	0,0029	0,0045	0,0047
0,0002	$-0,2\cdot 10^{-4}$	—0,0005	—0,0008	—0,0010	—0,0012	—0,0013
1,2150	1,1717	1,0405	0,9576	0,9051	0,8512	0,8446
0,5745	0,5806	0,6008	0,6139	0,6223	0,6311	0,6323
0,0080	0,0067	0,0018	—0,00229	—0,0056	—0,0105	—0,0116
—0,0012	—0,0008	0,0004	0,00122	0,0018	0,0027	0,0029
0,0001	$-0,9\cdot 10^{-5}$	—0,0003	—0,00042	—0,0005	0,0006	—0,0007
1,1644	1,1297	1,0203	0,9546	0,9145	0,8749	0,8703
0,5806	0,5861	0,6032	0,6137	0,6202	0,6268	0,6276
0,0054	0,0044	0,0007	—0,00214	—0,0044	—0,0076	—0,0083
—0,0007	—0,0005	0,0003	0,00081	0,0012	0,0017	0,0018
$10^{-4}\cdot 0,6$	$-0,5\cdot 10^{-5}$	—0,0002	—0,00023	—0,0003	—0,0003	—0,0004
1,1310	1,1004	1,0085	0,9554	0,9238	0,8939	0,8905
0,5852	0,5901	0,6047	0,61322	0,6184	0,6234	0,6240
0,0038	0,0030	0,0002	—0,00185	—0,0034	0,0056	—0,0061
—0,0004	—0,0003	0,0002	0,00054	0,0008	0,0011	0,0011
$10^{-4}\cdot 0,3$	$-0,3\cdot 10^{-5}$	$-0,8\cdot 10^{-4}$	—0,00013	—0,0002	—0,0002	—0,0002
1,1064	1,0796	1,0015	0,9578	0,9325	0,9091	0,9065
0,5913	0,5951	0,6059	0,61191	0,6154	0,6185	0,6185
0,0019	0,0015	—0,0001	—0,00127	—0,0021	—0,0032	—0,0034
—0,0002	$-0,1\cdot 10^{-3}$	0,0001	0,00025	0,0004	0,0005	0,0005
$10^{-4}\cdot 0,1$	$-0,1\cdot 10^{-5}$	$-0,3\cdot 10^{-4}$	$-0,4\cdot 10^{-4}$	$-0,5\cdot 10^{-4}$	$-0,7\cdot 10^{-4}$	$-0,7\cdot 10^{-4}$
1,0737	1,0529	0,9951	0,9642	—0,9469	0,9316	0,9300
0,5950	0,5980	0,6063	0,6107	0,6131	0,6153	0,6155
0,0011	0,0008	—0,0002	—0,00085	—0,0013	—0,0019	—0,0020
$-10^{-4}\cdot 0,8$	$-0,4\cdot 10^{-4}$	$0,6\cdot 10^{-4}$	0,00013	0,0002	0,0002	0,0002
$-10^{-5}\cdot 0,4$	$-0,4\cdot 10^{-5}$	$-0,1\cdot 10$	$-0,2\cdot 10^{-4}$	$-0,2\cdot 10^{-4}$	$-0,2\cdot 10^{-4}$	$-0,3\cdot 10^{-4}$
1,0539	1,0374	0,9931	0,9703	[illegible]	0,9470	0,9460

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
0,8	A_0	0,5970	0,5965	0,5961	0,5946	0,5947	0,5951
	A_1	0,0012	0,0012	0,0012	0,0011	0,0009	0,0009
	A_2	—0,0001	—0,0001	—0,0001	—0,00009	—0,00007	$-0{,}6\cdot10^{-4}$
	A_3	0,00001	$10^{-5}\cdot0{,}98$	$10^{-5}\cdot0{,}95$	$10^{-5}\cdot0{,}75$	$10^{-5}\cdot0{,}5$	$0{,}4\cdot10^{-5}$
	ξ	1,0428	1,0457	1,0475	1,0563	1,0559	1,0522
0,9	A_0	0,5985	0,5981	0,5979	0,5966	0,5968	0,5974
	A_1	0,0008	0,0008	0,0008	0,0007	0,0006	0,0005
	A_2	—0,00006	—0,00006	—0,00006	—0,00005	—0,00004	$-0{,}3\cdot10^{-4}$
	A_3	$0{,}10^{-5}\cdot0{,}46$	$10^{-5}\cdot0{,}44$	$10^{-5}\cdot0{,}42$	$10^{-5}\cdot0{,}3$	$10^{-5}\cdot0{,}2$	$0{,}2\cdot10^{-5}$
	ξ	1,0347	1,0370	1,0384	1,0453	1,0447	—1,0415
1	A_0	0,5997	0,5993	0,5991	0,5981	0,5983	0,5988
	A_1	0,0006	0,0005	0,0005	0,0005	0,0004	0,0004
	A_2	—0,00004	—0,00003	—0,00003	—0,00003	—0,00002	—0,00002
	A_3	$10^{-5}\cdot0{,}22$	$10^{-5}\cdot0{,}21$	$10^{-5}\cdot0{,}20$	$10^{-5}\cdot0{,}2$	$10^{-5}\cdot0{,}1$	$0{,}8\cdot10^{-6}$
	ξ	1,0286	1,0305	1,0317	1,0372	1,0365	1,0338
1,1	A_0	0,6005	0,6003	0,6001	0,5993	0,5994	0,5999
	A_1	0,0004	0,0004	0,0004	0,0003	0,0003	0,0003
	A_2	—0,00002	—0,00002	$-10^{-4}\cdot0{,}2$	$-10^{-4}\cdot0{,}2$	$-10^{-4}\cdot0{,}1$	$-0{,}1\cdot10^{-4}$
	A_3	$10^{-5}\cdot0{,}11$	$10^{-5}\cdot0{,}10$	$10^{-5}\cdot0{,}1$	$10^{-6}\cdot0{,}8$	$10^{-6}\cdot0{,}5$	$0{,}4\cdot10^{-6}$
	ξ	1,0240	1,0255	1,0265	1,0311	1,0303	1,0280
1,2	A_0	0,6012	0,6010	0,6008	0,6002	0,6003	0,6007
	A_1	0,0003	0,0003	0,0003	0,0002	0,0002	0,0002
	A_2	—0,00001	$-10^{-4}\cdot0{,}1$	$-10^{-4}\cdot0{,}1$	$-10^{-4}\cdot0{,}1$	$10^{-5}\cdot0{,}8$	$-0{,}6\cdot10^{-5}$
	A_3	$10^{-6}\cdot0{,}52$	$10^{-6}\cdot0{,}55$	$10^{-6}\cdot0{,}54$	$10^{-6}\cdot0{,}42$	$10^{-6}\cdot0{,}28$	$0{,}21\cdot10^{-6}$
	ξ	1,0204	1,0217	1,0225	1,0263	1,0256	1,0236
							$\lambda=6,$
0,025	A_0	0,3694	0,3623	0,3577	0,3257	0,2933	0,2811
	A_1	0,0896	0,0954	0,0991	0,1237	0,1504	0,1618
	A_2	—0,0998	—0,0993	—0,0988	—0,0949	—0,0891	—0,0863
	A_3	0,0866	0,0817	0,0784	0,0571	0,0350	0,0261
	ξ	1,7216	1,8023	1,8575	2,2675	2,7428	2,9346
0,05	A_0	0,3785	0,3725	0,3685	0,3417	0,3151	0,3053
	A_1	0,0798	0,0838	0,0865	0,1041	0,1232	0,1313
	A_2	—0,0758	—0,0754	—0,0752	—0,0722	—0,0677	—0,0653
	A_3	0,0593	0,0560	0,0539	0,0397	0,0247	0,0186
	ξ	1,6231	1,6905	1,7364	2,0697	2,4356	2,5751
0,075	A_0	0,3867	0,3817	0,3783	0,3557	0,3342	0,3267
	A_1	0,0691	0,0720	0,0738	0,0806	0,0994	0,1049
	A_2	—0,0573	—0,0570	—0,0568	—0,0544	—0,0508	—0,0487
	A_3	0,0405	0,0383	0,0369	0,0275	0,0173	0,0131
	ξ	1,5413	1,5979	1,6362	1,9086	2,1907	2,2912
0,1	A_0	0,3911	0,3898	0,3860	0,3680	0,3507	0,3450
	A_1	0,0589	0,0608	0,0620	0,0704	0,0794	0,0830

1,4	1,8	3	4	5	8	12
0,5974	0,5999	0,6063	0,6097	0,6115	0,6131	0,6132
0.0006	0,0005	—0,0002	—0,0006	—0,0008	—0,0012	—0,0013
$-10^{-4}\cdot 0,4$	$-0,2\cdot 10^{-4}$	$0,3\cdot 10^{-4}$	$0,7\cdot 10^{-4}$	$0,8\cdot 10^{-4}$	0,0007	0.0001
$-10^{-6}\cdot 0,2$	$-0,2\cdot 10^{-6}$	$-0,4\cdot 10^{-5}$	$-0,7\cdot 10^{-5}$	$-0,8\cdot 10^{-5}$	$-0,1\cdot 10^{-4}$	$-0,1\cdot 10^{-4}$
1,0410	1,0277	0,9929	0,9753	0,9658	0,9579	0,9572
0,5990	0,6010	0,6063	0,6089	0,6103	0,6115	0,6116
0,0004	0,0003	—0,0001	—0,0004	—0,0006	—0,0008	—0,0008
$-10^{-4}\cdot 0,2$	$-0,9\cdot 10^{-5}$	$0,2\cdot 10^{-4}$	$0,37\cdot 10^{-4}$	$0,4\cdot 10^{-4}$	$10^{-4}\cdot 0,6$	$0.6\cdot 10^{-4}$
$10^{-6}\cdot 0,8$	$-0,7\cdot 10^{-7}$	$-0,2\cdot 10^{-5}$	$-0,3\cdot 10^{-5}$	$0,4\cdot 10^{-5}$	$-0,4\cdot 10$	$-0,5\cdot 10^{-5}$
1,0322	1,0214	0,9933	0,9794	0,9720	0,9659	0,9653
0,6002	0,6019	0,6061	0,6083	—0,6094	0,6103	0,6104
0,0003	0,0002	—0,0001	—0,0003	—0,0004	—0,0005	—0,0006
$-10^{-4}\cdot 0,1$	$-0,4\cdot 10^{-5}$	$0,1\cdot 10^{-4}$	$0,2\cdot 10^{-4}$	$0,3\cdot 10^{-4}$	$0,3\cdot 10^{-4}$	$0,4\cdot 10^{-4}$
$10^{-6}\cdot 0,4$	$-0,3\cdot 10^{-7}$	$-0,9\cdot 10^{-6}$	$-0,15\cdot 10^{-5}$	$-0,2\cdot 10^{-5}$	$-0,2\cdot 10^{-5}$	$-0,2\cdot 10^{-5}$
1,0259	1,0169	0,9939	0,9826	0,9766	0,9718	0,9714
0,6011	0,6025	0,6060	0,6078	0,6087	0,6095	0,6095
0,0002	0,0001	$-0,7\cdot 10^{-4}$	—0,0002	—0,0003	—0,0010	—0,0004
$-10^{-5}\cdot 0,6$	$-0,2\cdot 10^{-5}$	$0,7\cdot 10^{-5}$	$0,13\cdot 10^{-4}$	$0,2\cdot 10^{-4}$	$0,2\cdot 10^{-4}$	$0,2\cdot 10^{-4}$
$10^{-6}\cdot 0,2$	$-0,2\cdot 10^{-7}$	$-0,5\cdot 10^{-6}$	$-0,7\cdot 10^{-6}$	$-0,9\cdot 10^{-6}$	$-0,1\cdot 10^{5}$	$-0,1\cdot 10^{-5}$
1,0213	1,0138	0,9945	0,9851	0,9803	0,9763	0,9760
0,6017	0,6029	0,6059	0,60738	0,6082	0,6088	0,6088
0,0001	$10^{-4}\cdot 9,0$	$-0,6\cdot 10^{-4}$	—0,00014	—0,0002	—0,0003	—0,0003
$-10^{-5}\cdot 0,4$	$-0,1\cdot 10^{-5}$	$0,5\cdot 10^{-5}$	$0,8\cdot 10^{-5}$	$0,1\cdot 10^{-4}$	$0,1\cdot 10^{-4}$	$0,1\cdot 10^{-5}$
$10^{-4}\cdot 1,0$	$-0,9\cdot 10^{-8}$	$-0,3\cdot 10^{-6}$	$-0,4\cdot 10^{-6}$	$0,5\cdot 10^{-6}$	$-0,6\cdot 10^{-6}$	$-0,6\cdot 10^{-6}$
1,0179	1,0114	0,9951	0,9872	0,9832	0,9799	0,9796
$a=2,313053$						
0,2637	0,2542	0,2627	0,3016	0,3585	0,5183	0.5733
0,1813	0,1971	0,2215	0,2005	0,1590	—0,0734	—0,1950
—0,0808	—0,0753	—0,0543	—0,0284	0,0151	0,1413	0,1919
0,0112	—0,0010	—0,0298	—0,0721	—0,0708	—0,1105	—0,1207
3,2119	3,3431	3,0157	2,3788	1,6099	0,4704	0,2513
0,2922	0,2863	0,3014	0,34054	0,3917	0,5138	0,5499
0,1447	0,1547	0,1626	0,13764	0,0947	—0,0698	—0,1439
—0,0603	—0,0549	—0,0337	—0,0102	0,0231	0,1022	0,1290
0,0080	—0,0007	—0,0218	—0,0476	—0,0499	—0,0733	—0,0787
2,7618	2,8295	2,4915	1,94742	1,3680	0,5145	0,3586
—0,3173	0,3144	0,3341	0,3714	0,4156	0,5080	0.5324
0,1137	0,1196	0,1178	0,0926	0,0546	—0,0627	—0,1090
—0,0444	—0,0395	—0,0206	—0,0010	0,0232	0,0735	0,0586
—0,0057	—0,0005	—0,0157	—0,0314	—0,0318	—0,0188	—0,0517
2,4121	2,4358	2,1010	1,6433	1,1989	0,5583	0,4163
0,3388	0,3383	0,3607	0,3951	0,4321	0,5020	0,5188
0,0884	0,0914	0,0844	0,0612	0,0297	—0,0534	—0,0841

h	ω/τ	0	0,06	0,1	0,4	0,8	1
	A_2	—0,0432	—0,0429	—0,0427	—0,0408	—0,0377	—0,0361
	A_3	0,0277	0,0262	0,0253	0,0190	0,0121	0,0092
	ξ	1,4728	1,5206	1,5528	1,7766	1,9946	2,0662
0,125	A_0	0,4006	0,3969	0,3945	0,3786	0,3647	0,3606
	A_1	0,0495	0,0510	0,0516	0,0571	0,0630	0,0653
	A_2	—0,0325	—0,0322	—0,0321	—0,0304	—0,0279	—0,0265
	A_3	0,0190	0,0180	0,0174	0,0131	0,0084	0,0061
	ξ	1,4151	1,4557	1,4829	1,6680	1,8368	1,8871
0,15	A_0	0,4062	0,4031	0,4010	0,3876	0,3766	0,3737
	A_1	0,0413	0,0421	0,0426	0,0462	0,0499	0,0513
	A_2	—0,0245	—0,0242	—0,0241	—0,0227	—0,0206	—0,0195
	A_3	0,0130	0,0124	0,0120	0,0091	0,0059	0,0045
	ξ	1,3662	1,4008	1,4240	1,5800	1,7093	1,7440
0,175	A_0	0,4112	0,4085	0,4067	0,3954	0,3866	0,3846
	A_1	0,0342	0,0348	0,0351	0,0374	0,0396	0,0402
	A_2	—0,0184	—0,0182	—0,0181	—0,0169	0,0152	—0,0143
	A_3	0,0090	0,0085	0,0082	0,0063	0,0041	0,0031
	ξ	1,3244	1,3543	1,3741	1,5031	1,6055	1,6288
0,2	A_0	0,4154	0,4131	0,4115	0,4019	0,3950	0,3937
	A_1	0,0283	0,0287	0,0289	0,0302	0,0314	0,0317
	A_2	—0,0139	—0,0137	—0,0136	—0,0126	—0,0112	—0,0105
	A_3	0,0062	0,0059	0,0057	0,0044	0,0029	0,0022
	ξ	1,2887	1,3145	1,3316	1,4404	1,5206	1,5356
0,225	A_0	0,4191	0,4171	0,4157	0,4076	0,4020	0,4012
	A_1	0,0235	0,0237	—0,0238	0,0245	0,0250	0,0250
	A_2	—0,0105	—0,0104	—0,0103	—0,0095	—0,0083	—0,0077
	A_3	0,0043	0,0041	0,0040	0,0031	0,0020	0,0015
	ξ	1,2580	1,2804	1,2952	1,3876	1,4507	1,4596
0,26	A_0	0,4223	0,4205	0,4193	0,4123	0,4079	0,4075
	A_1	0,0194	0,0195	0,0196	0,0200	0,0200	0,0199
	A_2	—0,0080	0,0079	—0,0078	0,0071	—0,0062	0,0057
	A_3	0,0030	0,0030	0,0028	0,0022	0,0014	0,0011
	ξ	1,2314	1,2510	1,2639	1,3430	1,3927	1,3972
0,3	A_0	0,4275	0,4261	0,4252	0,4198	0,4171	0,4170
	A_1	0,0134	0,0134	0,0134	0,0134	0,0131	0,0127
	A_2	0,0047	—0,0046	—0,0046	—0,0041	—0,0036	—0,0032
	A_3	0,0015	0,0014	0,0014	0,0011	0,0007	0,0005
	ξ	1,1883	1,2035	1,2135	1,2725	1,3037	1,3027
0,35	A_0	0,4314	0,4303	0,4295	0,4254	0,4237	0,4240
	A_1	0,0094	0,0094	0,0094	0,0092	0,0087	0,0084
	A_2	—0,0028	—0,0028	—0,0027	—0,0024	—0,0020	—0,0018
	A_3	0,0008	0,0007	0,0007	0,0006	0,0004	0,0003
	ξ	1,1553	1,1674	1,1753	1,2204	1,2402	1,2365

1,4	1,8	3	4	5	8	12
−0,0324	−0,0282	−0,0124	0,00314	0,0204	0,0529	0,0618
0,0040	−0,0004	−0,0112	0,0209	−0,0240	−0,0325	−0,0343
2,1401	2,1342	1,8127	1,4297	1,0840	0,6002	0,5189
0,3569	0,3581	0,3818	0,4126	0,4434	0,4963	0,5080
0,0684	0,0694	0,0599	0,0397	0,0144	−0,0452	−0,0658
−0,0235	−0,0201	−0,0073	0,0047	0,0168	0,0383	0,0437
0,0028	−0,0003	−0,0080	−0,0139	−0,0165	−0,0218	−0,0228
1,9239	1,9030	1,6008	1,2803	1,0078	0,6391	0,5796
0,3719	0,3745	0,3982	0,4254	0,4507	0,4910	0,4994
0,0527	0,0526	0,0423	0,0251	0,0052	−0,0379	−0,0519
−0,0169	−0,0142	−0,0042	0,0049	0,0135	0,0278	0,0313
0,0020	−0,0182	−0,0055	−0,0094	−0,0114	−0,0147	0,0153
1,7615	1,7250	1,4451	1,1760	0,9586	0,6745	0,6302
0,3842	0,3877	0,4108	0,4345	0,4552	0,4863	0,4924
0,0407	0,0398	0,0297	0,0154	−0,0001	−0,0316	−0,0414
−0,0123	−0,0100	−0,0023	0,00453	0,0106	0,0204	0,0226
0,0014	−0,0001	−0,0039	−0,0064	−0,0078	−0,0100	−0,0104
1,6305	1,5872	1,3305	1,1034	0,9279	0,7066	0,6731
0,3944	0,3983	0,4203	0,4409	0,4580	0,4821	0,4866
0,0314	0,0302	0,0209	0,0091	−0,0031	−0,0263	0,0332
−0,0089	−0,0071	−0,0011	0,0039	0,0083	0,0151	0,0166
0,0009	$-0{,}9 \cdot 10^{-4}$	−0,0027	−0,0044	−0,0054	−0,0068	−0,0070
1,5265	1,4800	1,2459	1,0531	0,9078	0,7352	0,7097
0,4026	0,4069	0,4274	0,4453	0,4595	0,4781	0,4818
0,0244	0,0237	0,0146	0,0049	−0,0046	−0,0218	0,0268
$-0{,}6 \cdot 10^{-2}$	−0,0051	−0,0004	0,0033	0,0065	0,0112	0,0122
$0{,}7 \cdot 10^{-3}$	$-0{,}6 \cdot 10^{-4}$	−0,0019	−0,0030	−0,0038	−0,0047	−0,0049
1,4434	1,3959	1,1833	1,0185	0,9001	0,7007	0,4410
0,4094	0,4137	0,4328	0,4483	0,4601	0,4752	0,4778
0,0191	0,0176	0,0102	0,0023	−0,0052	−0,0182	−0,0218
−0,0047	−0,0036	$-0{,}6 \cdot 10^{-4}$	0,0027	−0,0050	0,0084	0,0091
0,00049	$-0{,}4 \cdot 10^{-4}$	−0,0013	−0,0021	−0,0026	−0,0032	−0,0034
1,3765	1,3295	1,1366	0,9948	0,8960	0,7834	0,7679
0,4196	0,4238	0,4399	0,4517	0,4601	0,4700	0,4715
0,0118	−0,0105	0,0049	−0,0004	−0,0051	−0,0127	−0,0147
−0,0025	−0,0019	0,0002	0,0018	0,0031	0,0048	0,0052
0,00025	$-0{,}2 \cdot 10^{-4}$	−0,0007	−0,0010	−0,0013	−0,0016	−0,0016
1,2779	1,2341	1,0754	0,9683	0,8946	0,8213	0,8115
0,4266	0,4304	0,4439	0,4531	0,4592	0,4660	0,4670
0,00757	0,0064	0,0023	−0,0012	−0,0045	−0,0089	−0,0101
0,0014	−0,0010	0,0003	0,0012	0,0019	0,0023	[illegible]
0,00013	$0{,}1 \cdot 10^{-4}$	−0,0003	−0,00053	−0,0007	−0,0008	−0,0008
1,2111	1,1717	1,0400	0,9576	0,9051	0,8513	0,8446

h	ω/τ	0	0,06	0,1	0,4	0,8	1
0,4	A_0	0,4344	0,4335	0,4329	0,4296	0,4285	0,42[illegible]
	A_1	0,0067	—0,0067	0,0066	0,0064	0,0060	0,0057
	A_2	—0,0017	—0,0017	—0,0017	—0,0015	—0,0012	—0,0011
	A_3	0,0004	0,0004	0,0004	0,0003	0,0002	0,0001
	ξ	1,1297	1,1395	1,1458	1,1811	1,1937	1,1887
0,45	A_0	0,4367	0,4360	0,4355	0,4328	0,4321	0,4327
	A_1	0,0048	0,0048	0,0048	0,0045	0,0041	0,0039
	A_2	—0,0011	—0,0011	—0,0010	—0,0009	—0,0007	—0,0006
	A_3	0,0002	0,0002	0,0002	0,0002	0,00001	$0{,}8\cdot10^{-4}$
	ξ	1,1095	1,1176	1,1228	1,1509	1,1590	1,1535
0,5	A_0	0,4385	0,4379	0,4375	0,4353	0,4349	0,4355
	A_1	0,0036	0,0035	0,0035	0,0033	0,0030	0,0028
	A_2	—0,0007	—0,0007	—0,0007	—0,0006	—0,0005	—0,0004
	A_3	0,0001	0,0001	0,0001	0,00009	0,00006	$0{,}5\cdot10^{-4}$
	ξ	1,0935	1,1002	1,1045	1,1274	1,1325	1,1270
0,6	A_0	0,4412	0,4407	0,4404	0,4389	0,4388	0,4392
	A_1	0,0020	0,0020	0,0020	0,0018	0,0016	0,0015
	A_2	—0,0003	—0,0003	—0,0003	—0,0002	—0,0002	—0,0002
	A_3	0,0004	0,00004	0,00004	0,00003	0,00002	$0{,}2\cdot10^{-4}$
	ξ	1,0700	1,0748	1,0779	1,0938	1,0956	1,0905
0,7	A_0	0,4429	0,4425	0,4423	0,4412	0,4412	0,4417
	A_1	0,0012	0,0012	0,0012	0,0011	0,0009	0,0009
	A_2	—0,0001	—0,0001	—0,0001	—0,0001	—0,00009	$-0{,}8\cdot10^{-4}$
	A_3	0,00002	0,00092	0,00001	0,00001	$10^{-5}\cdot0{,}8$	$0{,}6\cdot10^{-5}$
	ξ	1,0540	1,0577	1,0600	1,0716	1,0719	1,0676
0,8	A_0	0,4441	0,4438	0,4437	0,4428	0,4429	0,4432
	A_1	0,0008	0,0007	—0,0007	0,0007	0,0006	0,0005
	A_2	—0,00007	—0,00007	—0,00007	—0,00005	—0,00004	$-0{,}4\cdot10^{-4}$
	A_3	$10^{-5}\cdot0{,}68$	$10^{-5}\cdot0{,}65$	$10^{-5}\cdot0{,}63$	$10^{-5}\cdot0{,}5$	$10^{-5}\cdot0{,}3$	$0{,}2\cdot10^{-5}$
	ξ	1,0428	1,0457	1,0475	1,0563	1,0559	1,0522
0,9	A_0	0,4450	0,4448	0,4446	0,4439	0,4440	0,4443
	A_1	0,0005	0,0005	0,0005	0,0004	0,0004	—0,0003
	A_2	—0,00004	—0,00004	—0,00004	—0,00003	—0,00002	$-0{,}2\cdot10^{-4}$
	A_3	$10^{-5}\cdot0{,}3$	$10^{-6}\cdot0{,}3$	$10^{-6}\cdot0{,}3$	$10^{15}\cdot0{,}2$	$10^{-6}\cdot0{,}1$	$10^{-6}\cdot0{,}1$
	ξ	1,0347	1,0369	1,0384	1,0453	1,0447	1,0415
1	A_0	0,4456	0,4454	0,4453	0,4448	0,4449	0,4451
	A_1	0,0003	0,0003	0,0003	0,0009	0,0002	0,0002
	A_2	—0,00002	—0,00002	—0,00002	—0,00002	—0,00001	$-10^{-4}\cdot0{,}1$
	A_3	$10^{-5}\cdot0{,}14$	$10^{-5}\cdot0{,}14$	$10^{-6}\cdot0{,}13$	$10^{-6}\cdot0{,}10$	$10^{-6}\cdot0{,}7$	$10^{-6}\cdot0{,}5$
	ξ	1,0286	1,0305	1,0317	1,0472	1,0365	1,0338
1,1	A_0	0,4461	0,4459	0,4459	0,4454	0,4455	0,4457
	A_1	0,0002	0,0002	0,0002	0,0002	0,0002	0,0002
	A_2	$-10^{-4}\cdot0{,}14$	$-10^{-4}\cdot0{,}13$	$-10^{-4}\cdot0{,}13$	—0,00001	$-0{,}8\cdot10^{-6}$	$-0{,}7\cdot10^{-5}$
	A_3	$10^{-6}\cdot0{,}72$	$01^{-6}\cdot0{,}69$	$10^{-6}\cdot0{,}67$	$10^{-6}\cdot0{,}53$	$-0{,}3\cdot10^{-6}$	$0{,}3\cdot10^{-6}$

1,4	1,8	3	4	5	8	12
0,4315	0,4349	0,4462	0,4534	0,4581	0,4629	0,4636
0,00498	0,0041	0,0011	—0,00142	—0,0034	—0,0064	—0,0071
$-0{,}8\cdot10^{-3}$	$-0{,}5\cdot10^{-3}$	0,0002	$0{,}79\cdot10^{-3}$	0,0019	0,0017	—0,0018
$0{,}7\cdot10^{-4}$	$-0{,}6\cdot10^{-5}$	—0,0002	—0,00028	—0,00034	—0,0004	—0,0004
1,1645	—1,1297	1,0203	0,9546	0,9145	0,8750	0,8703
0,4350	0,4380	0,4476	0,4534	0,4570	0,4605	0,4610
0,0034	0,0027	0,0004	—0,00132	—0,0027	—0,0046	—0,0051
$-0{,}5\cdot10^{-3}$	—0,0003	0,0002	0,00052	0,0008	0,0011	0,0011
$0{,}4\cdot10^{-4}$	$-0{,}3\cdot10^{-5}$	—0,0001	—0,00015	—0,0002	—0,0002	—0,0002
1,1310	1,1004	1,0085	0,9554	0,9238	0,8939	0,8905
0,4376	0,4403	0,4484	0,4531	0,4559	0,4587	0,4590
0,00234	0,0018	0,0001	—0,00113	—0,0021	—0,0034	—0,0037
$-0{,}3\cdot10^{-3}$	—0,0002	0,0001	0,00035	0,0005	0,0007	0,0007
$0{,}2\cdot10^{-5}$	$-0{,}2\cdot10^{-5}$	$-0{,}6\cdot10^{-4}$	$-0{,}8\cdot10^{-4}$	—0,0001	—0,0001	—0,0001
1,1064	1,0796	1,0015	0,9578	0,9325	0,9091	0,9065
0,4410	0,4431	0,4491	0,4524	0,4543	0,4560	0,4562
0,0012	0,0009	$-0{,}8\cdot10^{-4}$	—0,00077	—0,0013	—0,0019	—0,0021
—0,00012	$-0{,}6\cdot10^{-4}$	$0{,}7\cdot10^{-4}$	0,00016	0,0002	0,0003	0, 0003
$0{,}7\cdot10^{-5}$	$-0{,}7\cdot10^{-5}$	$-0{,}2\cdot10^{-4}$	$-0{,}3\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$
1,0738	1,0529	0,9951	0,9642	0,9469	0,9316	0,9300
0,4430	0,4471	0,4492	0,4517	0,4530	0,4542	0,4544
$0{,}7\cdot10^{-3}$	0,0005	—0,0001	—0,00052	—0,0005	—0,0012	—0,0012
$-0{,}5\cdot10^{-4}$	$-0{,}3\cdot10^{-4}$	$0{,}4\cdot10^{-4}$	$0{,}8\cdot10^{-4}$	0,0001	0,0001	0,0001
$0{,}3\cdot10^{-5}$	$-0{,}2\cdot10^{-5}$	$-0{,}7\cdot10^{-5}$	$-0{,}1\cdot10^{-4}$	$-0{,}13\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$
1,0539	1,0374	0,9931	0,9703	0,9577	0,9470	0,9460
0,4448	0,4457	0,4493	0,45114	0,4522	0,4530	0,4531
0,00041	0,0003	—0,0001	—0,00035	—0,0005	—0,0007	—0,0008
$-0{,}25\cdot10^{-4}$	$-0{,}1\cdot10^{-4}$	$0{,}2\cdot10^{-4}$	$0{,}4\cdot10^{-4}$	$0{,}6\cdot10^{-4}$	$0{,}7\cdot10^{-4}$	$0{,}8\cdot10^{-4}$
$0{,}1\cdot10^{-5}$	$-0{,}1\cdot10^{-5}$	$-0{,}3\cdot10^{-5}$	$0{,}45\cdot10^{-5}$	$-0{,}5\cdot10^{-5}$	$-0{,}7\cdot10^{-5}$	$-0{,}7\cdot10^{-5}$
1,0409	1,0277	0,9929	0,9753	0,9658	0,9579	0,9572
0,4453	0,4464	0,4493	0,4507	0,4515	0,4521	0,4522
0,00024	0,0002	—0,00017	—0,00024	—0,0004	—0,0005	—0,0005
$-0{,}1\cdot10^{-4}$	$-0{,}6\cdot10^{-5}$	$0{,}1\cdot10^{-4}$	$0{,}24\cdot10^{-4}$	$0{,}3\cdot10^{-4}$	$0{,}4\cdot10^{-4}$	$0{,}4\cdot10^{-4}$
$0{,}5\cdot10^{-6}$	$-0{,}5\cdot10^{-7}$	$-0{,}1\cdot10^{-5}$	$-0{,}2\cdot10^{-5}$	$-0{,}24\cdot10^{-5}$	$-0{,}3\cdot10^{-5}$	$-0{,}3\cdot10^{-5}$
1,0322	1,0214	0,9933	0,9794	0,9720	0,9659	0,9663
0,4459	0,4468	0,4492	0,4504	0,4510	0,4515	0,4515
0,00017	0,0001	$-0{,}6\cdot10^{-4}$	—0,00017	—0,0002	—0,0003	—0,0003
$-0{,}7\cdot10^{-5}$	$-0{,}3\cdot10^{-5}$	$0{,}8\cdot10^{-5}$	$0{,}14\cdot10^{-4}$	$0{,}2\cdot10^{-4}$	$0{,}2\cdot10^{-4}$	$0{,}2\cdot10^{-4}$
$0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-7}$	$-0{,}6\cdot10^{-6}$	$-0{,}9\cdot10^{-6}$	$-0{,}1\cdot10^{-5}$	$-0{,}1\cdot10^{-5}$	$-0{,}1\cdot10^{-5}$
1,0259	1,0169	0,9939	0,9826	0,9766	0,9718	0,9714
0,4464	0,4472	0,4491	0,4501	0,4506	[illegible]	0,4510
—0,00012	$0{,}8\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	—0,00012	—0,0002	—0,0002	—0,0002
$-0{,}4\cdot10^{-5}$	$-0{,}2\cdot10^{-5}$	$0{,}5\cdot10^{-5}$	$0{,}8\cdot10^{-5}$	$0{,}1\cdot10^{-4}$	$0{,}1\cdot10^{-4}$	$0{,}1\cdot10^{-4}$
$0{,}1\cdot10^{-6}$	$-0{,}1\cdot10^{-7}$	$-0{,}3\cdot10^{-6}$	$-0{,}5\cdot10^{-6}$	$-0{,}6\cdot10^{-6}$	$-0{,}7\cdot10^{-6}$	$-0{,}7\cdot10^{-6}$

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
	ξ	1,0240	1,0255	1,0265	1,0311	1,0303	1,0280
1,2	A_0	0,4465	0,4463	0,4463	0,4459	0,4460	0,4462
	A_1	0,0002	0,0002	0,0002	0,0001	0,0001	0,0001
	A_2	$10^{-6}\cdot-0{,}86$	$-10^{-6}\cdot0{,}83$	$-10^{-6}\cdot0{,}8$	$-10^{-6}\cdot0{,}67$	$-0{,}5\cdot10^{-6}$	$-0{,}4\cdot10^{-6}$
	A_3	$10^{-6}\cdot0{,}38$	$10^{-6}\cdot0{,}37$	$-10^{-6}\cdot0{,}35$	$10^{-6}\cdot0{,}28$	$0{,}2\cdot10^{-6}$	$0{,}1\cdot10^{-6}$
	ξ	1,0204	1,0217	1,0225	1,0263	1,0256	1,0236
							$\lambda=1$
0,025	A_0	0,3037	0,2989	0,2957	0,2734	0,2503	0,2414
	A_1	0,0656	0,0699	0,0727	0,0913	0,1117	0,1203
	A_2	−0,0752	−0,0749	−0,0747	−0,0723	−0,0681	−0,0659
	A_3	0,0638	0,0603	0,0580	0,0428	0,0266	0,0199
	ξ	1,7233	1,8043	1,8597	−2,2713	2,7493	2,9428
0,05	A_0	0,3100	0,3060	0,3033	0,2848	0,2660	0,2591
	A_1	0,0575	0,0606	0,0625	0,0757	0,0901	0,0961
	A_2	−0,0566	−0,0564	−0,0562	−0,0543	−0,0510	−0,0492
	A_3	0,0435	0,0412	0,0397	0,0296	0,0186	0,0140
	ξ	1,6244	1,6920	1,7380	2,0723	2,4397	2,5801
0,075	A_0	0,3157	0,3123	0,3100	0,2947	0,2797	0,27440
	A_1	0,0493	0,0514	0,0527	0,0619	0,0718	0,0758
	A_2	−0,0424	−0,0422	−0,0421	−0,0405	−0,0379	−0,0363
	A_3	0,0296	0,0281	0,0271	0,0204	0,0130	0,0098
	ξ	1,5422	1,5989	1,6373	1,9103	2,1933	2,2941
0,1	A_0	0,3207	0,3178	0,3159	0,3032	0,2913	0,2874
	A_1	0,0416	0,0430	0,0439	0,0501	0,0567	0,0594
	A_2	−0,0317	−0,0316	−0,0314	−0,0301	−0,0279	−0,0266
	A_3	0,0202	0,0192	0,0185	0,0140	0,0090	0,0068
	ξ	1,4735	1,5213	1,5536	1,7777	1,9961	2,0680
0,125	A_0	0,3250	0,3226	0,3210	0,3104	0,3010	0,2982
	A_1	0,0347	0,0357	0,0363	0,0404	0,0447	0,0463
	A_2	−0,0237	−0,0236	−0,0234	−0,0223	−0,0205	−0,0194
	A_3	0,0138	0,0131	0,0127	0,0097	0,0062	0,0048
	ξ	1,4156	1,4562	1,4835	1,6687	1,8378	1,8882
0,15	A_0	0,3288	0,3268	0,3254	0,3165	0,3091	0,3071
	A_1	0,0288	0,0294	0,0298	0,0325	0,0352	0,0361
	A_2	−0,0178	−0,0176	−0,0175	−0,0165	−0,0150	−0,0142
	A_3	0,0095	0,0090	0,0087	0,0067	0,0043	0,0033
	ξ	1,3665	1,4012	1,4244	1,5785	1,7098	1,7446
0,75	A_0	0,3321	0,3303	0,3292	0,3217	0,3159	0,3145
	A_1	0,0238	0,0242	0,0244	0,0261	0,0277	0,0282
	A_2	−0,0133	−0,0132	−0,0131	−0,0123	−0,0110	−0,0103
	A_3	0,0065	0,0062	0,0060	0,0046	0,0030	0,0023
	ξ	1,3247	1,3545	1,3743	1,5034	1,6059	1,6292

1,4	1,8	3	4	5	8	12
1,0213	1,0138	0,9945	0,98517	—0,9803	0,9763	0,9760
0,4467	0,4474	0,4491	0,4499	0,4503	0,4506	0,4507
$0,8\cdot10^{-4}$	$10^{-4}\cdot0,5$	$-0,3\cdot10^{-4}$	$-0,9\cdot10^{-4}$	—0,0001	—0,00025	0,0002
$-0,2\cdot10^{-6}$	$-0,9\cdot10^{-6}$	$0,3\cdot10^{-6}$	$0,5\cdot10^{-6}$	$0,7\cdot10^{-6}$	$0,8\cdot10^{-6}$	$0,9\cdot10^{-6}$
$0,6\cdot10^{-7}$	$-0,6\cdot10^{-8}$	$-0,2\cdot10^{-6}$	$-0,25\cdot10^{-6}$	$-0,3\cdot10^{-6}$	$-0,4\cdot10^{-6}$	$-0,4\cdot10^{-6}$
1,0179	1,0114	0,9951	0,98725	0,9832	0,9799	0,9796
$\varepsilon=1,734789$						
0,2286	0,2216	0,2282	0,2568	0,2966	0,3995	0,4324
0,1347	0,1459	0,1597	0,1413	0,1064	—0,0486	—0,1232
$-0,6\cdot10^{-4}$	—0,0565	—0,0378	—0,0162	0,0155	0,0974	0,1274
0,0086	—0,0008	—0,0228	—0,0512	—0,0519	—0,0769	—0,0829
3,2233	3,3575	3,0334	2,3791	1,6166	0,4707	0,2543
0,2497	0,2454	0,2565	0,2842	0,3190	0,3969	0,4188
0,1059	0,1128	0,1158	0,0958	0,0632	—0,0462	—0,0927
—0,0452	—0,0407	—0,0233	—0,0048	0,0188	0,0705	0,0869
0,0061	—0,0005	—0,0164	—0,034	—0,0362	—0,0512	—0,0514
2,7684	1,2837	2,5002	1,9475	1,3705	0,5148	0,3604
0,2678	0,2658	0,2797	0,3054	0,3348	0,3935	0,4084
0,0822	0,08611	0,0830	0,0639	0,0363	—0,0412	—0,0712
—0,033	—0,0290	—0,0142	—0,0007	0,0176	0,0508	0,0603
0,0043	—0,0004	—0,0116	—0,0223	—0,0251	—0,1341	—0,0360
2,4159	1,2440	2,1052	1,6433	1,1998	0,5586	0,4475
0,2831	0,2827	0,2982	0,3214	0,3457	0,3898	0,4002
0,0632	0,06514	0,05895	0,0419	0,0196	—0,0356	—0,0554
—0,0238	—0,0206	—0,0085	0,0030	0,0150	0,0366	0,0424
0,00303	—0,0003	—0,0082	—0,0148	—0,0173	—0,0228	—0,0230
2,1422	1,2137	1,8148	1,4297	1,0108	0,6004	0,5197
0,2956	0,2965	0,3126	0,3330	0,35293	0,3863	0,3936
0,0484	0,0490	0,0416	0,0270	0,0094	—0,0302	—0,0436
—0,0171	—0,0145	—0,0050	0,00377	0,0122	0,0266	0,0301
0,00212	—0,0002	—0,0058	0,0099	—0,0118	—0,0153	—0,0160
1,9291	1,1904	1,6018	1,2803	1,0079	0,6393	—0,5799
0,305	0,3077	0,3235	0,3414	0,3576	0,3831	0,3883
0,0371	0,0369	0,0292	0,0170	0,0033	—0,0254	—0,0346
—0,0123	—0,0102	—0,0028	0,0037	0,0097	0,0194	0,0216
0,00148	—0,0001	—0,0040	—0,0067	—0,0081	—0,0104	—0,0108
1,7623	1,1726	1,4456	1,1760	0,9586	0,6747	0,6305
0,3143	0,3166	0,3319	0,3473	0,3606	0,3801	0,3840
0,0284	0,0277	0,0204	0,0104	—0,0002	—0,0212	—0,0276
—0,0080	—0,0072	—0,0015	0,0533	0,0075	0,0112	0,0157
0,00103	$-0,9\cdot10^{-4}$	—0,0028	—0,0046	—0,0056	—0,0070	—0,0073
1,6309	1,1588	1,3308	1,1034	0,9270	0,7067	0,6734

h		ω/τ = 0	0,06	0,1	0,4	0,8	1
0,2	A_0	0,3349	0,3334	0,3324	0,3261	0,3215	0,3206
	A_1	0,0196	0,0200	0,0200	0,0210	0,0219	0,0221
	A_2	—0,0100	—0,0099	—0,0098	—0,0091	—0,0081	—0,0076
	A_3	0,0045	0,0043	0,0042	0,0032	0,0021	0,0016
	ξ	1,2889	1,3147	1,3318	1,4406	1,5208	1,5358
0,225	A_0	0,3374	0,3360	0,3352	0,3298	0,3262	0,3257
	A_1	0,0162	0,0164	0,0164	0,0170	0,0174	0,0174
	A_2	—0,0076	—0,0075	—0,0074	—0,0068	—0,0060	—0,0056
	A_3	0,0031	0,0030	0,0029	0,0022	0,0015	0,0011
	ξ	1,2581	1,2805	1,2953	1,3878	1,4508	1,4597
0,25	A_0	0,3395	0,3383	0,3375	0,3329	0,3301	0,3299
	A_1	0,0134	0,0135	0,0135	0,0138	0,0139	0,0137
	A_2	—0,0057	—0,0056	—0,0056	—0,0051	—0,0045	—0,0041
	A_3	0,0022	0,0021	0,0020	0,0017	0,0012	0,0008
	ξ	1,2315	1,2511	1,2640	1,3431	1,3928	1,3973
0,3	A_0	0,3429	0,3419	0,3414	0,3379	0,3361	0,3362
	A_1	0,0092	0,0092	0,0092	0,0092	0,0090	0,0088
	A_2	—0,0033	—0,0033	—0,0032	—0,0029	—0,0025	—0,0023
	A_3	0,0011	0,0010	0,0010	0,0008	0,0005	0,0004
	ξ	1,1883	1,2036	1,2135	1,2725	1,3038	1,3028
0,35	A_0	0,3454	0,3447	0,3442	0,3415	0,3404	0,3407
	A_1	0,0064	0,0064	0,0064	0,0063	0,0060	0,0057
	A_2	0,0020	—0,0020	—0,0019	—0,0017	—0,0014	—0,0013
	A_3	0,0006	0,0005	0,0005	0,0004	0,0003	0,0002
	ξ	1,1553	1,1674	1,1753	1,2204	1,2402	1,2365
0,4	A_0	0,3473	0,3468	0,3464	0,3443	0,3436	0,3439
	A_1	0,0046	0,0045	0,0045	0,0044	0,0041	0,0039
	A_2	—0,0012	—0,0012	—0,0012	—0,0010	—0,0009	—0,0008
	A_3	0,0003	0,0003	0,0003	0,0002	0,0001	0,0001
	ξ	1,1297	1,1395	1,1458	1,1811	1,1937	1,1887
0,45	A_0	0,3488	0,3484	0,3480	0,3464	0,3459	0,3463
	A_1	0,0033	0,0033	0,0032	0,0031	0,0028	0,0027
	A_2	—0,0008	—0,0007	—0,0007	—0,0006	—0,0005	—0,0005
	A_3	0,0002	0,0002	0,0001	0,0001	$10^{-4}\cdot 0,8$	$0,6\cdot 10^{-4}$
	ξ	1,1095	1,1176	1,1228	1,1509	1,1590	1,1535
0,5	A_0	0,3500	0,3496	0,3494	0,3480	0,3477	0,3481
	A_1	0,0024	0,0024	0,0024	0,0022	0,0020	0,0019
	A_2	—0,0005	—0,0005	—0,0005	—0,0004	—0,0003	—0,0003
	A_3	0,0009	0,00009	0,00008	0,00007	0,00004	$0,3\cdot 10^{-4}$
	ξ	1,0935	1,1002	1,1045	1,1274	1,1325	1,1270
0,6	A_0	0,3517	0,3514	0,3512	0,3503	0,3502	0,5505
	A_1	0,0014	0,0013	0,0013	0,0012	0,0011	0,0010
	A_2	—0,0002	—0,0002	—0,0002	—0,0002	—0,0001	—0,0001
	A_3	0,00003	0,00003	0,00003	0,00002	0,00001	$0,1\cdot 10^{-4}$

1,4	1,8	3	4	5	8	12
0,3211	0,3237	0,3381	0,3514	0,3623	0,3775	0,3803
0,0219	0,0209	0,0142	0,0001	−0,0022	−0,0176	−0,0222
−0,0064	−0,0051	−0,0007	0,0029	0,0059	0,0105	0,0115
$0{,}7\cdot10^{-3}$	$-0{,}7\cdot10^{-4}$	−0,0019	−0,0031	−0,0038	−0,0048	−0,0050
1,5267	1,1480	1,2461	1,0531	0,9098	0,7353	0,7098
0,3266	0,3234	0,3428	0,3543	0,3633	0,3752	0,3773
0,0169	0,0159	0,0099	0,0033	−0,0032	−0,0147	−0,0180
−0,0046	−0,0036	−0,0003	0,0024	0,0046	0,0078	0,0085
0,00051	$-0{,}4\cdot10^{-4}$	−0,0014	−0,0022	−0,0027	−0,0033	−0,0034
1,4435	1,1396	1,1833	1,0185	0,9001	0,7608	0,7411
0,3311	0,3339	0,3463	0,3562	0,3637	0,3732	0,3748
0,0132	0,0121	0,0069	0,0015	−0,0035	−0,0122	−0,0146
−0,0034	−0,0026	$-0{,}9\cdot10^{-5}$	0,0029	0,0036	0,0059	0,0064
$0{,}4\cdot10^{-3}$	$-0{,}3\cdot10^{-4}$	−0,0009	−0,0015	−0,0019	−0,0023	−0,0024
1,3766	1,1329	1,1366	0,9948	0,8960	0,7834	0,7680
0,3378	0,3405	0,3508	0,3584	0,3637	0,3699	0,3709
0,00813	0,0072	0,0033	−0,00027	−0,0034	−0,0085	−0,0099
−0,00181	−0,0013	0,0002	0,0013	0,0022	0,0033	0,0036
0,00018	$-0{,}2\cdot10^{-4}$	−0,0005	−0,0007	−0,0009	−0,0011	−0,0012
1,2779	1,1234	1,0754	0,96832	0,8976	0,8214	0,8115
0,3423	0,3448	0,3535	0,3593	0,3631	0,3674	0,3680
0,00518	0,0044	0,0016	−0,00086	−0,0029	−0,0060	−0,0068
−0,00100	−0,0007	0,0002	0,00085	0,0013	0,0020	0,0021
$0{,}9\cdot10^{-4}$	$-0{,}8\cdot10^{-5}$	−0,0002	−0,00038	−0,0005	−0,0006	−0,0006
1,2111	1,1172	1,0405	0,9576	0,9051	0,8513	0,8447
0,3455	0,3477	0,3549	0,3595	0,3624	0,3655	0,3659
0,00340	0,0028	0,0007	−0,00097	0,0023	−0,0043	−0,0048
$-0{,}6\cdot10^{-3}$	−0,0004	0,0002	0,00056	0,0008	0,0012	0,0013
$0{,}5\cdot10^{-4}$	$-0{,}4\cdot10^{-5}$	−0,0001	−0,0002	−0,0002	−0,0003	−0,0003
1,1645	1,1129	1,0203	0,9546	0,9145	0,8750	0,8703
0,3477	0,3497	0,3558	0,3594	0,3617	0,3640	0,3643
0,00229	0,0018	0,0003	−0,00089	−0,0018	−0,0032	−0,0034
−0,00034	−0,0002	0,0001	0,00037	0,0005	0,0008	0,0008
$0{,}3\cdot10^{-4}$	$-0{,}2\cdot10^{-5}$	$-0{,}7\cdot10^{-4}$	−0,00011	−0,0001	−0,0002	−0,0002
1,13102	1,1100	1,0085	0,95541	0,9238	0,8939	0,8905
0,3494	0,3511	0,3562	0,3592	0,3611	0,3628	0,3630
0,0016	0,0012	$0{,}9\cdot10^{-4}$	−0,00077	−0,0014	−0,0023	−0,0025
−0,0002	−0,0001	0,0001	0,00021	0,0004	0,0005	0,0005
$0{,}2\cdot10^{-4}$	$-0{,}1\cdot10^{-5}$	$-0{,}4\cdot10^{-4}$	$-0{,}6\cdot10^{-4}$	$-0{,}8\cdot10^{-4}$	$-0{,}9\cdot10^{-4}$	$-0{,}9\cdot10^{-4}$
1,1064	1,1080	1,0015	0,9578	0,9325	0,9091	0,9066
0,3516	0,3529	0,3567	0,3588	0,3600	0,3611	0,3612
0,0008	0,0006	$-0{,}6\cdot10^{-4}$	−0,00052	−0,0009	−0,0013	−0,0014
$-0{,}8\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	$0{,}5\cdot10^{-4}$	0,00011	0,0002	0,0002	0,0002
$0{,}5\cdot10^{-5}$	$-0{,}5\cdot10^{-6}$	$-0{,}1\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$-0{,}25\cdot10^{-4}$	$-10^{-4}\cdot0{,}3$	$-0{,}3\cdot10^{-4}$

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
	ξ	1,0700	1,0748	1,0779	1,0938	1,0956	1,0905
0,7	A_0	0,3528	0,3526	0,3524	0,3517	0,3517	0,3520
	A_1	0,0008	0,00008	0,0008	0.0007	0,0006	0,0006
	A_2	−0,0001	−0,0001	−0,0001	−0,00008	$-10^{-4}\cdot 0,64$	$-0,55\cdot 10^{-4}$
	A_3	$10^{-4}\cdot 0,12$	0,00001	$10^{-4}\cdot 0,11$	$10^{-5}\cdot 0,8$	$10^{-5}\cdot 0,67$	$-0,44\cdot 10^{-5}$
	ξ	1,0541	1,0577	1,0600	1,0716	1,0719	1,0676
0,8	A_0	0,3536	0,3534	0,3533	0,3528	0,3528	0,3530
	A_1	0,0005	0,0005	0,0005	0,0005	0,0004	0,0004
	A_2	$-10^{-4}\cdot 0,5$	$10^{-4}\cdot 0,51$	$-10^{-4}\cdot 0,50$	$-10^{-4}\cdot 0,4$	$10^{-4}\cdot 0,3$	$-0,3\cdot 10^{-4}$
	A_3	$10^{-6}\cdot 0,4$	$10^{-5}\cdot 0,47$	$-10^{-5}\cdot 0,45$	$-10^{-5}\cdot 0,35$	$10^{-5}\cdot 0,2$	$0,18\cdot 10^{-5}$
	ξ	1,0428	1,0457	1,0475	1,0563	1,0559	1,0522
0,9	A_0	0,3541	0,3540	0,3539	0,3535	0,3535	0,3537
	A_1	0,0003	0,0003	0,0003	0,0003	0,0002	0,0002
	A_2	$-10^{-4}\cdot 0,29$	$-10^{-4}\cdot 0,3$	$-10^{-4}\cdot 0,3$	$-10^{-4}\cdot 0,2$	$-10^{-4}\cdot 0,2$	$-0,1\cdot 10^{-4}$
	A_3	$10^{-5}\cdot 0,22$	$10^{-5}\cdot 0,2$	$10^{-5}\cdot 0,2$	$10^{-5}\cdot 0,2$	$10^{-5}\cdot 0,1$	$0,8\cdot 10^{-6}$
	ξ	1,0347	1,0370	1,0384	1,0453	1,0447	1,0415
1	A_0	0,3545	0,3544	0,3543	0,3540	0,3541	0,3542
	A_1	0,0002	0,0002	0,0002	0,0002	0,0002	0,0001
	A_2	$-10^{-4}\cdot 0,16$	$16\cdot 10^{-4}$	$-10^{-4}\cdot 0,16$	$-10^{-4}\cdot 0,1$	$-10^{-5}\cdot 0,96$	$-0,8\cdot 10^{-5}$
	A_3	$10^{-5}\cdot 0,1$	$10^{-6}\cdot 0,99$	$10^{-6}\cdot 0,98$	$10^{-6}\cdot 0,75$	$10^{-6}\cdot 0,50$	$0,4\cdot 10^{-6}$
	ξ	1,0286	1,0305	1,0517	1,0372	1,0365	1,0338
1,1	A_0	0,3548	0,3547	0,3547	0,3544	0,3545	0,3546
	A_1	0,0002	0,0002	0,0002	0,0001	0,0001	0,0001
	A_2	$-10^{-5}\cdot 0,97$	$-10^{-5}\cdot 0,94$	$-10^{-5}\cdot 0,92$	$-10^{-5}\cdot 0,76$	$-10^{-5}\cdot 0,56$	$-0,5\cdot 10^{-5}$
	A_3	$10^{-6}\cdot 0,52$	$10^{-6}\cdot 0,5$	$10^{-6}\cdot 0,5$	$10^{-6}\cdot 0,38$	$10^{-6}\cdot 0,25$	$0,1\cdot 10^{-6}$
	ξ	1,0240	1,0255	1,0265	1,0311	1,0303	1,0280
1,2	A_0	0,3551	0,3550	0,3550	0,3547	0,3548	0,3549
	A_1	0,0001	0,0001	0,0001	0,0001	0,00008	0,00008
	A_2	$-10^{-5}\cdot 0,6$	$-10^{-5}\cdot 0,6$	$-10^{-5}\cdot 0,6$	$-10^{-5}\cdot 0,5$	$-0,3\cdot 10^{-5}$	$-0,\cdot 10^{-5}$
	A_3	$10^{-6}\cdot 0,27$	$-10^{-6}\cdot 0,26$	$10^{-6}\cdot 0,26$	$10^{-6}\cdot 0,2$	$0,1\cdot 10^{-6}$	$0,1\cdot 10^{-6}$
	ξ	1,0204	1,0217	1,0225	1,0263	1,0256	1,0236

$\lambda = 10,$

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
0,025	A_0	0,2580	0,2545	0,2522	0,2357	0,2183	0,2115
	A_1	0,0502	0,0536	0,0557	0,0703	0,0862	0,0930
	A_2	−0,0588	−0,0587	−0,0586	−0,0569	−0,0538	−0,0520
	A_3	0,0491	0,0465	0,0448	0,0333	0,0209	0,0157
	ξ	1,7247	1,8059	1,8613	2,2742	2,7543	2,9490
0,05	A_0	0,2626	0,2597	0,2578	0,2443	0,2303	0,2252
	A_1	0,0436	0,0459	0,0474	0,0576	0,0688	0,0735
	A_2	−0,0440	−0,0438	−0,0437	−0,0424	−0,0399	−0,0385
	A_3	0,0334	0,0317	0,0306	0,0229	0,0145	0,0110
	ξ	1,6254	1,6931	1,7392	2,0742	2,4428	2,5838

1,4	1,8	3	4	5	8	12
1,0738	1,1053	0,9951	0,9642	0,9469	0,9316	0,9300
0,35291	0,3540	0,3569	0,3584	0,3592	0,3600	0,3607
0,00046	0,0003	$-0{,}7\cdot10^{-4}$	−0,00035	−0,0005	−0,0008	−0,0008
$-0{,}4\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$0{,}3\cdot10^{-4}$	$0{,}57\cdot10^{-4}$	$0{,}8\cdot10^{-4}$	0,0001	0,0001
$0{,}9\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}5\cdot10^{-6}$	$-0{,}78\cdot10^{-6}$	$0{,}1\cdot10^{-4}$	$-10^{-4}\cdot0{,}1$	$-0{,}1\cdot10^{-4}$
1,0539	1,1037	0,9931	0,9703	0,9577	0,9470	0,9460
0,35376	0,3546	0,3569	0,35804	0,3587	0,3592	0,3593
0,00028	0,0002	$0{,}6\cdot10^{-4}$	−0,00024	−0,0004	−0,0005	−0,0005
$-0{,}2\cdot10^{-4}$	$-0{,}8\cdot10^{-5}$	$0{,}2\cdot10^{-4}$	$0{,}3\cdot10^{-4}$	$0{,}4\cdot10^{-4}$	$0{,}5\cdot10^{-4}$	$0{,}5\cdot10^{-4}$
$0{,}8\cdot10^{-6}$	$-0{,}7\cdot10^{-7}$	$-0{,}2\cdot10^{-6}$	$-0{,}3\cdot10^{-6}$	$-0{,}4\cdot10^{-6}$	$-0{,}5\cdot10^{-6}$	$-0{,}5\cdot10^{-5}$
1,0410	1,1028	0,9929	0,9759	0,9658	0,9579	0,9572
0,3543	0,3550	0,3568	0,3578	0,3582	0,3587	0,3587
0,00018	0,0001	$0{,}5\cdot10^{-4}$	−0,00016	−0,0002	$-0{,}3\cdot10^{-3}$	−0,0003
$-0{,}8\cdot10^{-6}$	$-0{,}4\cdot10^{-5}$	$0{,}9\cdot10^{-6}$	$0{,}16\cdot10^{-4}$	$0{,}2\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$0{,}3\cdot10^{-4}$
$0{,}4\cdot10^{-6}$	$-0{,}3\cdot10^{-7}$	$-0{,}9\cdot10^{-6}$	$-0{,}14\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$
1,0322	1,1021	0,9933	0,97936	0,9720	0,9659	0,9653
0,3547	0,3553	0,3568	0,3575	0,3579	0,3583	0,3583
0,0001	$0{,}7\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	−0,00011	−0,0002	$-0{,}2\cdot10^{-3}$	−0,0002
$-0{,}5\cdot10^{-6}$	$-0{,}2\cdot10^{-5}$	$0{,}5\cdot10^{-6}$	$0{,}9\cdot10^{-6}$	$0{,}1\cdot10^{-4}$	$0{,}1\cdot10^{-4}$	$10^{-4}\cdot0{,}2$
$+0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-7}$	$-0{,}4\cdot10^{-6}$	$-0{,}7\cdot10^{-6}$	$-0{,}8\cdot10^{-6}$	$-0{,}1\cdot10^{-6}$	$-0{,}1\cdot10^{-5}$
1,0259	1,1017	0,9939	0,9826	0,9760	0,9718	0,9714
0,3550	0,3555	0,3568	0,3574	0,3577	0,3580	0,3580
$0{,}7\cdot10^{-4}$	$0{,}5\cdot10^{-4}$	$-0{,}3\cdot10^{-4}$	$-0{,}8\cdot10^{-4}$	0,0001	$-0{,}2\cdot10^{-3}$	−0,0002
$-0{,}3\cdot10^{-6}$	$-0{,}1\cdot10^{-5}$	$0{,}3\cdot10^{-6}$	$0{,}6\cdot10^{-6}$	$0{,}7\cdot10^{-6}$	$0{,}9\cdot10^{-6}$	$0{,}1\cdot10^{-4}$
$0{,}9\cdot10^{-7}$	$-0{,}7\cdot10^{-8}$	$-0{,}2\cdot10^{-6}$	$-0{,}3\cdot10^{-6}$	$-0{,}4\cdot10^{-6}$	$-0{,}5\cdot10^{-6}$	$-0{,}5\cdot10^{-6}$
1,02135	1,0138	0,9945	0,9852	0,9803	0,9763	0,9760
0,35526	0,3557	0,3567	0,3572	0,3545	0,3677	0,3577
$0{,}6\cdot10^{-4}$	$0{,}4\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$-0{,}6\cdot10^{-4}$	$-0{,}8\cdot10^{-4}$	$-0{,}1\cdot10^{-3}$	−0,0001
$-0{,}2\cdot10^{-5}$	$-0{,}6\cdot10^{-6}$	$0{,}2\cdot10^{-6}$	$0{,}4\cdot10^{-5}$	$0{,}5\cdot10^{-5}$	$0{,}6\cdot10^{-6}$	$0{,}6\cdot10^{-6}$
$0{,}5\cdot10^{-7}$	$-0{,}4\cdot10^{-8}$	$-0{,}1\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}3\cdot10^{-6}$	$-0{,}3\cdot10^{-6}$
1,01787	1,0114	0,9951	0,98724	0,9832	0,9799	0,9796
$a=1{,}387831$						
0,20177	0,1964	0,2017	0,2236	0,2531	0,3249	0,3168
0,10414	0,1125	0,1208	0,1047	0,0761	−0,0347	−0,08615
−0,0482	−0,0439	−0,0277	−0,0097	0,0143	0,0717	−0,00148
0,0068	−0,0006	−0,0179	−0,0383	−0,398	−0,0570	−0,06094
3,2320	1,3684	3,0465	2,3804	1,6215	0,4709	0,2564
0,21804	0,2148	0,2233	0,2439	0,2692	0,3232	0,3380
0,0809	0,0860	0,0867	0,0701	0,0151	−0,0330	[illegible]
−0,0351	−0,0314	−0,0170	−0,0022	0,0151	[illegible]	[illegible]
0,0048	0,0004	−0,0128	−0,0025	−0,0276	−0,0380	−0,0104
2,7731	2,8134	2,6066	1,0480	1,3721	0,5150	0,3615

h	ω/τ	0	0,06	0,1	0,4	0,8	1
0,075	A_0	0,2668	0,2643	0,2627	0,2516	0,2406	0,2367
	A_1	0,0370	0,0386	0,0397	0,0467	0,0543	0,0574
	A_2	—0,0327	—0,0326	—0,0325	—0,0314	—0,0294	—0,0282
	A_3	0,0227	0,0216	0,0208	0,0157	0,0101	0,0076
	ξ	1,5429	1,5997	1,6382	1,9115	2,1951	2,2964
0,1	A_0	0,2704	0,2683	0,2670	0,2578	0,2492	0,2463
	A_1	0,0310	0,0321	0,0328	0,0375	0,0426	0,0446
	A_2	—0,0244	—0,0242	—0,0241	—0,0232	—0,0215	—0,0205
	A_3	0,0155	0,0147	0,0142	0,0108	0,0070	0,0053
	ξ	1,4740	1,5219	1,5542	1,7785	1,9973	2,0693
0,125	A_0	0,2735	0,2718	0,2707	0,2631	0,2563	0,2542
	A_1	0,0258	0,0265	0,0270	0,0301	0,0334	0,0346
	A_2	—0,0181	—0,0180	—0,0179	—0,0171	—0,0157	—0,0149
	A_3	0,0105	0,0100	0,0097	0,0074	0,0049	0,0037
	ξ	1,4159	1,4566	1,4839	1,6692	1,8385	1,8890
0,15	A_0	0,2763	0,2748	0,2738	0,2675	0,2622	0,2608
	A_1	0,0213	0,0218	0,0221	0,0241	0,0261	0,0268
	A_2	—0,0136	—0,0134	—0,0133	—0,0126	—0,0115	—0,0108
	A_3	0,0072	0,0069	0,0067	0,0051	0,0033	0,0025
	ξ	1,3667	1,4015	1,4247	1,5788	1,7103	1,7451
0,175	A_0	0,2786	0,2774	0,2765	0,2712	0,2671	0,2661
	A_1	0,0175	0,0178	0,0180	0,0193	0,0205	0,0208
	A_2	—0,0101	—0,0100	—0,0099	—0,0093	—0,0084	—0,0079
	A_3	0,0050	0,0047	0,0046	0,0035	0,0023	0,0018
	ξ	1,3248	1,3547	1,3745	1,5036	1,6061	1,6295
0,2	A_0	0,2806	0,2795	0,2788	0,2744	0,2711	0,2705
	A_1	0,0144	0,0146	0,0147	0,0155	0,0161	0,0163
	A_2	—0,0076	—0,0075	—0,0074	—0,0069	—0,0061	—0,0057
	A_3	0,0034	0,0033	0,0032	0,0024	0,0016	0,0012
	ξ	1,2890	1,3148	1,3319	1,4408	1,5210	1,5360
0,225	A_0	0,2823	0.2814	0,2808	0,2770	0,2745	0,2741
	A_1	0,0119	0,0120	0,0121	0,0125	0,0128	0,0128
	A_2	—0,0057	—0,0056	—0,0056	—0,0052	—0,0045	—0,0042
	A_3	0,0024	0,0023	0,0022	0,0017	0,0011	0,0009
	ξ	1,2582	1,2806	1,2854	1,3879	1,4509	1,4598
0,25	A_0	0,2838	0,2830	0,2825	0,2792	0,2772	0,2771
	A_1	0,0098	0,0099	0,0099	0,0101	0,0102	0,0101
	A_2	—0,0043	—0,0043	—0,0042	—0,0039	—0,0034	—0,0031
	A_3	0,0017	0,0016	0,0016	0,0012	0,0008	0,0006
	ξ	1,2316	1,2512	1,2641	1,3431	1,3929	1,3973
0,3	A_0	0,2862	0,2856	0,2852	0,2827	0,2815	0,2816
	A_1	0,0067	0,0067	0,0067	0,0067	0,0066	0,0064
	A_2	—0,0025	—0,0025	—0,0024	—0,0022	0,0019	—0,0017
	A_3	0,0008	0,0008	0,0008	0,0006	0,0004	0,0003

1,4	1,8	3	4	5	8	12
0,2318	0,2302	0,2406	0,2594	0,2804	0,3210	0,3311
0,0622	0,0649	0,0617	0,0467	0,0258	—0,0295	—0,0503
—0,0254	—0,0223	—0,0103	0,0014	0,0139	0,0374	0,0439
0,0034	—0,0003	—0,0091	—0,0167	0,01904	—0,0254	—0,0266
2,4187	2,4431	2,1083	1,6434	1,2005	0,5588	0,4484
0,24313	0,2423	0,2542	0,2709	0,2881	0,3185	0,3256
0,0475	0,0488	0,0435	0,0304	0,0139	—0,0255	—0,0394
—0,0183	—0,0157	—0,0061	0,00275	0,0116	0,0270	0,0310
0,00235	—0,0002	—0,0064	—0,0111	—0,0131	—0,0170	—0,0178
2,1439	2,1384	1,8163	1,4297	0,0108	0,6006	0,5203
0,2524	0,2530	0,2646	0,2792	0,2932	0,3162	0,3212
0,03614	0,0365	0,0305	0,0195	0,0066	0,0217	0,0311
—0,0131	—0,0110	—0,0036	0,00308	0,0093	0,0196	0,0222
0,0016	—0,0001	—0,0044	—0,0075	—0,0089	—0,0114	—0,0119
1,9300	1,9053	1,6025	1,2803	1,0080	0,6394	0,5803
0,2599	0,2612	0,2725	0,2851	0,2965	0,3140	0,3176
0,0275	0,0273	0,02133	0,0123	0,0023	—0,0182	—0,0248
—0,0094	—0,0077	—0,0020	0,0029	0,0073	0,0143	0,0160
0,00114	—0,0001	—0,0031	—0,0051	—0,0061	—0,0077	0,00804
1,7628	1,7262	1,4459	1,1760	0,9586	0,6748	0,6308
0,2639	0,2676	0,2784	0,2893	0,2985	0,3120	0,3145
0,0210	0,0204	0,0149	0,0075	—0,0002	—0,0152	—0,0198
—0,0067	—0,0054	—0,0011	0,0026	0,0057	0,0105	0,0116
$0{,}8 \cdot 10^{3}$	$—0{,}7 \cdot 10^{4}$	—0,0021	—0,0034	—0,0042	—0,0053	—0,00546
1,6312	1,5879	1,3309	1,1034	0,9279	0,7068	0,6735
0,2708	0,2727	0,2829	0,2922	0,2997	0,3102	0,3121
0,0161	0,0154	0,0104	0,0043	—0,0016	—0,0127	—0,0160
—0,0048	—0,0038	—0,0005	0,0022	0,0044	0,0078	0,00853
$0{,}6 \cdot 10^{-3}$	$—0{,}5 \cdot 10^{-4}$	—0,0015	—0,0024	—0,0029	—0,0036	—0,00373
1,5269	1,4804	1,2462	1,05314	0,9098	0,7354	0,70997
0,2748	0,2767	0,2862	0,2942	0,3004	0,3086	0,3101
0,0124	0,0116	0,0072	0,00233	—0,0023	—0,0106	—0,0129
—0,0035	—0,0027	—0,0002	0,00182	0,0034	0,0057	0,00631
0,0004	$—0{,}4 \cdot 10^{-4}$	—0,0010	—0,00163	—0,0020	—0,0025	—0,00257
1,4437	1,3961	—1,1834	1,0185	0,9001	0,7608	0,7412
0,2779	0,2799	0,2886	0,2955	0,3007	0,3072	0,3024
0,0096	0,0080	0,0050	0,00106	—0,0025	—0,0088	—0,0105
—0,0025	0,0019	$0{,}1 \cdot 10^{-4}$	0,0015	0,0027	0,0043	0,00471
0,00027	$—0{,}2 \cdot 10^{-4}$	—0,0007	—0,0012	—0,0014	—0,0017	—0,00178
1,3767	1,3296	1,1366	0,9948	0,8960	0,7835	0,7681
0,2827	0,2846	0,2918	0,2970	0,3007	0,3050	[illegible]
0,0059	0,0053	0,0024	—0,00021	—0,0025	[illegible]	[illegible]
—0,0011	0,0010	0,0001	0,00097	0,0016	[illegible]	[illegible]
0,00014	$—0{,}1 \cdot 10^{-4}$	—0,0004	—0,0006	—0,0007	—0,0008	[illegible]

Table 8 (cont.) [342

h		ω/τ = 0	0,06	0,1	0,4	0,8	1
	ξ	1,1883	1,2036	1,2156	1,2725	1,3038	1,3028
0,35	A_0	0,2880	0,2873	0,2872	0,2853	0,2846	0,2847
	A_1	0,0047	0,0047	0,0047	0,0046	0,0044	0,0042
	A_2	—0,0015	—0,0015	0,0014	0,0013	—0,0011	—0,0010
	A_3	0,0004	0,0004	0,0004	0,0003	0,0002	0,0002
	ξ	1,1553	1,1674	1,1753	1,2204	1,2402	1,2365
0,4	A_0	0,2894	0,2890	0,2886	0,2872	[illegible]2868	0,2870
	A_1	0,0033	0,0033	0,0033	0,0032	0,0030	0,0028
	A_2	—0,0009	—0,0009	—0,0009	—0,0008	—0,0006	—0,0006
	A_3	0,0002	0,0002	0,0002	0,0002	0,0001	$0{,}8 \cdot 10^{-4}$
	ξ	1,1297	1,1395	1,1458	1,1811	1,1937	1,1887
0,45	A_0	0,2904	0,2900	0,2899	0,2887	0,2884	0,2887
	A_1	0,0024	0,0024	0,0024	0,0023	0,0021	0,0019
	A_2	—0,0006	—0,0006	—0,0005	—0,0005	—0,0004	—0,0003
	A_3	0,0001	0,0001	0,0001	$0{,}9 \cdot 10^{-4}$	0,00006	$0{,}4 \cdot 10^{-4}$
	ξ	1,1095	1,1176	1,1228	1,1510	1,1590	1,1535
0,5	A_0	0,2912	0,2909	0,2908	0,2898	0,2896	0,2900
	A_1	0,0018	0,0017	0,0017	0,0016	0,0015	0,0014
	A_2	—0,0004	—0,0004	—0,0003	—0,0003	0,0002	—0,0002
	A_3	$10^{-4} \cdot 0{,}7$	$-10^{-4} \cdot 0{,}7$	$10^{-4} \cdot 0{,}6$	$0{,}5 \cdot 10^{-4}$	$10^{-4} \cdot 0{,}3$	$0{,}3 \cdot 10^{-4}$
	ξ	1,0935	1,1002	1,1045	1,1274	1,1325	1,1270
0,6	A_0	0,2924	0,2922	0,9221	0,2314	0,2914	0,2916
	A_1	0,0010	0,0010	0,0010	0,0009	0,0008	0,0007
	A_2	—0,0002	—0,0002	—0,0001	—0,0001	—0,0001	$-0{,}9 \cdot 10^{-4}$
	A_3	$10^{-4} \cdot 0{,}24$	$10^{-4} \cdot 0{,}23$	$10^{-4} \cdot 0{,}22$	$10^{-4} \cdot 0{,}171$	$10^{-4} \cdot 0{,}11$	$0{,}9 \cdot 10^{-5}$
	ξ	1,0700	1,0748	1,0779	1,0938	1,0956	1,0905
0,7	A_0	0,2932	0,2930	0,2929	0,2024	0,2924	0,2926
	A_1	0,0006	0,0006	0,0006	0,0005	0,0005	0,0004
	A_2	$-10^{-4} \cdot 0{,}8$	$-10^{-4} \cdot 07$	$-10^{-4} \cdot 0{,}7$	$-0{,}6 \cdot 10^{-4}$	$0{,}5 \cdot 10^{-4}$	$-0{,}4 \cdot 10^{-4}$
	A_3	$-10^{-5} \cdot 0{,}9$	$10^{-5} \cdot 0{,}9$	$10^{-5} \cdot 0{,}9$	$0{,}6 \cdot 10^{-5}$	$0{,}4 \cdot 10^{-5}$	$0{,}3 \cdot 10^{-5}$
	ξ	1,0541	1,0577	1,0600	1,0716	1,0719	1,0676
0,8	A_0	0,2937	0,2936	0,2935	0,2931	0,2932	0,2933
	A_1	0,0004	0,0004	0,0004	0,0003	0,0003	0,0003
	A_2	$-10^{-4} \cdot 0{,}4$	$-10^{-4} \cdot 0{,}4$	$-10^{-4} \cdot 0{,}4$	$0{,}3 \cdot 10^{-4}$	$-0{,}2 \cdot 10^{-4}$	$-0{,}2 \cdot 10^{-4}$
	A_3	$10^{-5} \cdot 0{,}4$	$10^{-5} \cdot 0{,}3$	$10^{-5} \cdot 0{,}3$	$0{,}3 \cdot 10^{-5}$	$0{,}2 \cdot 10^{-5}$	$0{,}1 \cdot 10^{-5}$
	ξ	1,0428	1,0457	1,0475	1,0563	1,0559	1,0522
0,9	A_0	0,2941	0,2940	0,2939	0,2936	0,2937	0,2938
	A_1	0,0002	0,0002	0,0002	0,0002	0,0002	0,0002
	A_2	$10^{-4} \cdot 0{,}2$	$-10^{-4} \cdot 0{,}2$	$-10^{-4} \cdot 0{,}2$	$-0{,}2 \cdot 10^{-4}$	$-0{,}1 \cdot 10^{-4}$	$-0{,}1 \cdot 10^{-4}$
	A_3	$10^{-5} \cdot 0{,}2$	$10^{-5} \cdot 0{,}2$	$10^{-2} \cdot 0{,}1$	$0{,}1 \cdot 10^{-5}$	$0{,}8 \cdot 10^{-6}$	$0{,}6 \cdot 10^{-6}$
	ξ	1,0347	1,0369	1,0384	1,0453	1,0447	1,0415
1	A_0	0,2944	0,2943	0,2942	0,2940	0,2940	0,2942
	A_1	0,0002	0,0002	0,0002	0,0001	0,0001	0,0001

Table 8 (cont.)

1,4	1,8	3	4	5	8	12
1,2780	1,2342	1,0754	0,9683	0,8976	0,8214	0,8115
0,2853	0,2876	0,2236	0,2976	0,3003	0,3032	0,3037
0,00377	0,0032	0,0011	—0,00063	—0,0021	—0,0013	—0,0049
—0,0007	—0,0005	0,0002	0,00063	0,0010	0,0015	0,00158
0,7·10^{-4}	—0,6·10^{-5}	—0,0002	—0,00029	—0,0003	—0,0004	—10^{-3}·0,447
1,2111	1,1718	1,0406	0,95760	0,9051	0,8513	0,8447
0,2881	0,2896	0,2946	0,2978	0,2998	0,3019	0,3022
0,0025	0,0020	0,0005	—0,00070	—0,0017	—0,0031	—0,00344
—0,0004	—0,0003	0,0001	0,00041	0,0006	0,0009	0,00096
0,4·10^{-4}	—0,3·10^{-3}	—0,0001	—0,00015	—0,0002	—0,0002	—0,00023
1,1645	1,1297	1,0203	0,9546	0,9145	0,8750	0,8703
0,2897	0,2910	0,2952	0,2978	0,2993	0,3009	0,3011
0,00166	0,0013	0,0002	—0,00065	—0,0013	—0,0022	—0,00246
—0,00025	—0,0002	0,0001	0,00027	0,0004	0,0006	10^{-3}·0,595
0,2·10^{-4}	—0,2·10^{-5}	—0,5·10^{-4}	—0,8·10^{-4}	—0,0001	—0,0001	—01^{-3}·0,127
1,13102	1,1005	1,0085	0,9354	0,9239	0,8939	0,8905
0,29081	0,2920	0,2956	0,29764	0,2989	0,3001	0,3002
0,0012	0,0009	0,6·10^{-2}	—0,00056	—0,0010	—0,0017	—0,00180
—0,00015	—0,9·10^{-4}	0,7·10	0,00018	0,0003	0,0004	0,00038
0,1·10^{-4}	—0,1·10^{-5}	—0,3·10^{-4}	—0,5·10^{-4}	—0,6·10^{-4}	—0,7·10^{-4}	—10^{-4}·0,71
1,1064	1,0796	1,0015	0,9578	0,9325	0,9091	0,9066
0,2923	0,2933	0,2959	0,2973	0,2981	0,2989	0,2990
0,6·10^{-3}	0,0004	—0,4·10^{-4}	—0,00038	—0,0006	—0,0009	0,00101
—0,6·10^{-4}	—0,3·10^{-4}	0,4·10^{-4}	0,8·10^{-4}	0,0001	0,0002	0,00017
0,4·10^{-3}	—0,4·10^{-6}	—0,1·10^{-4}	—0,16·10^{-4}	—0,2·10^{-4}	—0,2·10^{-4}	—10^{-4}·0,241
1,0738	1,0529	0,9951	0,9642	0,9469	0,9376	0,9301
0,2932	0,2939	0,2959	0,29702	0,2976	0,2981	0,2982
0,00033	0,0002	—0,5·10^{-4}	—0,00025	—0,0004	—0,0569	10^{-3}·0,602
—0,3·10^{-4}	—0,1·10^{-4}	0,2·10^{-4}	0,4·10^{-4}	0,6·10^{-4}	0,7·10^{-4}	10^{-4}·0,785
0,15·10^{-3}	—0,1·10^{-6}	0,4·10^{-5}	—0,6·10^{-5}	—0,7·10^{-5}	—0,9·10^{-5}	—10^{-5}·0,907
1,0539	1,0374	0,9931	0,9703	0,9577	0,9470	0,9460
0,2938	0,2944	0,2959	0,2968	0,2972	0,2076	0,2976
0,0002	0,0001	—0,5·10^{-4}	—0,00017	—0,0002	—0,4·10^{-3}	10^{-3}·0,378
—0,1·10^{-4}	—0,6·10^{-5}	0,1·10^{-4}	0,2·10^{-4}	0,3·10^{-4}	0,4·10^{-4}	10^{-4}·0,399
0,6·10^{-6}	—0,6·10^{-7}	—0,2·10^{-5}	—0,2·10^{-5}	—0,3·10^{-5}	—0,4·10^{-5}	10^{-5}·0,373
1,0410	1,0278	0,9929	0,9753	0,9658	0,9579	0,9572
0,2942	0,2947	0,2960	0,2963	0,2969	0,2972	0,2973
0,00013	0,8·10^{-4}	—0,4·10^{-4}	—0,00012	—0,0002	—0,0001	—10^{-3}·0,247
—0,6·10^{-6}	—0,3·10^{-5}	0,6·10^{-5}	0,12·10^{-4}	0,2·10^{-4}	0,2·10^{-4}	10^{-4}·0,215
0,3·10^{-6}	0,2·10^{-7}	—0,7·10^{-6}	—0,1·10^{-5}	—0,1·10^{-5}	—0,2·10^{-5}	10^{-5}·0,165
1,0322	1,0214	0,9933	0,9794	0,9720	0,9659	0,9653
0,2945	0,2949	0,2959	0,29645	0,2967	0,2969	0,2970
0,8·10^{-4}	0,5·10^{-4}	—0,3·10^{-4}	—0,8·10^{-4}	—0,0001	—0,0002	10^{-4}·0,17

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
	A_2	$-10^{-4}\cdot 0,12$	$-10^{-4}\cdot 0,12$	$-10^{-4}\cdot 0,1$	$-0,9\cdot 10^{-5}$	$-0,7\cdot 10^{-5}$	$-0,6\cdot 10^{-}$
	A_3	$10^{-6}\cdot 0,8$	$10^{-6}\cdot 0,7$	$10^{-6}\cdot 0,7$	$0,6\cdot 10^{-6}$	$0,4\cdot 10^{-6}$	$0,3\cdot 10^{-}$
	ξ	1,0286	1,0305	1,0317	1,0372	1,0365	1,0338
1,1	A_0	0,2946	0,2945	0,2945	0,2943	0,2943	0,2944
	A_1	0,0002	0,0001	0,0001	0,0001	$0,9\cdot 10^{-4}$	$0,8\cdot 10^{-}$
	A_2	$-10^{-5}\cdot 0,7$	$-10^{-5}\cdot 0,7$	$-10^{-5}\cdot 0,7$	$-0,6\cdot 10^{-5}$	$-0,4\cdot 10^{-5}$	$-0,3\cdot 10^{-}$
	A_3	$10^{-6}\cdot 0,4$	$10^{-6}\cdot 0,4$	$10^{-6}\cdot 0,4$	$0,3\cdot 10^{-6}$	$0,2\cdot 10^{-6}$	$0,1\cdot 10^{-}$
	ξ	1,0240	1,0255	1,0265	1,0311	1,0352	1,0280
1,2	A_0	0,2948	—0,2947	0,2947	0,2945	0,2945	0,2946
	A_1	$10^{-4}\cdot 0,85$	$10^{-4}\cdot 0,83$	$10^{-4}\cdot 0,8$	$0,7\cdot 10^{-4}$	$0,6\cdot 10^{-4}$	$0,5\cdot 10^{-}$
	A_2	$-10^{-5}\cdot 0,45$	$-10^{-5}\cdot 0,43$	$-10^{-5}\cdot 0,4$	$-0,3\cdot 10^{-5}$	$-0,2\cdot 10^{-5}$	$-0,2\cdot 10^{-}$
	A_3	$10^{-6}\cdot 0,21$	$10^{-6}\cdot 0,2$	$10^{-6}\cdot 0,2$	$0,1\cdot 10^{-6}$	$0,1\cdot 10^{-6}$	$0,8\cdot 10^{-}$
	ξ	1,0204	1,0217	1,0225	1,0263	1,0256	1,0235
							$\lambda = 12,$
0,025	A_0	0,2243	0,2217	0,2197	0,2072	0,1937	0,1884
	A_1	0,0398	0,0424	0,0442	0,0558	0,0687	0,0741
	A_2	—0,0474	—0,0473	—0,0472	—0,0460	—0,0436	—0,0421
	A_3	0,0390	0,0370	0,066	0,0267	0,0169	0,0127
	ξ	1,7258	1,8071	1,8627	2,2765	2,7583	2,9539
0,05	A_0	0,2279	0,2257	0,2242	0,2139	0,2031	0,1991
	A_1	0,0342	0,0360	0,0372	0,0454	0,0543	0,0580
	A_2	—0,0352	—0,0351	—0,0350	—0,0341	—0,0321	—0,0309
	A_3	0,0265	0,0251	0,0243	0,0183	0,0117	0,0089
	ξ	1,6262	1,6940	1,7401	2,0757	2,4453	2,5868
0,075	A_0	0,2310	0,2292	0,2280	0,2195	0,2111	0,2081
	A_1	0,0289	0,0301	0,0309	0,0365	0,0426	0,0450
	A_2	—0,0262	—0,0260	—0,0259	—0,0251	—0,235	—0,0225
	A_3	0,0180	0,0171	0,0165	0,0125	0,0081	0,0061
	ξ	1,5435	1,6003	1,6388	1,9125	2,1966	2,2981
0,1	A_0	0,2338	0,2322	0,2312	0,2243	0,2177	0,2155
	A_1	0,0241	0,0249	0,0255	0,0292	0,0332	0,0348
	A_2	—0,0193	—0,0192	—0,0192	—0,0184	—0,0171	—0,0163
	A_3	0,0122	0,0116	0,0113	0,0086	0,0056	0,0042
	ξ	1,4744	1,5223	1,5546	1,7792	1,9982	2,0705
0,125	A_0	0,2362	0,2349	0,2340	0,2283	0,2232	0,2216
	A_1	0,0199	0,0205	0,0208	0,0233	0,259	0,0268
	A_2	—0,0143	—0,0142	—0,0142	—0,0135	—0,0124	—0,0118
	A_3	0,0083	0,0079	0,0077	0,0059	0,0038	0,0029
	ξ	1,4162	1,4569	1,4842	1,6696	1,8390	1,8896
0,15	A_0	0,2382	0,2371	0,2364	0,2317	0,2277	0,2266
	A_1	0,0164	0,0168	0,0170	0,0186	0,0202	—0,0207
	A_2	—0,0107	—0,0106	—0,0105	—0,0099	—0,0090	—0,0085
	A_3	0,0057	0,0054	0,0053	0,0040	0,0026	0,0020

1,4	1,8	3	4	5	8	12
$-0{,}4\cdot10^{-5}$	$-0{,}1\cdot10^{-6}$	$0{,}4\cdot10^{-5}$	$0{,}7\cdot10^{-5}$	$0{,}9\cdot10^{-5}$	$+0{,}1\cdot10^{-4}$	$10^{-4}\cdot0{,}122$
$0{,}1\cdot10^{-6}$	$-0{,}1\cdot10^{-7}$	$-0{,}3\cdot10^{-6}$	$-0{,}5\cdot10^{-6}$	$-0{,}6\cdot10^{-6}$	$-0{,}8\cdot10^{-6}$	$-10^{-6}\cdot0{,}784$
1,0259	1,0169	0,9939	0,9826	0,9766	0,9718	0,9714
0,2947	0,2951	0,2959	0,2963	0,2965	0,2967	0,2967
$0{,}6\cdot10^{-4}$	$0{,}3\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$-0{,}6\cdot10^{-4}$	$-0{,}8\cdot10^{-4}$	$-0{,}0001$	$-10^{-3}\cdot0{,}118$
$-0{,}2\cdot10^{-5}$	$-0{,}8\cdot10^{-6}$	$0{,}2\cdot10^{-5}$	$0{,}4\cdot10^{-5}$	$0{,}5\cdot10^{-5}$	$0{,}7\cdot10^{-5}$	$10^{-5}\cdot0{,}726$
$0{,}6\cdot10^{-7}$	$-0{,}6\cdot10^{-8}$	$-0{,}2\cdot10^{-6}$	$-0{,}26\cdot10^{-6}$	$-0{,}3\cdot10^{-6}$	$-0{,}4\cdot10^{-6}$	$-10^{-6}\cdot0{,}394$
1,0214	1,0138	0,9945	0,9852	0,9803	0,9763	0,9760
0,2949	0,2952	−0,2959	0,29623	0,2964	0,2966	0,2966
$0{,}4\cdot10^{-4}$	$0{,}2\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	$-0{,}6\cdot10^{-4}$	$-0{,}8\cdot10^{-4}$	$-10^{-4}\cdot0{,}85$
$-0{,}1\cdot10^{-6}$	$-0{,}1\cdot10^{-4}$	$0{,}2\cdot10^{-6}$	$0{,}27\cdot10^{-6}$	$0{,}3\cdot10^{-6}$	$0{,}4\cdot10^{-5}$	$10^{-5}\cdot0{,}45$
$0{,}3\cdot10^{-7}$	$0{,}1\cdot10^{-4}$	$-0{,}9\cdot10^{-7}$	$-0{,}14\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-10^{-6}\cdot0{,}21$
1,01787	1,0050	0,9951	0,98725	0,9832	0,9799	0,9796
$a=1{,}156526$						
0,1806	1,1784	0,1807	1,1980	0,2207	0,2738	0,2894
0,0829	0,0893	0,0943	0,08065	0,0570	−0,0261	−0,0625
−0,0389	−0,0356	−0,02114	−0,00604	0,0128	0,0553	0,0692
0,0055	−0,0005	−0,01456	−0,0298	−0,0315	−0,0442	−0,0469
3,2389	3,3771	3,0567	2,3817	1,6251	0,4710	0,2579
0,19356	0,1010	0,19771	0,21373	0,2329	0,2726	0,2832
0,0638	0,0677	0,06732	0,0539	0,0338	−0,0248	−0,0481
−0,0282	−0,0250	−0,22129	$-0{,}8\cdot10^{-3}$	0,0128	0,0400	0,0479
0,0039	−0,0003	−0,00103	−0,0197	−0,0218	−0,0295	−0,0310
2,7772	2,8481	2,6114	1,9486	1,3738	0,5152	0,3627
0,20432	0,2031	0,21117	0,2255	0,2413	0,2710	0,2783
0,0487	0,0505	0,04761	0,0356	0,0193	−0,0222	−0,0375
−0,0207	−0,0172	−0,00780	0,00156	0,0112	0,0288	0,0336
0,0027	−0,0004	−0,00726	−0,0131	−0,0150	−0,0197	−0,0206
2,4209	2,4459	2,1107	1,6436	1,2010	0,5590	0,4491
0,21311	0,2129	0,22158	0,2342	0,2470	0,2693	0,2747
0,03696	0,0379	0,03349	0,0231	0,0103	−0,0192	−0,0295
−0,00145	−0,0123	−0,00463	0,00242	0,0092	0,0208	0,0238
0,00188	−0,0002	−0,00506	−0,0087	−0,0102	−0,0132	−0,0138
2,1451	2,1398	1,81744	1,4298	1,0847	0,6008	0,5207
0,2202	0,2207	0,22950	0,2404	0,2508	0,2675	0,2712
0,0280	0,0282	0,02336	0,0148	0,0049	−0,0163	0,0234
−0,0103	−0,0086	−0,00268	0,00256	0,0073	0,0151	0,0170
0,00131	−0,0001	−0,0035	−0,0058	−0,0070	−0,0089	−0,00925
1,93074	1,9360	1,6031	1,28031	1,0080	0,6395	0,5806
0,2259	0,2268	0,2354	0,24482	[illegible]	0,2560	0,2686
0,0212	0,0212	0,01628	0,0093	0,0016	−0,0137	−0,0186
−0,0074	−0,0070	−0,00149	0,0024	0,0067	0,0111	[illegible]
0,0009	$-0{,}2\cdot10^{-4}$	−0,00249	−0,0040	−0,0047	−0,0060	−0,00625

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
	ξ	1,3669	1,4017	1,4243	1,5791	1,7106	1,7455
0,775	A_0	0,2400	0,2390	0,2384	0,2345	0,2314	0,2306
	A_1	0,0135	0,0137	0,0139	0,0149	0,0158	0,0160
	A_2	−0,0079	−0,0079	−0,0078	−0,0073	−0,0066	−0,0062
	A_3	0,0039	0,037	0,0036	−0,0028	0,0018	0,0014
	ξ	1,3250	1,3548	1,3747	1,5038	1,6064	1,6297
0,2	A_0	0,2415	0,2407	0,2401	0,2368	0,2344	0,2339
	A_1	0,0111	0,0112	0,0113	0,0119	0,0124	0,0125
	A_2	−0,0059	−0,0059	−0,0058	−0,0054	−0,0048	−0,0045
	A_3	0,0027	0,0026	0,0025	0,0019	0,0013	0,0010
	ξ	1,2891	1,3149	1,3320	1,4409	1,5211	1,5361
0,225	A_0	0,2428	0,2421	0,2416	0,2388	0,2369	0,2366
	A_1	0,0091	0,0092	0,0092	0,0096	0,0098	0,0098
	A_2	−0,0045	−0,0044	−0,0044	−0,0040	−0,0035	−0,0033
	A_3	0,0019	0,0018	0,0017	0,0013	0,0009	0,0007
	ξ	1,2582	1,2807	1,2055	1,3879	1,4510	1,4599
0,25	A_0	0,2439	0,2434	0,2429	0,2405	0,2390	0,2383
	A_1	0,0075	0,0075	0,0076	0,0077	0,0078	0,0077
	A_2	−0,0034	−0,0033	−0,0033	−0,0030	−0,0026	−0,0024
	A_3	0,0013	0,0012	0,0012	0,0009	0,0006	0,0005
	ξ	1,2316	1,2512	1,2641	1,3432	1,3929	1,3974
0,3	A_0	0,2456	0,2452	0,2449	0,2431	0,2422	0,2422
	A_1	0,0051	0,0051	0,0051	0,0051	0,0050	0,0049
	A_2	−0,0020	−0,0019	−0,0019	−0,0017	−0,0015	−0,0013
	A_3	0,0006	0,0006	0,0006	0,0005	0,0003	0,0002
	ξ	1,1883	1,2306	1,2136	1,2726	1,3038	1,3028
0,35	A_0	0,2470	0,2466	0,2464	0,2450	0,2444	0,2446
	A_1	0,0036	0,0036	0,0036	0,0035	0,0033	0,0032
	A_2	−0,0012	−0,0011	−0,0011	−0,0010	−0,0008	−0,0008
	A_3	0,0003	0,0003	0,0003	0,0002	0,0002	0,0001
	ξ	1,1553	1,1674	1,1753	1,2204	1,2402	1,2365
0,4	A_0	0,2480	0,2476	0,2475	0,2464	0,2461	0,2462
	A_1	0,0025	0,0025	0,0025	0,0024	0,0022	0,0021
	A_2	−0,0007	−0,0007	−0,0007	−0,0006	−0,0005	−0,0004
	A_3	0,0002	0,0002	0,0002	0,0001	0,00008	$0{,}7 \cdot 10^{-4}$
	ξ	1,1297	1,1395	1,1458	1,1811	1,1937	1,1887
0,45	A_0	0,2488	0,2485	0,2483	0,2475	0,2473	0,2475
	A_1	0,0018	0,0018	0,0018	0,0017	0,0016	0,0016
	A_2	−0,0004	−0,0004	−0,0004	−0,0004	−0,0003	−0,0003
	A_3	$10^{-4} \cdot 0{,}97$	$10^{-4} \cdot 0{,}93$	$10^{-4} \cdot 0{,}9$	$0{,}7 \cdot 10^{-4}$	0,00005	$10^{-4} \cdot 0{,}4$
	ξ	1,1095	1,1176	1,1228	1,1510	1,1590	1,1535
0,5	A_0	0,2494	0,2492	0,2490	0,2483	0,2482	0,2484
	A_1	0,0013	0,0013	0,0013	0,0012	0,0011	0,0010

1,4	1,8	3	4	5	8	12
1,7632	1,7266	1,44621	1,1760	0,9586	0,6749	0,6310
0,2305	0,2327	0,2399	0,2479	0,2547	0,2645	0,2664
0,0162	0,0157	0,01132	0,00562	—0,0002	—0,0115	—0,0149
—0,0052	—0,0042	—0,00077	0,00206	0,0045	0,0081	0,00896
0,00063	$—0,1 \cdot 10^{-4}$	—0,00168	—0,00269	—0,0033	—0,0041	—0,00425
1,6314	1,5882	1,33107	1,10343	0,9279	0,7068	0,6737
0,2342	0,2356	0,24316	0,25004	0,2556	0,2632	0,2646
0,0124	0,0118	0,00787	0,00320	—0,0013	—0,0096	—0,0120
—0,0038	—0,0021	—0,00036	0,00173	0,0034	0,0060	0,00658
0,000	$—0,4 \cdot 10^{-4}$	—0,00117	—0,00185	—0,0023	—0,0028	—0,00291
1,5271	1,4505	1,24622	1,05314	0,9098	0,7354	0,7101
0,2371	0,2386	0,2466	0,25151	0,2561	0,2621	0,2631
0,0096	0,0089	0,00546	0,00175	—0,0018	—0,0080	—0,0098
—0,0027	—0,0020	—0,000111	0,00145	0,0027	0,0045	0,00487
0,00031	$—0,3 \cdot 10^{-4}$	—0,00082	—0,0012	—0,0016	—0,0019	—0,0020
1,4437	1,3662	1,18340	1,0185	0,9001	0,7609	0,7412
0,2395	0,2410	0,24742	0,2525	0,2563	0,2611	0,2619
0,0074	0,0068	0,00379	0,00079	—0,0020	—0,0067	—0,00796
—0,0020	—0,0014	0,000022	0,00116	0,0021	0,0034	0,00364
0,00021	$—0,2 \cdot 10^{-4}$	—0,00057	$—0,9 \cdot 10^{-3}$	—0,0011	—0,0013	—0,00139
1,3767	1,3297	1,13666	0,9948	0,8960	0,7835	0,7681
0,24303	0,2444	0,24978	0,25361	0,2562	0,2594	0,2599
0,0045	0,0041	0,001823	—0,00017	—0,0019	—0,0046	—0,00537
—0,0011	—0,0008	0,000119	0,00076	0,0012	0,0019	0,00208
0,00011	$—0,1 \cdot 10^{-4}$	—0,00029	$—0,4 \cdot 10^{-3}$	—0,0005	—0,0007	—0,00069
1,27798	1,2342	1,07544	0,9683	0,8976	0,8214	0,8116
0,2454	0,2467	0,25111	0,2540	0,2560	0,2581	0,2585
0,0029	0,0024	0,00086	$—0,5 \cdot 10^{-3}$	—0,0016	—0,0033	—0,0037
—0,00058	—0,0004	0,00012	$0,5 \cdot 10^{-3}$	0,0008	0,0011	0,00122
$0,5 \cdot 10^{-4}$	$—0,5 \cdot 10^{-5}$	—0,00015	$—0,2 \cdot 10^{-3}$	—0,0003	—0,0003	—0,00035
1,2111	1,1718	1,0405	0,9576	0,9051	0,8513	0,8447
0,2470	0,2481	0,2518	0,2542	0,2556	0,2572	0,2574
0,00187	0,0015	0,00039	$—0,5 \cdot 10^{-3}$	—0,0013	—0,0023	—0,0026
—0,0003	—0,0002	0,00010	$0,3 \cdot 10^{-3}$	0,0005	0,0007	0,00074
$0,3 \cdot 10^{-4}$	$—04 \cdot 10^{-5}$	$—0,8 \cdot 10^{-4}$	$—0,1 \cdot 10^{-3}$	—0,0001	—0,0002	—0,00018
1,1643	1,1299	1,0203	0,9546	0,9145	0,8750	0,8703
0,2482	0,2442	0,2523	0,2541	0,2553	0,2564	0,2566
0,0013	0,0010	0,00016	$—0,5 \cdot 10^{-3}$	—0,0010	—0,0017	—0,00186
—0,0002	$0,9 \cdot 10^{-4}$	$0,9 \cdot 10^{-4}$	$0,2 \cdot 10^{-3}$	0,0003	0,0004	0,00046
$0,16 \cdot 10^{-4}$	$—0,4 \cdot 10^{-4}$	$—0,4 \cdot 10^{-4}$	$—0,6 \cdot 10^{-4}$	$—0,8 \cdot 10^{-4}$	$—0,1 \cdot 10^{-3}$	$—10^{-4}$ · 0,995
1,13102	1,1004	1,0085	0,9554	0,9239	0,8939	0,8905
0,2491	0,2499	0,2525	0,2541	0,2550	0,2558	0,2559
0,00087	0,0007	$0,5 \cdot 10^{-4}$	$—0,4 \cdot 10^{-3}$	—0,0008	—0,0013	$—10^{-3}$ · 0,130

h \ ω/τ		0	0,06	0,1	0,4	0,8	1
	A_2	—0,0003	—0,0003	—0,0003	—0,0002	—0,0002	—0,0002
	A_3	$10^{-4}\cdot 0{,}54$	$10^{-4}\cdot 0{,}5$	$10^{-4}\cdot 0{,}5$	$0{,}4\cdot 10^{-4}$	0,00003	$0{,}2\cdot 10^{-4}$
	ξ	1,0935	1,1002	1,1045	1,1274	1,1325	1,1270
0,6	A_0	0,2502	0,2501	0,2500	0,2495	0,2494	0,2496
	A_1	0,0007	0,0007	0,0007	0,0007	0,0006	0,0006
	A_2	—0,0001	—0,0001	—0,0001	—0,0001	$-0{,}8\cdot 10^{-4}$	$-0{,}7\cdot 10^{-4}$
	A_3	$10^{-4}\cdot 0{,}19$	$10^{-4}\cdot 0{,}18$	$10^{-4}\cdot 0{,}17$	$-10^{-4}\cdot 0{,}1$	$0{,}9\cdot 10^{-5}$	$0{,}7\cdot 10^{-5}$
	ξ	1,0700	1,0748	0,0779	1,0938	1,0956	1,0905
0,7	A_0	0,2508	0,2507	0,2506	0,2502	0,2502	0,2504
	A_1	0,0004	0,0004	0,0004	0,0004	0,0003	0,0003
	A_2	$-10^{-4}\cdot 0{,}6$	$-10^{-4}\cdot 0{,}6$	$-10^{-4}\cdot 0{,}6$	$-10^{-4}\cdot 0{,}4$	$-0{,}4\cdot 10^{-4}$	$-0{,}3\cdot 10^{-4}$
	A_3	$10^{-5}\cdot 0{,}7$	$10^{-5}\cdot 0{,}7$	$10^{-5}\cdot 0{,}6$	$0{,}5\cdot 10^{-5}$	$0{,}3\cdot 10^{-5}$	$0{,}3\cdot 10^{-5}$
	ξ	1,0541	1,0577	1,0600	1,0716	1,0719	1,0676
0,8	A_0	0,2512	0,2511	0,2510	0,2507	0,2508	0,2509
	A_1	0,0003	0,0003	0,0003	0,0002	0,0002	0,0002
	A_2	$-10^{-4}\cdot 0{,}3$	$-10^{-4}\cdot 0{,}3$	$-10^{-4}\cdot 0{,}3$	$-0{,}2\cdot 10^{-4}$	$-0{,}2\cdot 10^{-4}$	$-0{,}1\cdot 10^{-4}$
	A_3	$10^{-5}\cdot 0{,}3$	$10^{-5}\cdot 0{,}3$	$10^{-5}\cdot 0{,}3$	$0{,}2\cdot 10^{-5}$	$0{,}1\cdot 10^{-5}$	$0{,}1\cdot 10^{-5}$
	ξ	1,0428	1,0457	1,0475	1,0563	1,0559	1,0522
0,9	A_0	0,2515	0,2514	0,2513	0,2511	0,2512	0,2513
	A_1	0,0002	0,0002	0,0002	0,0002	0,0001	0,0001
	A_2	$-10^{-4}\cdot 0{,}16$	$-10^{-4}\cdot 0{,}2$	$-10^{-4}\cdot 0{,}2$	$-0{,}1\cdot 10^{-4}$	$-0{,}1\cdot 10^{-4}$	$-0{,}8\cdot 10^{-5}$
	A_3	$10^{-5}\cdot 0{,}13$	$10^{-5}\cdot 0{,}12$	$10^{-5}\cdot 0{,}12$	$0{,}9\cdot 10^{-6}$	$0{,}6\cdot 10^{-6}$	$0{,}5\cdot 10^{-6}$
	ξ	1,0347	1,0370	1,0384	1,0453	1,0447	1,0415
1	A_0	0,2517	0,2516	0,2516	0,2514	0,2514	0,2515
	A_1	0,0001	0.0001	0,001	0,0001	$0{,}9\cdot 10^{-4}$	$0{,}8\cdot 10^{-4}$
	A_2	$-10^{-5}\cdot 0{,}94$	$-10^{-5}\cdot 0{,}91$	$-10^{-5}\cdot 0{,}9$	$-0{,}7\cdot 10^{-5}$	$-0{,}5\cdot 10^{-5}$	$-0{,}5\cdot 10^{-5}$
	A_3	$10^{-6}\cdot 0{,}61$	$10^{-6}\cdot 0{,}6$	$10^{-6}\cdot 0{,}6$	$0{,}4\cdot 10^{-6}$	$10^{-6}\cdot 0{,}3$	$0{,}2\cdot 10^{-6}$
	ξ	1,0287	1,0305	1,0317	1,0312	1,0365	1,0338
1,1	A_0	0,2518	0,2518	0,2517	0,2516	0,2516	0,2517
	A_1	$10^{-4}\cdot 0{,}9$	$10^{-4}\cdot 0{,}9$	$10^{-4}\cdot 0{,}9$	$10^{-4}\cdot 0{,}8$	$0{,}6\cdot 10^{-4}$	$0{,}6\cdot 10^{-4}$
	A_2	$-10^{-5}\cdot 0{,}6$	$-10^{-5}\cdot 0{,}5$	$-10^{-5}\cdot 0{,}53$	$-0{,}4\cdot 10^{-5}$	$-0{,}3\cdot 10^{-5}$	$-0{,}3\cdot 10^{-5}$
	A_3	$10^{-6}\cdot 0{,}3$	$10^{-6}\cdot 0{,}3$	$10^{-6}\cdot 0{,}3$	$0{,}2\cdot 10^{-6}$	$0{,}1\cdot 10^{-6}$	$0{,}1\cdot 10^{-6}$
	ξ	1,0240	1,0255	1,0265	1,0311	1,0303	1,0280
1,2	A_0	0,2519	0,2519	0,2519	0,2518	0,2518	0,2518
	A_1	$10^{-4}\cdot 0{,}6$	$10^{-4}\cdot 0{,}6$	$10^{-4}\cdot 0{,}6$	$10^{-4}\cdot 0{,}6$	$0{,}5\cdot 10^{-4}$	$0{,}4\cdot 10^{-4}$
	A_2	$-10^{-5}\cdot 0{,}3$	$-10^{-5}\cdot 0{,}3$	$-10^{-5}\cdot 0{,}3$	$-0{,}3\cdot 10^{-5}$	$-0{,}2\cdot 10^{-5}$	$-0{,}2\cdot 10^{-5}$
	A_3	$10^{-6}\cdot 0{,}2$	$10^{-6}\cdot 0{,}15$	$10^{-6}\cdot 0{,}15$	$0{,}1\cdot 10^{-6}$	$0{,}8\cdot 10^{-7}$	$0{,}6\cdot 10^{-7}$
	ξ	1,0204	1,0217	1,0225	1,0263	1,0256	1,0236

1,4	1,8	3	4	5	8	12
$-0{,}00012$	$-0{,}7\cdot10^{-4}$	$0{,}6\cdot10^{-4}$	$0{,}1\cdot10^{-3}$	$0{,}0002$	$0{,}0003$	$10^{-3}\cdot0{,}293$
$0{,}9\cdot10^{-5}$	$-0{,}8\cdot10^{-6}$	$-0{,}3\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	$-0{,}5\cdot10^{-4}$	$-10^{-4}\cdot0{,}56$
1,1064	1,0795	1,0015	0,9579	0,9325	0,9091	0,9066
0,2502	0,2552	0,2527	0,2538	0,2544	0,2550	0,2550
0,00045	0,0003	$-0{,}3\cdot10^{-4}$	$-0{,}3\cdot10^{-3}$	$-0{,}0005$	$-0{,}7\cdot10^{-3}$	$-10^{-3}\cdot0{,}76$
$-0{,}5\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$0{,}3\cdot10^{-4}$	$0{,}7\cdot10^{-4}$	$10^{-4}\cdot0{,}5$	$0{,}12\cdot10^{-3}$	$10^{-4}\cdot0{,}13$
$0{,}3\cdot10^{-5}$	$-0{,}4\cdot10^{-6}$	$-0{,}8\cdot10^{-6}$	$-0{,}1\cdot10^{-4}$	$-0{,}1\cdot10^{-4}$	$-0{,}18\cdot10^{-4}$	$-10^{-4}\cdot0{,}19$
1,0738	1,0623	0,9951	0,96422	0,9465	0,9316	0,9301
0,2508	0,2514	0,2528	0,2536	0,2540	0,2544	0,2545
0,00025	0,0002	$-0{,}4\cdot10^{-4}$	$-0{,}2\cdot10^{-3}$	$-0{,}3\cdot10^{-3}$	$-0{,}0004$	$-10^{-3}\cdot0{,}45$
$-0{,}2\cdot10^{-4}$	$-0{,}1\cdot10^{-4}$	$0{,}2\cdot10^{-4}$	$0{,}3\cdot10^{-4}$	$0{,}4\cdot10^{-4}$	$0{,}6\cdot10^{-4}$	$10^{-4}\cdot0{,}61$
$0{,}1\cdot10^{-6}$	$-0{,}1\cdot10^{-6}$	$-0{,}3\cdot10^{-6}$	$-0{,}5\cdot10^{-6}$	$-10^{-5}\cdot0{,}6$	$-0{,}7\cdot10^{-5}$	$-10^{-5}\cdot0{,}71$
1,0539	1,0374	0,9961	0,9703	0,9577	0,9470	0,9460
0,2513	0,2517	0,2528	0,2534	0,2537	0,2540	0,2541
0,00015	0,0001	$-0{,}36\cdot10^{-4}$	$-0{,}13\cdot10^{-3}$	$-0{,}2\cdot10^{-3}$	$-0{,}0003$	$-10^{-3}\cdot0{,}29$
$-0{,}9\cdot10^{-5}$	$-0{,}5\cdot10^{-5}$	$0{,}9\cdot10^{-5}$	$0{,}2\cdot10^{-4}$	$10^{-4}\cdot0{,}2$	$0{,}3\cdot10^{-4}$	$10^{-4}\cdot0{,}31$
$0{,}5\cdot10^{-6}$	$-0{,}4\cdot10^{-7}$	$-0{,}1\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}3\cdot10^{-5}$	$-10^{-6}\cdot0{,}29$
1,0410	1,0277	0,9929	0,97532	0,9658	0,9579	0,9572
0,2516	0,2519	0,2528	0,2533	0,2535	0,2537	0,2538
$0{,}9\cdot10^{-4}$	$0{,}6\cdot10^{-4}$	$-0{,}3\cdot10^{-4}$	$-0{,}9\cdot10^{-4}$	$-0{,}1\cdot10^{-3}$	$-0{,}2\cdot10^{-3}$	$10^{-3}\cdot0{,}19$
$-0{,}5\cdot10^{-5}$	$-0{,}2\cdot10^{-5}$	$0{,}6\cdot10^{-5}$	$0{,}9\cdot10^{-5}$	$0{,}1\cdot10^{-4}$	$0{,}2\cdot10^{-4}$	$10^{-4}\cdot0{,}17$
$0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-7}$	$-0{,}6\cdot10^{-6}$	$-0{,}8\cdot10^{-6}$	$-0{,}1\cdot10^{-5}$	$-0{,}1\cdot10^{-5}$	$-10^{-5}\cdot0{,}13$
1,0322	1,0214	0,9933	0,97937	0,9720	0,9659	0,9653
0,25177	0,2520	0,2528	0,25318	0,2534	0,2535	0,2536
$0{,}6\cdot10^{-4}$	$0{,}5\cdot10^{-4}$	$-0{,}2\cdot10^{-4}$	$-0{,}6\cdot10^{-4}$	$-0{,}9\cdot10^{-4}$	$-0{,}1\cdot10^{-3}$	$-10^{-3}\cdot0{,}13$
$-0{,}3\cdot10^{-5}$	$-0{,}1\cdot10^{-6}$	$0{,}3\cdot10^{-6}$	$0{,}5\cdot10^{-5}$	$0{,}7\cdot10^{-5}$	$0{,}9\cdot10^{-5}$	$10^{-5}\cdot0{,}94$
$0{,}1\cdot10^{-6}$	$-0{,}1\cdot10^{-7}$	$-0{,}26\cdot10^{-8}$	$-0{,}4\cdot10^{-6}$	$-0{,}5\cdot10^{-6}$	$-0{,}6\cdot10^{-6}$	$-10^{-6}\cdot0{,}61$
1,0259	1,0169	0,9938	0,982	0,9766	0,9718	0,9714
0,25192	0,2521	0,2528	0,2531	0,2532	0,2534	0,2534
$0{,}4\cdot10^{-4}$	$0{,}3\cdot10^{-4}$	$-0{,}16\cdot10^{-4}$	$-0{,}4\cdot10^{-4}$	$-0{,}6\cdot10^{-4}$	$-0{,}8\cdot10^{-4}$	$-10^{-4}\cdot0{,}89$
$-0{,}2\cdot10^{-5}$	$-0{,}8\cdot10^{-6}$	$0{,}2\cdot10^{-6}$	$0{,}3\cdot10^{-5}$	$0{,}4\cdot10^{-5}$	$0{,}5\cdot10^{-5}$	$10^{-5}\cdot0{,}56$
$0{,}5\cdot10^{-7}$	$-0{,}6\cdot10^{-8}$	$-0{,}1\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-0{,}3\cdot10^{-6}$	$-10^{-6}\cdot0{,}31$
1,02135	1,0152	0,99451	0,98517	0,9803	0,9763	0,9760
0,25204	0,2522	0,2528	0,25303	0,2531	0,2533	0,2533
$0{,}3\cdot10^{-4}$	$0{,}1\cdot10^{-4}$	$-0{,}1\cdot10^{-4}$	$-0{,}3\cdot10^{-4}$	$-0{,}5\cdot10^{-4}$	$-0{,}6\cdot10^{-4}$	$-10^{-4}\cdot0{,}61$
$-0{,}98\cdot10^{-6}$	$0{,}4\cdot10^{-6}$	$0{,}1\cdot10^{-5}$	$0{,}2\cdot10^{-5}$	$0{,}3\cdot10^{-5}$	$0{,}3\cdot10^{-6}$	$-10^{-5}\cdot0{,}35$
$0{,}3\cdot10^{-7}$	$-0{,}2\cdot10^{-8}$	$-0{,}7\cdot10^{-7}$	$-0{,}1\cdot10^{-6}$	$-0{,}1\cdot10^{-6}$	$-0{,}2\cdot10^{-6}$	$-10^{-6}\cdot0{,}16$
1,0179	1,0114	0,9951	0,98724	0,9832	0,9799	0,9798

8.12. The Iteration Method of Solving the Integro-Differential Equation for the Submerged Hydrofoil* [323

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*This paragraph was written together with V. A. Stepanov.

Equation (VIII.125) can be solved also by the iteration method.

Let us write equation (VIII.125) as follows:

$$\Gamma(\bar{y}) = \frac{a_{\bar{h}}}{2\lambda(\bar{y})}\left[\alpha(\bar{y}) - \frac{1}{2\pi}\int_{-1}^{+1}\Gamma'(\bar{\eta})\frac{1}{\bar{y}-\bar{\eta}}d\bar{\eta} + \right.$$

$$\left. + \frac{1}{2\pi}\int_{-1}^{+1}\Gamma(\bar{\eta})G'(\bar{y}-\bar{\eta})d\bar{\eta}\right]. \quad \text{(VIII.184)}$$

Designating

$$\frac{a_{\bar{h}}}{2\lambda(\bar{y})}\left[\alpha(\bar{y}) + \frac{1}{2\pi}\int_{-1}^{+1}\Gamma(\bar{\eta})G'(\bar{y}-\bar{\eta})d\bar{\eta}\right] = \Gamma_{\bar{h}\infty}, \quad \text{(VIII.185)}$$

then

$$\Gamma(\bar{y}) = \Gamma_{\bar{h}\infty} - \frac{a_{\bar{h}}}{4\lambda(\bar{y})\pi}\int_{-1}\frac{\Gamma'(\bar{\eta})}{\bar{y}-\bar{\eta}}d\bar{\eta},$$

$$G'(\bar{y}-\bar{\eta}) = \mathrm{Re}\left\{2\omega^2\int_0^\infty\frac{(v-1)^2}{v}\sqrt{\frac{v}{v+1}}e^{-\omega i(v+1)\left[\sqrt{\frac{v}{v+1}}(\bar{y}-\bar{\eta})-4\bar{H}i\right]}\times\right.$$

$$\left.\times dv + \frac{1}{[(\bar{y}-\bar{\eta})-4\bar{H}i]^2}\right\}. \quad \text{(VIII.186)}$$

Using the relationships in (VIII.185) and (VIII.186) [350
we may construct the iteration process as follows:

$$\Gamma_0(\bar{y}) = \frac{a_{\bar{h}}}{2\lambda(\bar{y})}\alpha(\bar{y}), \quad \Gamma_1(\bar{y}) = \frac{a_{\bar{h}}}{2\lambda(\bar{y})}\alpha(\bar{y}) -$$

$$-\frac{a_{\bar{h}}}{4\lambda(\bar{y})\pi}\int_{-1}^{+1}\frac{\Gamma_1'(\bar{\eta})}{\bar{y}-\bar{\eta}}d\bar{\eta},$$

$$\left.\begin{aligned}
\Gamma_{\bar{h}2\infty} &= \frac{a_{\bar{h}}}{2\lambda(\bar{y})}\left[\alpha(\bar{y}) + \frac{1}{2\pi}\int_{-1}^{+1}\Gamma_1(\bar{\eta})\,G'(\bar{y}-\bar{\eta})\,d\bar{\eta}\right] \\
\Gamma_2(\bar{y}) &= \Gamma_{\bar{h}2\infty} - \frac{a_{\bar{h}}}{4\lambda(y)\pi}\int_{-1}^{+1}\frac{\Gamma_2'(\bar{\eta})}{\bar{y}-\bar{\eta}}\,d\bar{\eta} \\
\Gamma_n(\bar{y}) &= \Gamma_{\bar{h}n\infty} - \frac{a_{\bar{h}}}{4\lambda(y)\pi}\int_{-1}^{+1}\frac{\Gamma_n'(\bar{\eta})}{\bar{y}-\bar{\eta}}\,d\bar{\eta} \\
\Gamma_{\bar{h}n\infty} &= \frac{a_{\bar{h}}}{2\lambda(\bar{y})}\left\{\alpha(\bar{y}) + \frac{1}{2\pi}\int_{-1}^{+1}\Gamma_{n-1}(\bar{\eta})\,G'(\bar{y}-\bar{\eta})\,d\bar{\eta}.\right.
\end{aligned}\right\} \qquad (\text{VIII.187})$$

Let us use this method to solve the problem of foil motion with the elliptical distribution of circulation along the span.

$$\left.\begin{aligned}
\Gamma_1(\bar{y}) &= \frac{A_1\pi\psi}{\lambda}\sqrt{1-\bar{y}^2} \\
\Gamma_{\bar{h}2\infty}(\bar{y}) &= \frac{B_2\pi\psi}{\lambda}\sqrt{1-\bar{y}^2} \\
\Gamma_2(\bar{y}) &= \frac{A_2\pi\psi}{\lambda}\sqrt{1-\bar{y}^2} \\
\Gamma_{\bar{h}3\infty}(\bar{y}) &= \frac{B_3\pi\psi}{\lambda}\sqrt{1-\bar{y}^2} \\
&\ldots\ldots\ldots\ldots \\
\Gamma_n(\bar{y}) &= \frac{A_n\pi\psi}{\lambda}\sqrt{1-\bar{y}^2} \\
\Gamma_{\bar{h}n\infty}(\bar{y}) &= \frac{B_n\pi\psi}{\lambda}\sqrt{1-\bar{y}^2}.
\end{aligned}\right\} \qquad (\text{VIII.188})$$

where [351

$$\left.\begin{aligned}
&B_1 = \alpha,\quad A_1 = \frac{4}{\pi\left(1+\frac{2\psi}{\lambda}\right)}B_1 \\
&B_2 = \left(\alpha + \frac{A_1\psi}{2\lambda}b\right),\quad A_2 = \frac{4}{\pi\left(1+\frac{2\psi}{\lambda}\right)}B_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&B_n = \left(\alpha + \frac{A_{n-1}\psi}{2\lambda}b\right),\quad A_n = \frac{4}{\pi\left(1+\frac{2\psi}{\lambda}\right)}B_n
\end{aligned}\right|$$

Hence we obtain the following relationship:

$$B_n = a + \frac{2\psi}{\lambda\left(1 + \frac{2\psi}{\lambda}\right)} b B_{n-1}.$$

With $h \to \infty$, $B_n \to B_{n-1}$ and

$$B = a + \frac{2\psi}{\lambda\left(1 + \frac{2\psi}{\lambda}\right)} bB, \qquad \text{(VIII.189)}$$

or

$$B = \frac{a}{1 - \frac{2\psi}{\lambda\left(1 + \frac{2\psi}{\lambda}\right)\pi} b}, \qquad \text{(VIII.190)}$$

where

$$b = \frac{2}{\pi^2} \int_{-1}^{+1} \sqrt{1 - \bar{y}^2} \int_{-1}^{+1} \sqrt{1 - \bar{\eta}^2}\, G'(\bar{y} - \bar{\eta})\, d\eta d\bar{y}.$$

From expression (VIII.190) it is easy to calculate the coefficients of lifting force and drag for the submerged hydrofoil. We obtain

$$\bar{\gamma} = \frac{\Gamma_{\bar{h}}}{\Gamma_\infty} = \frac{P_{\bar{h}}}{P_\infty} = \frac{1}{1 - \frac{2\psi}{\lambda\left(1 + \frac{2\psi}{\lambda}\right)\pi} b}.$$

The coefficients c_y and c_x will again be defined by expressions (VIII.100) and (VIII.101), in which function ζ is in the form

$$\zeta = 1 - \frac{l}{\pi^2} \int_{-1}^{+1} \sqrt{1 - \bar{y}^2} \int_{-1}^{+1} \sqrt{1 - \bar{\eta}^2}\, G'(\bar{y} - \bar{\eta})\, d\bar{\eta} d\bar{y}. \qquad \text{(VIII.191)}$$

Utilizing the expression (VIII.33) in the formula (VIII.191) we obtain the formula (VIII.40). [352

T. Nishiyama [213, 214], instead of the singular integro-differential equation (VIII.125), suggests the following approximate integral equation:

$$\Gamma(\bar{y}) = \Gamma_\infty(\bar{y}) + \frac{a_h}{4\lambda(\bar{y})\pi} \int_{-1}^{+1} \Gamma(\bar{\eta})\, G'(\bar{y} - \bar{\eta})\, d\bar{\eta}, \qquad \text{(VIII.192)}$$

where

$$\Gamma_\infty(y) = \Gamma_1(\bar{y}).$$

Evidently this equation corresponds to a certain degree to the second approximation of the iteration process (VIII.187).

Using Nistrom's method [43] one may transform the equation (VIII.192) into a system of algebraic equations

$$\Gamma(\bar{y}_i) = -\frac{a_h}{8\pi\lambda(\bar{y})}\sum^{n} a_i^n \Gamma(\eta_i)\,\bar{k}(y_i - \eta_i) + \Gamma_\infty(\bar{y}_i), \qquad \text{(VIII.193)}$$

where

$$K(\bar{y}) = -\frac{4\bar{h}^2\lambda_0 - \frac{1}{4}\lambda_0^4\bar{y}^2}{\left(4\bar{h}^2 + \frac{1}{4}\lambda_0^2\bar{y}^2\right)^2} + 2\omega_n\left(\frac{\lambda_0}{h}\right)^2 e^{-\omega_n}\times$$

$$\times\left\{\left(1 + \frac{4h^2}{4\bar{h}^2 + \frac{1}{4}\lambda_0^2\bar{y}^2}\right)\bar{K}_0\left(\frac{\omega_h}{2\bar{h}}\sqrt{4\bar{h}^2 + \frac{1}{4}\lambda_0^2\bar{y}^2}\right) + \right.$$

$$+ 2\left[\frac{2\bar{h}}{\sqrt{4\bar{h}^2 + \frac{1}{4}\lambda_0^2\bar{y}^2}} - \frac{1}{\omega_4}\,\frac{\bar{h}}{\sqrt{4h^2 + \frac{1}{4}\lambda_0^2\bar{y}^2}} + \right.$$

$$\left.\left. + \frac{8h^3}{\omega_h\left(4\bar{h}^2 + \frac{1}{4}\lambda_0^2\bar{y}^2\right)^{3/2}}\right]\bar{K}_1\left(\frac{\omega_h}{2\bar{h}}\sqrt{4h^2 + \frac{1}{4}\lambda_0^2\bar{y}^2}\right)\right\},$$

$$\bar{h} = \frac{h}{2a}, \quad \omega_n = \nu h$$

$K_i(x)$ is the Bessel function of the second kind.

Instead of the coefficient a_h, T. Nishiyama erroneously uses the coefficient a_∞ in the system (VIII.193).

The formula for the drag of the submerged hydrofoil is transformed by Nishiyama to the following form:

$$Q = -\frac{\varrho}{4\pi}\lim_{\xi\to 0}\int_0^\infty (P^2 + N^2)\lambda e^{-\xi\lambda}d\lambda - \frac{\varrho}{4\pi}\int_0^\infty (P^2 + N^2)\lambda e^{-2\lambda h}d\lambda +$$ [353

$$+ \frac{\nu^2\varrho}{\pi}\int_0^{\frac{\pi}{2}} (P_0 + N_0^2)\sec^5\theta\, e^{-2\nu h\sec^2\theta}d\theta,$$

$$p + iN = \int_{-b}^{+b}\Gamma(\eta)e^{i\lambda\eta}d\eta, \quad P_0 + iN_0 = \int_{-b}^{+b}\Gamma(\eta)e^{i\nu\eta\sec^2\theta\sin\theta}d\eta. \qquad \text{(VIII.194)}$$

If we designate the circulation $\Gamma(y)$ by the series (VIII.192), then the formula (VIII.194) transforms into the following form:

$$
\begin{aligned}
Q = {} & \frac{\pi\varrho}{4} \lim_{\delta\to 0} \int_{\delta}^{\infty} \Big\{ A_0 J_1(t) + A_2 \Big[J_1(t) - 3\frac{J_2(t)}{t} \Big] + \\
& + A_4 \Big[J_1(t) - 6\frac{J_2(t)}{t} + 15\frac{J_3(t)}{t} \Big] + \ldots + \Big\}^2 e^{-\delta t} \frac{dt}{t} - \\
& - \frac{\pi\varrho}{4} \int_0^{\infty} \Big\{ A_0 J_1(t) - A_2 \Big[J_1(t) - 3\frac{J_2(t)}{t} \Big] + A_4 \Big[J_1(t) - 6\frac{J_2(t)}{t} + \\
& + 15\frac{J_3(t)}{t} \Big] + \ldots + \Big\} e^{-4\bar{H}t} \frac{dt}{t} + \pi\varrho \int_0^{\infty} \Big\{ A_0 J_1\Big(\frac{\omega}{2} t\sqrt{t^2+1}\Big) + \\
& + A_2 \Bigg[J_1\Big(\frac{\omega}{2} t\sqrt{t^2+1}\Big) - 3\frac{J_2\Big(\frac{\omega}{2} t\sqrt{t^2+1}\Big)}{\frac{\omega}{2} t\sqrt{t^2+1}} \Bigg] + \ldots \\
& \ldots + \Big\}^2 \frac{\sqrt{t^2+1}}{t^2} e^{-2\omega\bar{h}(1+t^2)} dt,
\end{aligned}
\qquad \text{(VIII.195)}
$$

where $\omega = \nu b$;

$J_i(x)$ - the Bessel function of the first kind.

8.13. The Direct Method of Solving Integro-Differential Equations for the Submerged Hydrofoil*

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*This section was written by S. V. Koval'chuk. Calculations were performed on the Minsk-1 computer by I. P. Tkachenko.

A solution of the regular equation for the submerged hydrofoil by replacing the nucleus of the equation with a singular nucleus was given above. For practical calculations, this solution is somewhat cumbersome; therefore, the development of methods for solving singular integro-differential equations is of interest. In aerodynamics, the most widely used are the direct methods of solving singular equations, when the class of functions in which the solution is determined is known (in our case the functions which satisfy the condition of $\Gamma(-B) = \Gamma(B) = 0$). [354

These methods may be extended to include the case of

the integro-differential equation for the submerged hydrofoil. A rather comprehensive treatment of a number of methods is given in the well known texts on aerodynamics by V. V. Golubev [18] and Karafoli [39]. A convenient method for solving the Prandtl equation is given by Multhopp [197]. This method is being widely used in aerodynamics and recently in the theory of elasticity [37]. In the study by A. I. Kalandiya the singular integral equation is approximately solved using this method.

Let us solve the equation (VIII.15) by the Multhopp method, which uses a simple and convenient for practical applications procedure in obtaining the numerical solution. Multhopp operates with the Langrangian trigonometric interpolation polynomial in the search for the unknown function. To construct the polynomial one selects a positive integer n and takes the roots of the Chebyshev polynomial of the second order for the points of interpolation at (-1, +1) as follows:

$$x_k = \cos\theta_k, \ \theta_k = \frac{k\pi}{n+1} \quad k = 1, 2, 3 \ldots n.$$

It follows, then, that the odd trigonometric polynomial of the n-th order which at points θ_k coincides with $\Gamma(\theta)$ can be written as follows:

$$L(\Gamma, \bar{y}) = \sum_{k=1}^{n} (-1)^{k+1} \Gamma(\bar{y}_k) \frac{\sin(n+1)\theta \sin\theta_k}{\cos\theta - \cos\theta_k}, \qquad \text{(VIII.196)}$$

or, in a slightly differently form,

$$L(\Gamma, \bar{y}) = \frac{2}{n+1} \sum_{k=1}^{n} \Gamma(\bar{y}_k) \sum_{m=1}^{n} \sin m\theta_k \sin m\theta. \qquad \text{(VIII.197)}$$

Let us substitute into the equation (VIII.15) the polynomial (VIII.197). The expression obtained can be satisfied at every point of interpolation by a certain value of circulation unknown previously. By succesively assigning the values of θ_1, θ_2, θ_3... to the argument θ, we will obtain a system of linear algebraic equations for finding the approximate values of the unknown function. Integration of the singular part, in this case, is usually followed to the end; we cannot do the same with regard to the regular part of the nucleus because of the complexity, while the existing formulas of numerical integration cannot

give sufficiently accurate results. Therefore, for a more convenient use of the Multhopp's method, let us divide the equation (VIII.15) into a series of recurrent equations of singular type with the aid of the parameter τ as follows: [355

$$\tau = \sqrt{4\bar{H}^2 + 1} - 2\bar{H}, \quad \bar{H} = \frac{h}{l},$$

where h is the depth of submergence of the hydrofoil; l is the half-span.

The solution (in the form of a series) with respect to the parameter τ is given in the beginning of this chapter.

Since the characteristics of the profile composition for a foil of infinite span, which are not defined by the τ-parameter, are included in the equation, it is necessary to generalize that solution by introducing a parameter τ'. We will attempt to find a solution of equation (VIII.15) in the form of a double series:

$$\Gamma(\bar{y}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tau^{2n}\tau'^{2m}\Gamma_{mn}(\bar{y}). \qquad \text{(VIII.198)}$$

Then

$$\Gamma'(\bar{y}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \tau^{2n}\tau'^{2m}\Gamma'_{nm}(\bar{y}), \quad \tau' = \sqrt{4\bar{h}^2 + 1} - 2\bar{h}, \quad \bar{h} = \frac{h}{B}.$$

Let us write the expansion of the regular part of the nucleus as follows:

$$G(\bar{y} - \eta) \sum_{n=1}^{\infty} = G_n(\bar{y})\tau^{2n},$$

and, representing $a_{\bar{h}}$ and $\alpha(\bar{y})$ as

$$a_{\bar{h}} = \sum_{m=0}^{\infty} a_n \tau'^{2m}, \quad \alpha(\bar{y}) = \sum_{m=0}^{\infty} \alpha_n(\bar{y})\tau'^{2m},$$

we can substitute all the expanded values into the equation (VIII.15). Separating the terms with equal $\tau^{2n}\tau'^{2m}$, we obtain for the solution three types of singular equations as follows:

$$\text{I.} \quad \Gamma_{00} = \frac{a_0}{2\lambda(\bar{y})}\left(\alpha_0 + \frac{1}{2\pi}\int_{-1}^{+1} \frac{\Gamma'_{00}}{\bar{y} - \eta}\, d\eta\right)$$

$$\left.\begin{aligned}
&\Gamma_{10} = \frac{a_0}{2\lambda(\bar{y})}\left(-\frac{1}{2\pi}\int_{-1}^{+1}\Gamma'_{00}G_1 d\bar{\eta} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'_{10}}{\bar{y}-\bar{\eta}}\,d\bar{\eta}\right)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Gamma_{n0} = \frac{a_0}{2\lambda(\bar{y})}\left(-\frac{1}{2\pi}\int_{-1}^{+1}\sum_{i=1}^{n}\Gamma'_{(n-i)0}(\bar{\eta})\,G_i d\bar{\eta} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'_{n0}(\bar{\eta})}{\bar{y}-\eta}\,d\eta\right)
\end{aligned}\right\}. \qquad \text{(VIII.199)}$$

[356

$$\left.\begin{aligned}
&\text{II. } \Gamma_{01} = \frac{a_0}{2\lambda(\bar{y})}\left(\alpha_1 + \frac{a_1 2\lambda(\bar{y})}{a_0^2}\Gamma_{00} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'_{01}}{\bar{y}-\bar{\eta}}\,d\bar{\eta}\right)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Gamma_{0n} = \frac{a_0}{2\lambda(\bar{y})}\left(\alpha_n + f(a_{\bar{n}},\ \lambda(\bar{y}),\ \Gamma_{0,n-1}\ldots) - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'_{0n}}{\bar{y}-\bar{\eta}}\,d\bar{\eta}\right)
\end{aligned}\right\}. \qquad \text{(VIII.200)}$$

$$\left.\begin{aligned}
&\text{III. } \Gamma_{11} = \frac{a_0}{2\lambda(\bar{y})}\left(-\frac{1}{2\pi}\int_{-1}^{+1}\Gamma'_{01}G_1 d\bar{\eta} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'_{10}\frac{a_1}{a_0}}{\bar{y}-\bar{\eta}}\,d\bar{\eta} - \right.\\
&\qquad\left. - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'_{11}}{\bar{y}-\bar{\eta}}\,d\bar{\eta}\right)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\Gamma_{mn} = \frac{a_0}{2\lambda(\bar{y})}\left\{-\frac{1}{2\pi}\int_{-1}^{+1}\sum_{i=1}^{m}\Gamma'_{(m-i)n}G_i d\bar{\eta} - \right.\\
&\left. - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\sum_{i=1}^{n}\Gamma'_{m,n-i}\frac{a_i}{a_0}}{\bar{y}-\bar{\eta}}\,d\bar{\eta} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'_{nm}(\bar{\eta})}{\bar{y}-\eta}\,d\bar{\eta}\right\}
\end{aligned}\right\}. \qquad \text{(VIII.201)}$$

Obviously, all these equations are represented in the general form as

$$\frac{\Gamma(\bar{y})}{B(y)} + \frac{1}{2\pi}\int_{-1}^{+1}\frac{\Gamma'(\eta)}{y-\eta}\,d\bar{\eta} = f(\bar{y}). \qquad \text{(VIII.202)}$$

Let us obtain the formula of the mechanical quadrature for the particular integral equation (VIII.202). Performing substitution $y = \cos\theta_k$, $\eta = \cos\theta$, we can write this equation as follows: [357

$$\frac{\Gamma(\theta_k)}{B(\theta_k)} - \frac{1}{2\pi}\int_0^\pi \frac{\Gamma'(\theta)\,d\theta}{\cos\theta - \cos\theta_k} = f(\theta_k). \qquad \text{(VIII.203)}$$

Taking into account the inequality

$$\frac{1}{\pi}\int_0^\pi \frac{\cos m\theta\, d\theta}{\cos\theta - \cos\theta_k} = \frac{\sin m\theta_k}{\sin\theta_k},$$

we obtain the required mechanical quadrature

$$\frac{1}{2\pi}\int_0^\pi \frac{\Gamma'(\theta)}{\cos\theta_k - \cos\theta}\,d\theta \approx \frac{1}{2\pi}\int_0^\pi \frac{dL(\Gamma, \bar{\eta})}{d\theta}\cdot\frac{d\theta}{\cos\theta_k - \cos\theta} =$$

$$= -\frac{1}{n+1}\sum_{k=1}^{n}\Gamma(y_k)\sum_{m=1}^{n}\frac{m\sin m\theta_k \sin m\theta}{\sin\theta}. \qquad \text{(VIII.204)}$$

Substituting the integral term in the expression (VIII.203) by the expression (VIII.204) and following the Multhopp's method, we obtain a system of linear equations for determining the approximate values of the desired function at points x_k (designated through Γ_k):

$$b_\nu\Gamma_\nu = f_\nu + \sum_{k-1}^{n} b_{\nu k}\Gamma_k, \quad \nu = 1, 2\ldots n, \qquad \text{(VIII.205)}$$

where

$$b_\nu = \frac{1}{B_\nu} + b_{\nu\nu};\quad b_{\nu\nu} = \frac{n+1}{4\sin\theta_k};$$

$$b_{\nu k} = \begin{cases} \dfrac{1}{4(n+1)\sin\theta_\nu}\left(\dfrac{1}{\mathrm{tg}^2\dfrac{\theta_k - \theta_\nu}{2}} - \dfrac{1}{\mathrm{tg}^2\dfrac{\theta_k + \theta_\nu}{2}}\right) & \text{with } |\nu - k| = 1, 3, \ldots \\ 0 & \text{with } |\nu - k| = 2, 4\ldots \end{cases}$$

The coefficients $b_{\nu k}$ are distinguished by an interesting peculiarity. With a symmetrical distribution of circulation, the coefficients with odd ν can be expressed through

coefficients with even ν and vice versa. The values $b_{\nu\nu}$ are given by Multhopp in [197].

If we substitute the solution Γ_k of the system (VIII. 205) in the expression (VIII.196) we will obtain the following function: [358

$$\Gamma_n(\bar{y}) = \sum_{k=1}^{n} (-1)^{k+1} \Gamma_k \frac{\sin(n+1)\theta \sin\theta_k}{\cos\theta - \cos\theta_k}, \qquad \text{(VIII.206)}$$

which should serve as an approximate solution of equation (VIII.203).

The agreement between the solutions using this method has been proven by A. I. Kalandiya [37]. In general, this question reduces to the evaluation of the rate

$$\|\overline{H}_0\| = \max \sum_{k=1}^{n} \frac{b_{\nu k}}{b_\nu}.$$

For positive λ, as in our case,

$$\|\overline{H}_0\| < \frac{1}{(n+1)^2} \max \sum_{k=1}^{n} \frac{4}{(\theta_k - \theta_\nu)^2} = 1,$$

which proves the agreement.

For the symmetrical distribution of circulation along the span of the foil and with the number of interpolating points equal to 7, we obtain, for evaluating coefficients, an algebraic system of equations of the second order, which is solved in the general form as follows:

$$\left.\begin{aligned}
\Gamma^1 = \Gamma^7 &= \frac{f'}{a_1} + \frac{k_5(1{,}9142k_4 - 0{,}1464k_2) + k_6(0{,}1464k_1 - 1{,}9142k_3)}{a_1(k_1k_4 - k_2k_3)};\\
\Gamma^2 = \Gamma^6 &= \frac{k_5k_4 - k_6k_3}{k_1k_4 - k_2k_3};\\
\Gamma^3 = \Gamma^5 &= \frac{f^3}{a_3} + \frac{k_5(0{,}9142k_4 - 0{,}8536k_2) + k_6(0{,}8536k_1 - 0{,}9142k_3)}{a_3(k_1k_4 - k_2k_3)}\\
\Gamma^4 &= \frac{k_1k_6 - k_5k_2}{k_1k_4 - k_2k_3}
\end{aligned}\right\}, \text{(VIII.204) [sic]}$$

where

$$k_1 = a_2\frac{1{,}9831}{a_1} - \frac{1{,}092}{a_3}, \quad a_1 = \frac{2\lambda_1}{a_0} + \frac{2}{\sin\tau_1} = \frac{2\lambda_1}{a_0} + 5{,}226,$$

$$k_2 = -\frac{0,2144}{a_1} - \frac{1,4421}{a_3}, \quad a_2 = \frac{2\lambda_2}{a_0} + \frac{2}{\sin\tau_2} = \frac{2\lambda_2}{a_0} + 2,828,$$

$$k_3 = -\frac{0,1517}{a_1} - \frac{1,0196}{a_3}, \quad a_3 = \frac{2\lambda_3}{a_0} + \frac{2}{\sin\tau_3} = \frac{2\lambda_3}{a_0} + 2,165,$$

$$k_4 = \alpha_4 - \frac{0,0164}{a_1} - \frac{1,3465}{a_3},$$ [359

$$k_5 = f^2 + \frac{1,036f'}{a_1} + \frac{1,1945f^2}{a_3}, \quad a_4 = \frac{2\lambda_4}{a_0} + \frac{2}{\sin\tau_4} = \frac{2\lambda_4}{a_0} + 2.$$

$$k_6 = f^4 + \frac{0,112f'}{a_1} + \frac{1,5774f^2}{a_3}$$

The values in the formulas represent the trigonometric expressions constructed from the Multhopp's coefficients. They can be calculated to any degree of accuracy. The shape of the foil is defined by the relative span λ_n, and the effective angle of attack, which in addition accounts for the variation in the zero lifting force angle under the free surface, is described by the expression f(y).

In order to obtain the numerical results let us transform the functions $f(\bar{y})$ into a more convenient form. The system of singular equations (VIII.99) which takes into account the effect of the finiteness of the foil span submerged under the free surface, contains a function $f(\bar{y})$ which is equal to

$$-\frac{1}{2\pi}\int_{-1}^{+1}\sum_{l=1}^{n}\Gamma'_{(n-l)0}(\bar{\eta})\,G_l(\bar{y}-\bar{\eta})\,d\bar{\eta}.$$

Taking into account that the desired function Γ_{nm} assumes zero values at the ends of the segment (-1, +1), we obtain after integrating by parts

$$-\frac{1}{2\pi}\int_{-1}^{+1}\sum_{l=1}^{n}\Gamma'_{(n-l)0}(\bar{\eta})\,G_l\,d\bar{\eta} = \frac{1}{2\pi}\int_{-1}^{+1}\sum_{l=1}^{n}\Gamma_{(n-l)0}(\eta)\,G'_l\,d\bar{\eta}.$$

Taking the transformation for the circulation in the form of the trigonometric series $\sum A_k \sin k\theta$, we find the following integrals:

$$\frac{1}{2\pi}\int_{0}^{\pi}\sin\theta\,\Sigma\,A_k\sin k\theta\,d\theta = \frac{A_1}{4},$$

$$\frac{1}{2\pi}\int_0^\pi \cos^2\theta \sin\theta \Sigma A_k \sin k\theta d\theta = \frac{A_1 + A_3}{16},$$

$$\frac{1}{2\pi}\int_0^\pi \cos^4\theta \sin\theta \Sigma A_k \sin k\theta d\theta = \frac{2A_1 + 3A_3 + A_5}{64},$$

$$\frac{1}{2\pi}\int_0^\pi \cos^6\theta \sin\theta \Sigma A_k \sin k\theta d\theta = \frac{5A_1 + 9A_3 + 5A_5 + A_7}{256},$$

and the equations (VIII.199) can be written as follows: [360

$$\left.\begin{aligned}
&\bar{\Gamma}_{00} = \frac{a_0}{2\lambda(\bar{y})}\left(1 - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{00}}{\bar{y}-\bar{\eta}}d\bar{\eta}\right)\\
&\bar{\Gamma}_{10} = \frac{a_0}{2\lambda(\bar{y})}\left(-\frac{G_{2,0}\bar{A}^1_{00}}{4} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{10}}{\bar{y}-\bar{\eta}}d\bar{\eta}\right)\\
&\bar{\Gamma}_{20} = \frac{a_0}{2\lambda(\bar{y})}\left[-\frac{G_{4,1}\bar{A}^1_{00}}{2} + \frac{3G_{4,0}}{16}(\bar{A}^1_{00} + \bar{A}^3_{00}) - \frac{G_{2,0}\bar{A}^1_{10}}{4} + \right.\\
&\qquad\left. + \cos^2\theta_k \frac{3G_{4,0}}{4}\bar{A}^1_{00} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{20}}{\bar{y}-\bar{\eta}}d\bar{\eta}\right]\\
&\bar{\Gamma}_{30} = \frac{a_0}{2\lambda(\bar{y})}\left\{-\frac{3G_{6,2}}{4}\bar{A}^1_{00} + \frac{3G_{6,1}}{4}(\bar{A}^1_{00} + \bar{A}^3_{00}) - \right.\\
&\quad - \frac{5G_{6,0}}{64}(2\bar{A}^1_{00} + 3\bar{A}^3_{00} + \bar{A}^5_{00}) - \frac{G_{4,1}}{2}\bar{A}^1_{10} + \frac{3G_{4,0}}{16}(\bar{A}^1_{10} + \\
&\quad + \bar{A}^3_{10}) - \frac{G_{20}\bar{A}^1_{20}}{4} + \cos^2\theta_k\left[3G_{6,1}\bar{A}^1_{00} - \frac{30G_{6,0}}{16}(\bar{A}^1_{00} + \right.\\
&\qquad\left. + \bar{A}^3_{00}) + \frac{3G_{4,0}}{4}\bar{A}^1_{10}\right] - \cos^4\theta_k\frac{5G_{6,0}}{4}\bar{A}^1_{00} - \\
&\qquad\left. - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{3,0}}{\bar{y}-\bar{\eta}}d\bar{\eta}\right\}\\
&\bar{\Gamma}_{40} = \frac{a_0}{2\lambda(\bar{y})}\left\{-G_{8,3}\bar{A}^1_{00} + \frac{30}{16}G_{8,2}(\bar{A}^1_{00} + \bar{A}^3_{00}) - \right.\\
&\quad - \frac{30G_{8,1}}{64}(2\bar{A}^1_{00} + 3\bar{A}^3_{00} + \bar{A}^5_{00}) + \frac{7G_{8,0}}{256}(5\bar{A}^1_{00} +
\end{aligned}\right\} \quad \text{(VIII.208)}$$

$$+9\bar{A}_{00}^{3}+5\bar{A}_{00}^{5}+\bar{A}_{00}^{7}\Bigr)-\frac{3G_{6,2}}{4}\bar{A}_{10}^{1}+\frac{3G_{6,1}}{4}\left(\bar{A}_{10}^{1}+\bar{A}_{10}^{3}\right)-$$

$$-\frac{5G_{6,0}}{64}\left(2\bar{A}_{10}^{1}+3\bar{A}_{10}^{3}+\bar{A}_{10}^{5}\right)-\frac{G_{4,1}\bar{A}_{20}^{1}}{4}+$$

$$+\frac{3G_{4,0}}{16}\left(\bar{A}_{20}^{1}+\bar{A}_{20}^{3}\right)-\frac{G_{2,0}\bar{A}_{30}^{1}}{4}+\cos^{2}\theta_{k}\Bigl(\frac{30G_{8,2}}{4}\bar{A}_{00}^{1}-$$ [361

$$-\frac{180G_{8,1}}{16}\left(\bar{A}_{00}^{1}+\bar{A}_{00}^{3}\right)+\frac{105}{64}G_{8,0}\left(2\bar{A}_{00}^{1}+3\bar{A}_{00}^{3}+\bar{A}_{00}^{5}\right)+$$

$$+3G_{6,1}\bar{A}_{10}^{1}-\frac{30G_{6,0}}{16}\left(\bar{A}_{10}^{1}+A_{10}^{3}\right)+\frac{3}{4}G_{4,0}\bar{A}_{20}^{1}\Bigr]-$$

$$-\cos^{4}\theta_{k}\Bigl[\frac{30G_{8,1}}{4}\bar{A}_{00}^{1}-\frac{105}{16}G_{8,0}\left(\bar{A}_{00}^{1}+\bar{A}_{00}^{3}\right)+$$

$$+\frac{5}{4}G_{6,0}\bar{A}_{10}^{1}\Bigr]+\cos^{6}\theta_{k}\frac{7G_{8,0}}{4}\bar{A}_{00}^{1}-\frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}_{40}^{1}}{\bar{y}-\eta}d\bar{\eta}\Bigr\}$$

where

$$\bar{A}_{i}=\frac{A_{i}}{\alpha_{0}+\alpha_{k}},\quad \bar{\Gamma}=\frac{\Gamma}{\alpha_{0}+\alpha_{k}}$$

$$\bar{\bar{\Gamma}}_{01}=\frac{a_{0}}{2\lambda(\bar{y})}\left(\alpha_{1}+\frac{2a_{1}\lambda(\bar{y})}{a_{0}^{2}}\bar{\bar{\Gamma}}_{00}-\frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{01}}{\bar{y}-\eta}d\bar{\eta}\right)$$

$$\bar{\Gamma}_{02}=\frac{a_{0}}{2\lambda(\bar{y})}\Bigl[\alpha_{2}+\frac{2a_{1}\lambda(\bar{y})}{a_{0}^{2}}\bar{\Gamma}_{01}+2\lambda(\bar{y})\Bigl(\frac{a_{2}}{a_{0}^{2}}-$$

$$-\frac{a_{1}^{2}}{a_{0}^{3}}\Bigr)\bar{\bar{\Gamma}}_{00}-\frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{02}}{\bar{y}-\eta}d\bar{\eta}\Bigr]$$

$$\bar{\Gamma}_{03}=\frac{a_{0}}{2\lambda(\bar{y})}\Bigl[\alpha_{3}+\frac{2a_{1}\lambda(\bar{y})}{a_{0}^{2}}\bar{\Gamma}_{02}+2\lambda(\bar{y})\Bigl(\frac{a_{2}}{a_{0}^{2}}-$$

$$-\frac{a_{1}^{2}}{a_{0}^{3}}\Bigr)\bar{\Gamma}_{01}+2\lambda(\bar{y})\left(\frac{a_{3}}{a_{0}^{2}}-\frac{2a_{1}a_{2}}{a_{0}^{3}}+\frac{a_{1}^{3}}{a_{0}^{4}}\right)\bar{\bar{\Gamma}}_{00}-$$

$$-\frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{03}}{\bar{y}-\eta}d\bar{\eta}\Bigr] \quad ; \quad \text{(VIII.209)}$$

$$\bar{\Gamma}_{04}=\frac{a_{0}}{2\lambda(\bar{y})}\Bigl[\alpha_{4}+\frac{2a_{1}\lambda(\bar{y})}{a_{0}^{2}}\bar{\Gamma}_{03}+2\lambda(\bar{y})\Bigl(\frac{a_{2}}{a_{0}^{2}}-$$

$$-\frac{a_{1}^{2}}{a_{0}^{3}}\Bigr)\bar{\Gamma}_{02}+2\lambda(\bar{y})\left(\frac{a_{3}}{a_{0}^{2}}-\frac{2a_{1}a_{2}}{a_{0}^{3}}+\frac{a_{1}^{3}}{a_{0}^{4}}\right)\bar{\Gamma}_{01}+$$

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$$+ 2\lambda(\overline{y})\left(\frac{a_4}{a_0^2} - \frac{2(a_3 a_1 + a_2^2)}{a_0^3} + \frac{3a_2 a_1^2}{a_0^4} - \frac{a_1^4}{a_0^5}\right) -$$

$$- \frac{1}{2\pi}\int_{-1}^{+1}\frac{\overline{\Gamma}'_{04}}{\overline{y}-\overline{\eta}}\,d\overline{\eta}\Bigg]$$

where $\overline{\Gamma}_{0l} = \dfrac{\Gamma_{0l}}{a_0}$; $\overline{\overline{\Gamma}}_{00} = \overline{\Gamma}_{00}\left(\dfrac{a_0 + a_{\kappa}}{a_0}\right)$

$$\overline{\Gamma}_{11} = \frac{a_0}{2\lambda(\overline{y})}\left\{\frac{2a_1\overline{\overline{\Gamma}}_{10}\lambda(\overline{y})}{a_0^2} - \frac{\overline{A}^1_{01}G_{2,0}}{4} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\overline{\Gamma}'_{11}}{\overline{y}-\overline{\eta}}\,d\overline{\eta}\right\}$$

$$\overline{\Gamma}_{21} = \frac{a_0}{2\lambda(\overline{y})}\left\{\frac{2a_1\overline{\overline{\Gamma}}_{20}\lambda(\overline{y})}{a_0^2} - \frac{G_{4,1}\overline{A}^1_{01}}{2} + \frac{3G_{4,0}}{16}(\overline{A}^1_{01} + \right.$$

$$\left. + \overline{A}^3_{03}) - \frac{G_{2,0}\overline{A}^1_{11}}{4} + \cos^2\theta_k\frac{3G_{4,0}}{4}\overline{A}^1_{01} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\overline{\Gamma}'_{21}}{\overline{y}-\overline{\eta}}\,d\overline{\eta}\right\}$$

$$\overline{\Gamma}_{31} = \frac{a_0}{2\lambda(\overline{y})}\left\{\frac{2a_1\overline{\overline{\Gamma}}_{30}\lambda(\overline{y})}{a_0^2} - \frac{3G_{6,2}}{4}\overline{A}^1_{01} + \frac{3G_{6,1}}{4}(\overline{A}^1_{01} + \right.$$

$$+ \overline{A}^3_{01}) - \frac{5G_{6,0}}{64}(2\overline{A}^1_{01} + 3\overline{A}^3_{01} + \overline{A}^5_{01}) - \frac{G_{4,1}\overline{A}^1_{11}}{2} +$$

$$+ \frac{3G_{4,0}}{16}(\overline{A}^1_{11} + \overline{A}^3_{11}) - \frac{G_{2,0}}{4}\overline{A}^1_{21} + \cos^2\theta_k\left[3G_{6,1}\overline{A}^1_{01} + \right.$$

$$\left. + \frac{30G_{6,0}}{16}(\overline{A}^1_{01} + \overline{A}^3_{01}) + \frac{3G_{4,0}}{4}\overline{A}^1_{11}\right] -$$

$$\left. - \cos^4\theta_k\frac{5G_{6,0}}{4}\overline{A}^1_{01} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\overline{\Gamma}'_{31}}{\overline{y}-\overline{\eta}}\,d\overline{\eta}\right\} \qquad \text{(VIII.210)}$$

$$\overline{\Gamma}_{12} = \frac{a_0}{2\lambda(\overline{y})}\left[2\overline{\overline{\Gamma}}_{10}\lambda(\overline{y})\left(\frac{a_2}{a_0^2} - \frac{a_1^2}{a_0^3}\right) + \frac{2\overline{\Gamma}_{11}a_1\lambda(\overline{y})}{a_0^2} - \right.$$

$$\left. - \frac{\overline{A}^1_{02}G_{2,0}}{4} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{\overline{\Gamma}'_{12}}{\overline{y}-\overline{\eta}}\,d\overline{\eta}\right]$$

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$$\overline{\Gamma}_{22} = \frac{a_0}{2\lambda(\overline{y})}\left[2\overline{\overline{\Gamma}}_{20}\lambda(\overline{y})\left(\frac{a_2}{a_0^2} - \frac{a_1^2}{a_0^3}\right) + \cdots_{21}\lambda(\overline{y})\frac{a_1}{a_0^2} - \right.$$

$$- \frac{G_{4,1}\overline{A}^1_{02}}{2} + \frac{3G_{4,0}}{16}(\overline{A}^1_{02} + \overline{A}^3_{02}) - \frac{G_{2,0}\overline{A}^1_{12}}{4} +$$

$$+\cos^2\theta\,\frac{3G_{4,0}}{4}\bar{A}^1_{02}-\frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{22}}{\bar{y}-\bar{\eta}}\,d\bar{\eta}\Big]$$

$$\bar{\Gamma}_{13}=\frac{a_0}{2\lambda(\bar{y})}\Big[+\frac{\bar{A}^1_{03}G_{2,0}}{4}+2\bar{\Gamma}_{10}\lambda(\bar{y})\left(\frac{a_3}{a_0^2}-\frac{2a_1a_2}{a_0^3}+\frac{a_1^3}{a_0^4}\right)+$$

$$+2\bar{\Gamma}_{11}\lambda(\bar{y})\left(\frac{a_2}{a_0^2}-\frac{a_1^2}{a_0^3}\right)+2\bar{\Gamma}_{12}\lambda(\bar{y})\frac{a_1}{a_0^2}-$$

$$-\frac{1}{2\pi}\int_{-1}^{+1}\frac{\bar{\Gamma}'_{13}}{\bar{y}-\bar{\eta}}\,d\bar{\eta}\Big]$$

If the functions a_h and $(\alpha_0 + \alpha_k - \Delta\alpha_0)$ are assigned numerical values, then even the first system of singular equations can produce the solution to the problem.

Once we obtain the distribution of circulation along the submerged hydrofoil span in the form of a power series with respect to parameter τ, we can determine the coefficients of the lifting force C_y and the coefficients of the induced drag c_{xi} as follows:

$$C_y=\lambda\int_{-1}^{+1}\Gamma(\bar{y})\,d\bar{y}=\frac{\pi\lambda}{m+1}\left(\Gamma_{\frac{m+1}{2}}+2\sum_{n=1}^{\frac{m-1}{2}}\Gamma_n\sin\theta_n\right) \quad \text{(VIII.211)}$$

$$C_{x_i}=\lambda\int_{-1}^{+1}\alpha_i\bar{\Gamma}(y)\,d\bar{y}. \quad \text{(VIII.212)}$$

By determining these coefficients using formulas (VIII.109) and (VIII.110), the functions ξ_1 and ξ_2 that describe the effect of finiteness of the span under the free surface will be as follows:

$$\zeta_1=\zeta\left(\frac{h}{\lambda}\right)(1+\tau_1)\doteq\frac{2\alpha_{\text{геом}}}{A_0+A_1\tau^2+A_2\tau^4+A_3\tau^6+\ldots}-\frac{\pi\lambda}{a_0}, \quad \text{(VIII.213)}$$

$$\zeta_2=\xi\left(\frac{h}{\lambda}\right)(1+\delta)=\frac{\pi\lambda^3}{C_y^2}\int_{-1}^{+1}\alpha_i\Gamma(\bar{y})\,d\bar{y}. \quad \text{(VIII.214)}$$

Let us consider some special shapes of foils. [365

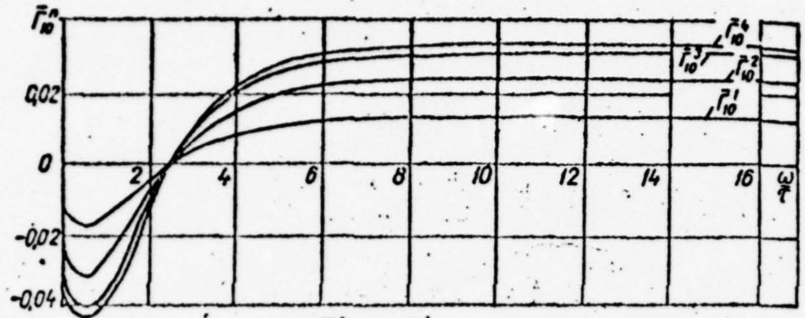

Fig. 24

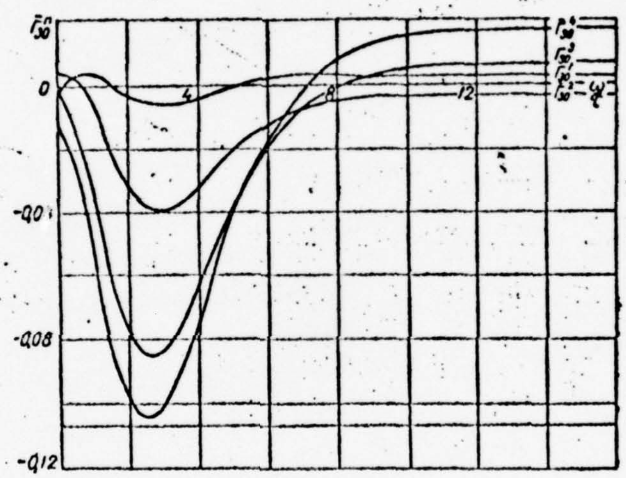

Fig. 25

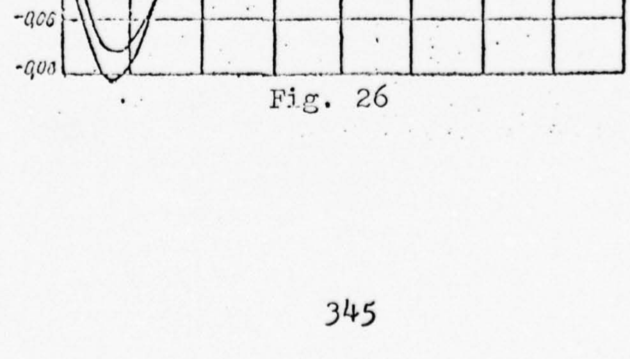

Fig. 26

1. The elliptical foil: [365

$$\frac{1}{\lambda(\bar{y})} = \frac{1}{\lambda_0}\sqrt{1-\bar{y}^2}.$$

The first four approximations of the desired circulation in the expansion with respect to parameter τ have been obtained for this foil as follows:

$$\Gamma(\bar{y}) = \Gamma_{00} + \tau^2\Gamma_{10} + \tau^4\Gamma_{20} + \tau^6\Gamma_{30} + \ldots$$

The curves showing circulation as a function of parameter $|\frac{\omega}{\tau}|$ are given in Figures 24, 25 and 26, while the numerical values are given in Table 9.

Using the correction ξ_1 one may compare the results of the solution of the regular integral equation and the solutions for the optimum foil with the solutions obtained by utilizing the Multhopp methods and those obtained with respect to parameter τ. The results of computations of function ξ_1 for $\omega \to 0$ are in close agreement (Table 10).

2. Using formula (VIII.13) the values of the corrections were calculated for several types of hydrofoils with different geometric parameters and $\omega = 0$ (Fig. 27-31). The obtained results for the rectangular, trapezoidal, and the double trapezoidal foils make it possible to draw certain conclusions in regard to the choice of the shape of the submerged hydrofoil. From Fig. 27 it is evident that the variation of the wing span has a significantly smaller effect on the value of the correction under the free surface than at a considerable distance from it. For example, an increase in the span from 2.7 to 9.5 at a depth of 0.6 changes ξ by 14%, while at depths between 0.04 and 0.06 the difference is only 6%. Computations of the trapezoidal and the double trapezoidal foils (Fig. 28-31) indicate that the reverse trapezoidal shape has a positive effect on the quality of the hydrofoil submerged under a free surface. This becomes clear if we consider that the optimum shape of the hydrofoil submerged under a free surface is fuller than the elliptical. Therefore, the correction ξ_1 for hydrofoils with the positive trapezoidal shape, i.e., with that approaching the elliptical shape, drastically increases with a decrease in depth. The utilization of hydrofoils with the reverse trapezoidal shape is of interest from the point of view of stability of craft with submerged hydrofoils.

Table 9 [366

$\frac{\omega}{\tau}$	Γ^1_{10}	Γ^2_{10}	Γ^3_{10}	Γ^4_{10}	Γ^1_{20}	Γ^2_{20}
0	—0,0131	—0,0242	—0,0316	—0,0342	0,0120	0,0034
0,01	—0,0132	—0,0245	—0,0320	—0,0346	0,0117	0,0030
0,02	—0,0134	—0,0247	—0,0323	—0,0349	0,0115	0,0026
0,03	—0,0135	—0,0250	—0,0326	—0,0353	0,0113	0,0023
0,04	—0,0136	—0,0252	—0,0329	—0,0356	0,0110	0,0019
0,05	—0,0137	—0,0254	—0,0333	—0,0360	0,0108	0,0015
0,06	—0,0139	—0,0257	—0,0336	—0,0363	0,0105	0,0011
0,07	—0,0140	—0,0259	—0,0339	—0,0367	0,0103	0,0008
0,08	—0,0142	—0,0262	—0,0342	—0,0370	0,0101	0,0004
0,09	—0,0143	—0,0264	—0,0345	—0,0373	0,0098	0,0000
0,1	—0,0144	—0,0266	—0,0348	—0,0376	0,0096	—0,0004
0,2	—0,0154	—0,0286	—0,0373	—0,0404	0,0070	—0,0046
0,3	—0,0162	—0,0299	—0,0391	—0,0423	0,0042	—0,0090
0,4	—0,0166	—0,0307	—0,0401	—0,0434	0,0014	—0,0135
0,5	—0,0168	—0,0309	—0,0405	—0,0438	—0,0014	—0,0180
0,6	—0,0166	—0,0308	—0,0402	—0,0435	—0,0042	—0,0225
0,7	—0,0163	—0,0301	—0,0394	—0,0426	—0,0069	—0,0267
0,8	—0,0158	—0,0292	—0,0381	—0,0413	—0,0095	—0,0306
0,9	—0,0151	—0,0280	—0,0366	—0,0396	—0,0118	—0,0341
1	—0,0143	—0,0266	—0,0347	—0,0376	—0,0139	—0,0374
1,2	—0,0125	—0,0232	—0,0304	—0,0328	—0,0177	—0,0426
1,4	—0,0105	—0,0194	—0,0254	—0,0275	—0,0206	—0,0461
1,6	—0,0084	—0,0155	—0,0203	—0,0219	—0,0226	—0,0482
1,8	—0,0063	—0,0116	—0,0151	—0,0164	—0,0239	—0,0489
2	—0,0043	—0,0078	—0,0102	—0,0110	—0,0246	—0,0485
2,2	—0,0023	—0,00425	—0,00555	—0,00601	—0,0248	—0,0471
2,4	—0,0005	—0,0009	—0,0012	—0,0013	—0,0245	—0,0451
2,6	0,0011	0,0021	0,0028	0,0030	—0,0240	—0,0425
2,8	0,0026	0,0049	0,0064	0,0069	—0,0233	—0,0397
3	0,0040	0,0074	0,0096	0,0104	—0,0224	—0,0366
3,2	0,0052	0,0096	0,0125	0,0136	—0,0214	—0,0355
3,4	0,0062	0,0116	0,0151	0,0163	—0,0204	—0,0304
3,6	0,0072	0,0133	0,0174	0,0188	—0,0193	—0,0274
3,8	0,0080	0,0148	0,0193	0,0209	—0,0183	—0,0245
4	0,0087	0,0161	0,0211	0,0228	—0,0174	—0,0217
4,2	0,0093	0,0173	0,0226	0,0244	—0,0165	—0,0192
4,4	0,0099	0,0183	0,0239	0,0259	—0,0157	—0,0169
4,6	0,0103	0,0191	0,0250	0,0271	—0,0149	—0,0147
4,8	0,0108	0,0199	0,0260	0,0281	—0,0142	—0,0128
5	0,0111	0,0205	0,0268	0,0290	—0,0136	—0,0111
6	0,0122	0,0226	0,0295	0,0319	—0,0115	—0,0049
7	0,0127	0,0235	0,0307	0,0333	—0,0104	—0,0017
8	0,0129	0,0239	0,0313	0,0338	—0,0099	—0,0002
9	0,0130	0,0241	0,0315	0,0341	—0,0097	—0,0006
10	0,0131	0,0242	0,0316	0,0342	—0,0096	0,0009
11	0,0131	0,0242	0,0316	0,0342	—0,0096	0,0010
12	0,0131	0,0242	0,0316	0,0342	—0,0096	0,0010
13	0,0131	0,0242	0,0316	0,0342	—0,0096	0,0011
14	0,0131	0,0242	0,0316	0,0342	—0,0096	0,0011
15	0,0131	0,0242	0,0316	0,0342	—0,0096	0,0011

Γ_{20}^{3}	Γ_{20}^{4}	Γ_{30}^{1}	Γ_{30}^{2}	Γ_{30}^{3}	Γ_{30}^{4}
—0,0201	—0,0327	—0,0035	0,0054	—0,00007	—0,0105
—0,0205	—0,0331	—0,0033	0,0055	—0,00026	—0,0108
—0,0208	—0,0335	—0,0031	0,0056	—0,0004	—0,0110
—0,0212	—0,0339	—0,0029	0,0056	—0,0006	—0,0114
—0,0217	—0,0343	—0,0027	0,0057	—0,0008	—0,0116
—0,0221	—0,0347	—0,0025	0,0057	—0,0010	—0,0119
—0,0224	—0,0351	—0,0023	0,0058	—0,0012	—0,0122
—0,0229	—0,0355	—0,0021	0,0058	—0,0014	—0,0125
—0,0233	—0,0359	—0,0019	0,0059	—0,0016	—0,0128
—0,0238	—0,0364	—0,0017	0,0059	—0,0018	—0,0131
—0,0242	—0,0368	—0,0015	0,0060	—0,0020	—0,0134
—0,0288	—0,0414	—0,0003	0,0063	—0,0042	—0,0164
—0,0337	—0,0463	—0,0019	0,0063	—0,0068	—0,0200
—0,0387	—0,0513	0,0032	0,0059	—0,0098	—0,0239
—0,0437	—0,0563	0,0043	0,0051	—0,0132	—0,0284
—0,0485	—0,0611	0,0051	0,0038	—0,0172	—0,0333
—0,0530	—0,0655	0,0056	0,0021	—0,0217	—0,0387
—0,0571	—0,0695	0,0059	0,0000	—0,0266	—0,0446
—0,0607	—0,0729	0,0058	—0,0025	—0,0317	—0,0507
—0,0638	—0,0758	0,0055	—0,0053	—0,0371	—0,0570
—0,0684	—0,0798	0,0045	—0,0114	—0,0483	—0,0698
—0,0709	—0,0815	0,0029	—0,0179	—0,0593	—0,0821
—0,0713	—0,0809	0,0011	—0,0242	—0,0694	—0,0931
—0,0700	—0,0784	—0,0007	—0,0299	—0,0780	—0,1024
—0,0672	—0,0745	—0,0023	—0,0348	—0,0850	—0,1095
—0,0632	—0,0692	—0,0036	—0,0387	—0,0900	—0,1143
—0,0585	—0,0631	—0,0046	—0,0414	—0,0931	—0,1169
—0,0531	—0,0564	—0,0052	—0,0431	—0,0943	—0,1173
—0,0475	—0,0494	—0,0056	—0,0439	—0,0940	—0,1158
—0,0416	—0,0423	—0,0056	—0,0438	—0,0922	—0,1127
—0,0359	—0,0353	—0,0054	—0,0430	—0,0892	—0,1082
—0,0302	—0,0284	—0,0050	—0,0416	—0,0854	—0,1027
—0,0248	—0,0219	—0,0045	—0,0398	—0,0808	—0,0964
—0,0196	—0,0157	—0,0038	—0,0376	—0,0757	—0,0895
—0,0148	—0,0099	—0,0031	—0,0352	—0,0702	—0,0823
—0,0104	—0,0046	—0,0024	—0,0328	—0,0647	—0,0749
—0,0063	—0,0003	—0,0017	—0,0303	—0,0590	—0,0676
—0,0025	0,0048	—0,0010	—0,0278	—0,0535	—0,0604
0,0009	0,0088	—0,0003	—0,0254	—0,0481	—0,0533
0,0039	0,0125	—0,0003	—0,0230	—0,0430	—0,0466
0,0149	0,0256	0,0025	—0,0137	—0,0218	[illegible]
0,0206	0,0326	0,0034	—0,0082	—0,0082	[illegible]
0,0234	0,0360	0,0036	—0,0053	—0,0005	0,0094
0,0248	0,0377	0,0034	—0,0041	—0,0036	0,0149
0,0254	0,0384	0,0032	—0,0035	0,0056	0,0178
0,0237	0,0388	0,0030	—0,0034	0,0065	0,0191
0,0258	0,0389	0,0029	—0,0033	0,0069	[illegible]
0,0259	0,0390	0,0028	—0,0034	0,0071	0,0201
0,0259	0,0390	0,0028	—0,0034	0,0072	0,0202
0,0259	0,0390	0,0028	—0,0034	0,0072	0,0202

Table 10 [368

$\bar{h}$		0,05	0,1	0,15	0,20	0,25	0,30	0,35
$\xi_{opt.}$		1,6153	1,4689	1,3641	1,2877	1,2309	1,1880	1,1551
$\xi_{ellipt.}$		1,6477	1,4827	1,3702	1,2904	1,2321	1,1836	1,1554
	with λ							
ξ from formula (VIII.183)	4	1,6213	1,4720	1,3657	1,2885	1,2313	1,1882	1,1552
	6	1,6231	1,4728	1,3662	1,2887	1,2314	1,1883	1,1553
	8	1,6244	1,4735	1,3665	1,2889	1,2315	1,1883	1,1553
	10	1,6254	1,4740	1,3667	1,2890	1,2316	1,1883	1,1553

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0,40	0,45	0,50	0,55	0,60	0,70	0,80	1,0	1,2
1,1296	1,1095	1,0934	1,0805	1,0699	1,0540	1,0428	1,0286	1,0204
1,1297	1,1096	1,0935	1,0805	1,0699	1,0540	1,0428	1,0286	1,0204
1,1297	1,1095	1,0935	1,0851	1,0700	1,0540	1,0428	1,0286	1,0204
1,1297	1,1095	1,0935	1,0851	1,0700	1,0540	1,0428	1,0286	1,0204
1,1297	1,1095	1,0935	1,0851	1,0700	1,0540	1,0428	1,0286	1,0204
1,1297	1,1095	1,0935	1,0851	1,0700	1,0540	1,0428	1,0286	1,0204

[368

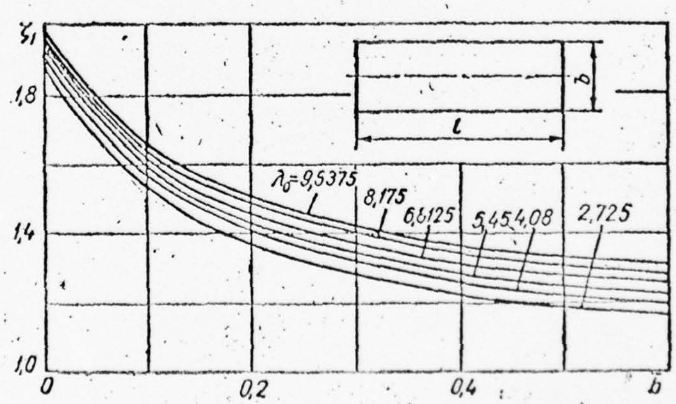

Fig. 27

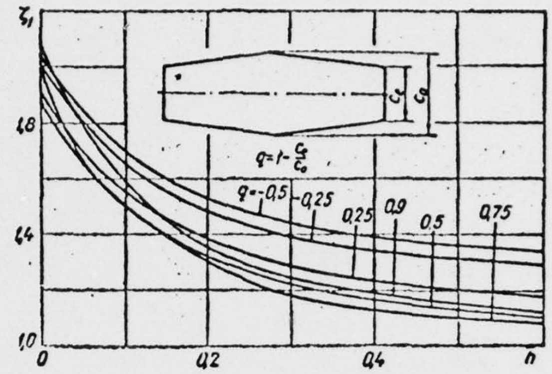

Fig. 28

[369

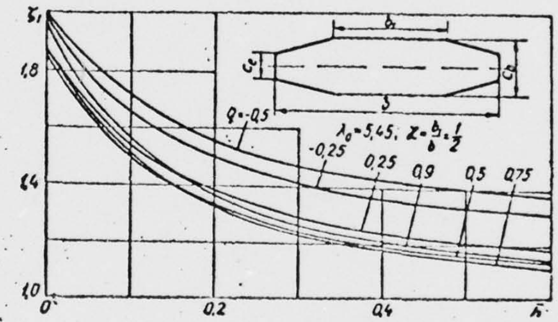

Fig. 29

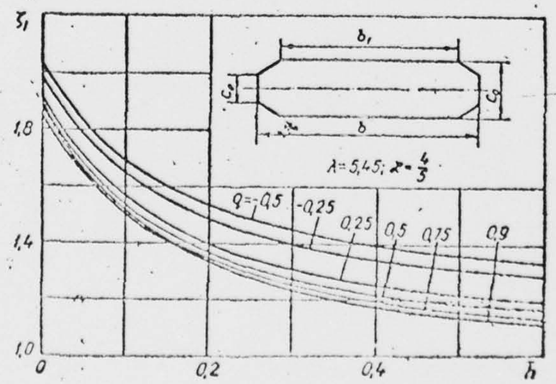

Fig. 30

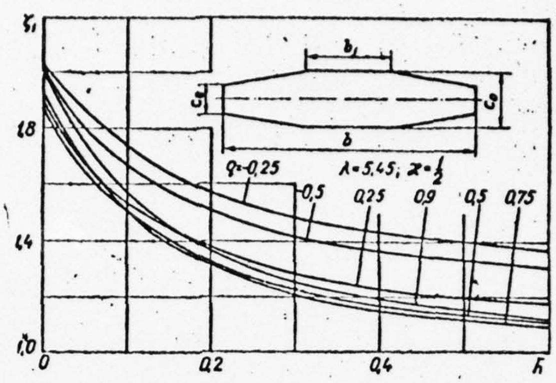

Fig. 31 [370

8.14. Practical Computations of Hydrodynamic Coefficients for Submerged Hydrofoils [365

An accurate evaluation of the hydrodynamic characteristics for submerged hydrofoils can be achieved by solving the basic integro-differential equation. However, for certain problems, it is necessary to follow the less difficult approach resulting in a lesser accuracy. In addition we need formulas for coefficients c_y and c_x which [370
yield approximate, averaged results. The lifting force coefficients for a flat and a slightly curved submerged hydrofoil of finite span are determined by the formula (VIII.110).

$$C_y = \frac{\psi a_\infty}{1 + \frac{\psi a_\infty}{\pi\lambda}\zeta_1}(\alpha_0 + \alpha_к - \Delta\alpha_0).$$

The functions Ψ, ζ, $\Delta\alpha_0$ are determined with the aid of the expressions derived in Chapters I, II and VIII. Combining the results in Chapters I and II, the function Ψ may be found as follows:

$$\psi = 1 + (1+\mu)^2\left\{-\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]\tau^2 + \right.$$
$$+ \left\{\frac{3}{4}\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]^2 - 2\left[\mathrm{Re}\,F_2\left(\frac{\omega}{2\tau}\right) + 1\right] + \right.$$
$$\left. + \frac{9}{4}\left[1 + 2\,\mathrm{Re}\,F_3\left(\frac{\omega}{2\tau}\right)\right]\right\}\tau^4 - \left\{\frac{1}{2}\left[1 + 2\,\mathrm{Re}\,F_1\left(\frac{\omega}{2\tau}\right)\right]^3 - \right.$$

$$-3\left[1+2\operatorname{Re}F_2\left(\frac{\omega}{2\tau}\right)\right]\times\left[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]+$$

$$+3\left[1+2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\right]\left[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]+3\left[1+2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\right]-$$

$$-9\left[1+2\operatorname{Re}F_4\left(\frac{\omega}{2\tau}\right)\right]+\frac{25}{4}\left[1+2\operatorname{Re}F_5\left(\frac{\omega}{2\tau}\right)\right]\Bigg\}\tau^6-$$

$$-\pi\omega e^{-2\pi\omega}\left[1-\frac{3}{16}\omega^2+\frac{15}{5376}\omega^4-\frac{171}{737280}\omega^6+\ldots\right]\Bigg\}, \quad \text{(VIII.215)}$$ [371

$$\psi=\psi_0\left[1-(\alpha_0+\alpha_\kappa)\tau\left[1+2\operatorname{Re}F_0\left(\frac{\omega}{2\tau}\right)\right]\right],$$

where $Fn(\lambda)$ is determined using formulas and tables in Chapter I.

For small angles of attack, the relation $\Psi = f(\alpha)$ may be assumed to be linear and then formula (VIII.215) will contain the Ψ_0 value. The correction $\Delta\alpha_0$ will consist of two terms which account for the curvature and profile thickness, respectively:

$$\Delta\alpha_0=-\frac{\varkappa\alpha_0}{\psi}-\frac{k\delta\tau^3}{2\psi}\left[1+2\operatorname{Re}F_2\left(\frac{\omega}{2\tau}\right)\right],$$

where

$$\varkappa=\frac{1}{2}\left[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]\tau^2+\left\{-\frac{1}{2}\left[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]^2+\right.$$

$$\left.+\left[1+2\operatorname{Re}F_2\left(\frac{\omega}{2\tau}\right)\right]-\frac{3}{2}\left[1+2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\right]\right\}\tau^4-$$

$$-\left\{-\frac{3}{8}\left[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]^3+2\left[1+2\operatorname{Re}F_2\left(\frac{\omega}{2\tau}\right)\right]\left[1+\right.\right.$$

$$\left.+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]-\frac{9}{4}\left[1+2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\right]\left[1+2\operatorname{Re}F_1\left(\frac{\omega}{2\tau}\right)\right]-$$

$$-\frac{3}{2}\left[1+2\operatorname{Re}F_3\left(\frac{\omega}{2\tau}\right)\right]+6\left[1+2\operatorname{Re}F_4\left(\frac{\omega}{2\tau}\right)\right]-$$

$$\left.-\frac{15}{16}\left[1+2\operatorname{Re}F_5\left(\frac{\omega}{2\tau}\right)\right]\right\}\tau^6-\pi\omega e^{-2\pi\omega}\left[\frac{3}{32}\omega^2-\right.$$

$$\left.-\frac{7}{768}\omega^4+\frac{259}{1474360}\omega^6+\ldots\right].$$

The argument μ is determined by the formula $\mu = \frac{0.77\delta}{1 - 0.6\delta}$; α_∞ and α_0 are determined by the results obtained from wind tunnel tests or from other experimental data. In the event the values for the given profile are unknown, then it is usually assumed that $\alpha_\infty = 5.45$ and $\alpha_0 = (105\text{-}112)\delta_{cp}$.

The function ζ_1 has to be determined from the integro-differential equation; however, in determining functions Ψ and $\Delta\alpha_0$, which are assumed to be constant along the span, by means of formulas (VIII.215) and (VIII.216), it is sufficient to determine the function ζ_1 by means of formula [372
(VIII.40). We may attempt to consider also the shape in the plan view by assuming that $\zeta_1 = \zeta + \tau_1$, where ζ is determined from formula (VIII.40) and τ_1, depending on the shape of the foil, from the aerodynamic data. However, this approach for determining the function ζ_2 is not necessarily more valid, because, for the hydrofoil submerged under a free surface, the Glauert correction τ_1 also depends on the mode of motion (see Section 13), and its evaluation from the aerodynamic data will be only approximate.

Since the most interesting for practical applications is the case of $Fr_b \to \infty$, let us examine it in greater detail. For this case Ψ and $\Delta\alpha_0$ will be determined by the formulas

$$\psi_0 = 1 + (1 + \mu)^2\left(-\tau^2 + \tau^4 - \frac{3}{4}\tau^6 + \frac{5}{8}\tau^8 - \frac{3}{8}\tau^{10}\right), \quad \text{(VIII.218) [sic]}$$

$$\psi = \psi_0[1 - (\alpha_0 + \alpha_\kappa)\tau], \quad \text{(VIII.219)}$$

$$\Delta\alpha_0 = -\frac{\varkappa\alpha_0}{\psi} + \frac{K\delta\tau^3}{2\psi}, \quad \text{(VIII.220)}$$

$$\zeta = 1 + 0{,}5\tau_\lambda^2 + 0{,}25\tau_\lambda^4 + 0{,}0625\tau_\lambda^6 + 0{,}0469\tau_\lambda^8 + 0{,}0237\tau_\lambda^{10} + 0{,}0188\tau_\lambda^{12},$$

$$\varkappa = \frac{1}{2}\tau^2 - \tau^4 + \frac{13}{16}\tau^6 - \frac{5}{8}\tau^8 + \frac{5}{16}\tau^{10}. \quad \text{(VIII.221)}$$

From the equations given in Chapter I and from physical considerations, it follows that functions Ψ_0 and $\varkappa$ for a thin foil with $h \to 0$ approach the values of $\Psi_0 = 0.5$ and $\varkappa = 0$. However, the asymptotic formulas in Chapter I

produce different boundary values. Because of this, the remainder terms of the τ^{10} order, correcting the boundary values, are introduced into the formulas (VIII.218) and (VIII.221). Their values for finite submergences are small, hardly affecting the results of the solution obtained for submergence depths of $\bar{h} > 0.10$, i.e., those of practical interest.

The total drag of a hydrofoil of finite span near the free surface is divided into three components: induced, wave, and profile drag. It is clear from this chapter that it is more convenient not to separate the induced drag from the wave drag, but to treat them together.

The coefficient of the induced/wave drag can be determined by the formula (VIII.110)

$$C_{xi} = \frac{C_y^2}{\pi\lambda}\zeta_2,$$

where function ζ_2 should also be determined from the solution of the equation (VIII.25). However, in the first approximation, it may be assumed to equal ζ according to the formula (VIII.40).

This book does not consider the questions dealing with the flow around the submerged hydrofoil in a viscous fluid; therefore, it is not possible to give a sufficiently [373 accurate method for evaluating the profile drag for the submerged hydrofoil.

An approximate value for the profile drag can be determined by employing the experimental methods used in aerodynamics [16, 17] and introducing a correction for the effect of the free surface when $Fr \to \infty$:

$$C_{xp}^{min} = C_{xf}[1 + nc_*(0,54 + 0,46\varphi)], \qquad \text{(VIII.222)}$$

where $c_* = \delta + 0.17\delta_{cp}$;

C_{xj} - the friction coefficient of the flat foil;

$\varphi = 1 - \tau^2$.

The value n can be assumed to be constant and equal to 0.03 when $Re > 5 \cdot 10^6$.

In addition to formula (VIII.223) for computing the profile drag, one can use the F. G. Glass grid system

[16, 17], in which c_* should be replaced by $c_*(0.54 + 0.46\varphi)$.

The formulas (VIII.109) and (VIII.110) for the coefficients C_y and C_{xi} can be extended to include other more complex cases of motion of lifting systems. In Chapters XI-XII below, other cases of the submerged hydrofoil motion will be examined. The results which will be obtained there and which will deal with the evaluation of coefficients C_y and C_{xi} should be used in the actual practical problems. The extensive reference material available in this book in the form of function tables will facilitate considerably the practical utilization of the results obtained.

Let us consider one simple approximate generalization of formulas (VIII.109) and (VIII.110) for the V-shaped submerged hydrofoil moving at high velocities.

The coefficients C_{yv} and C_{xiv} for the V-shaped submerged hydrofoil can be expressed as follows:

$$C_{yv} = \frac{\psi_v a_\infty}{1 + \frac{\psi_v a_\infty}{\pi\lambda}\zeta_1}(\alpha_0 + \alpha_\kappa \cos\beta - \Delta\alpha_0), \qquad \text{(VIII.223)}$$

$$C_{xi} = \frac{C^2_{yv}}{\pi\lambda}\zeta_2.$$

For the hydrofoil that intersects the free surface these may be rewritten as follows:

$$C_{yv} = \frac{\psi_v a_\infty}{1 + \frac{\psi_v \operatorname{tg}\beta}{2\pi\bar{h}}\zeta_1}(\alpha_0 + \alpha_k - \Delta\alpha_{0v}), \qquad \text{(VIII.224)}$$

$$C_{xi} = \frac{C^2_{yv}\operatorname{tg}\beta}{2\lambda\bar{h}}\zeta_2, \qquad \text{(VIII.225)}$$

where β - angle of the V-shape;

$\bar{h}$ - depth of submergence of the central cross section of the hydrofoil.

Functions Ψ_v and $\Delta\alpha_{0v}$ can be approximately determined by formulas (VIII.215) and (VIII.220). [374

By disregarding the vortex trail behind the V-shaped

hydrofoil, the lifting force of the hydrofoil will be expressed by the formula

$$Y = C_{y\infty} \bar{\gamma} \frac{\varrho v^2}{2} s,$$

where $\bar{\gamma}$, when being determined by the hypothesis of plane cross sections, will be found by the formula

$$\bar{\gamma} = \frac{\int\limits_{\bar{h}_1}^{\bar{h}_2} \gamma dh}{\bar{h}_1 - \bar{h}_2}.$$

Let us find the function γ using formulas in Chapter II. Performing integration we obtain

$$\gamma = 1 - \sin(\alpha_0 + \alpha_к)\bar{\tau} - \frac{(1+\mu)^2}{2} \cos \alpha_к \bar{\tau}^2 - \frac{k\delta}{2(\alpha_0 + \alpha_к)} \bar{\tau}^3, \qquad \text{(VIII.226)}$$

where

$$\bar{\tau} = \frac{1}{2} \left\{ \frac{(\bar{h}_1 + \bar{h}_2)[1 + 4(\bar{h}_1^2 + \bar{h}_2^2)]}{\bar{h}_1 \sqrt{4\bar{h}_1^2 + 1} + \bar{h}_2^2 \sqrt{4\bar{h}_2^2 + 1}} + \sqrt{4\bar{h}_1^2 + 1} - \right.$$
$$\left. - 2\bar{h}_1 \left(1 + \frac{2(\bar{h}_1 + \bar{h}_2)}{\sqrt{4\bar{h}_1^2 + 1} + \sqrt{4\bar{h}_2^2 + 1}} \right) - (\bar{h}_1 + \bar{h}_2) \right\}, \qquad \text{(VIII.227)}$$

$$\bar{\tau}^2 = \frac{8}{3}(\bar{h}_1^2 + \bar{h}_1 \bar{h}_2 + \bar{h}_2^2) - \frac{1}{3} \left(\frac{4(\bar{h}_1 + \bar{h}_2)}{\sqrt{4\bar{h}_1^2 + 1} + \sqrt{4\bar{h}_2^2 + 1}} + \right.$$
$$\left. + \frac{4(\bar{h}_1^2 + \bar{h}_2^2)(\bar{h}_1 + \bar{h}_2)}{\bar{h}_1^2 \sqrt{4\bar{h}_1^2 + 1} + \bar{h}_2^2 \sqrt{4\bar{h}_2^2 + 1}} + \frac{16(\bar{h}_1^3 + \bar{h}_2^3)(\bar{h}_1^2 + \bar{h}_1 \bar{h}_2 + \bar{h}_2^2)}{\bar{h}_1^2 \sqrt{4\bar{h}_1^2 + 1} + \bar{h}_2^2 \sqrt{4\bar{h}_2^2 + 1}} \right], \qquad \text{(VIII.228)}$$

$$\bar{\tau}^3 = 2 \left[\frac{8(\bar{h}_1^4 + \bar{h}_2^4)(\bar{h}_1^2 + \bar{h}_2^2)(\bar{h}_1 + \bar{h}_2) + 2(\bar{h}_1^3 + \bar{h}_2^3)(\bar{h}_1^2 + \bar{h}_1 \bar{h}_2 + \bar{h}_2^2)}{\bar{h}_1^3 \sqrt{4\bar{h}_1^2 + 1} + \bar{h}_2^3 \sqrt{4\bar{h}_2^2 + 1}} + \right.$$
$$+ \frac{1}{2} \frac{4(\bar{h}_1^2 + \bar{h}_2^2)(\bar{h}_1 + \bar{h}_2) + (\bar{h}_1 + \bar{h}_2)}{\bar{h}_1 \sqrt{4\bar{h}_1^2 + 1} + \bar{h}_2 \sqrt{4\bar{h}_2^2 + 1}} - (\bar{h}_1 + \bar{h}_2)[3 +$$
$$\left. + 8(\bar{h}_2^2 + \bar{h}_1^2)] \right], \qquad \text{(VIII.229)}$$

where $\bar{h}_1$ - depth of submergence of the center cross section;
$\bar{h}_2$ - depth of submergence of the end cross section.

For a hydrofoil intersecting a free surface ($h_2 = 0$), the formulas become simplified and γ is as follows: [375

$$\bar{\gamma} = 1 - \sin(\alpha_0 + \alpha_\kappa)\tau\left(1 - \frac{\bar{h}}{1 + \sqrt{4\bar{h}^2 + 1}}\right) -$$

$$- \frac{(1+\mu)^2}{2}\cos\alpha_\kappa\left[1 - \frac{4}{3}\bar{h}\left(\frac{2 + \tau(1 - 2\bar{h})}{1 + \sqrt{4\bar{h}^2 + 1}}\right)\right] -$$

$$- \frac{k\delta}{(\alpha_0 + \alpha_\kappa)}[2(1 + 4\bar{h}^2) - h]. \qquad \text{(VIII.230)}$$

The functions Ψ_V and $\Delta\alpha_{0V}$ are determined by the formulas

$$\psi = 1 - \sin(\alpha_0 + \alpha_\kappa \cos\beta)\bar{\tau} - \frac{(1+\mu)^2}{2}\cdot\frac{\cos\alpha_\kappa}{\cos\alpha_0}\bar{\tau}^2, \qquad \text{(VIII.231)}$$

$$\Delta\alpha_1 = \frac{k\delta\bar{\tau}^3}{2\psi}.$$

The $\bar{\tau}^i$ curves are given in Figures 32-34.

For determining function ζ for the V-shaped hydrofoil near the free surface we can utilize the approximate Prandtl condition on the proportionality between the induced drag and a certain arbitrary front surface [179]. [376

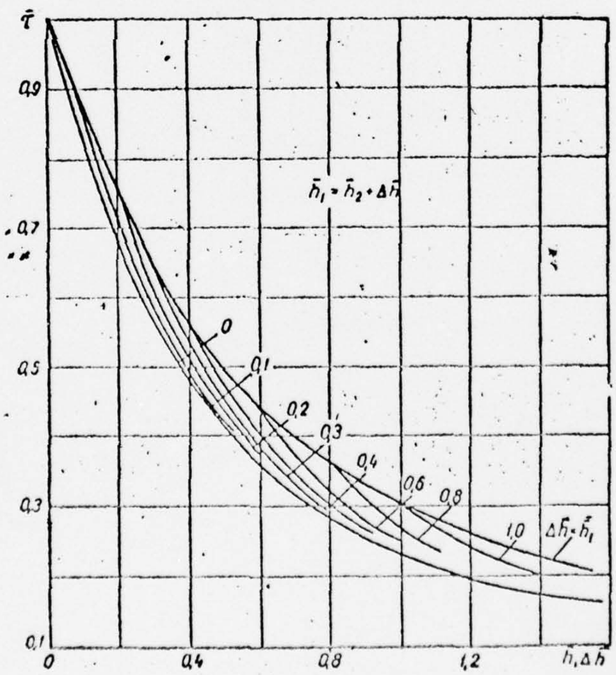

Fig. 32

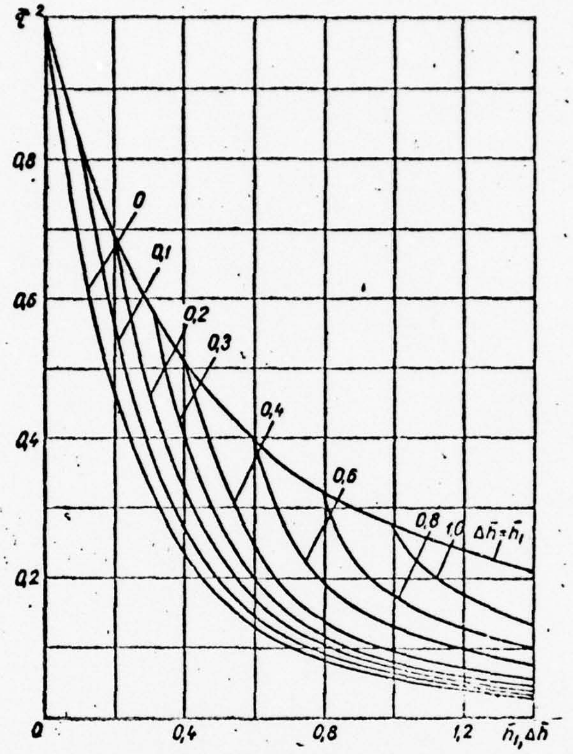 [377

Fig. 33

Applying this concept to function ζ, we obtain [375

$$\zeta_0 = \zeta_0\left(\frac{F}{l^2}\right) = \zeta_0\left(\frac{\bar{h}^2}{\lambda} + \frac{\Delta F}{l^2}\right) = \zeta_0\left(\frac{\bar{h}^2}{\lambda} + \frac{\operatorname{tg}\beta}{4}\right). \quad \text{(VIII.232)}$$

A series of computations for hydrofoils using formulas (VIII.109) and (VIII.110) and the comparison of the results obtained with the experimental data has fully confirmed the validity of the calculated theoretical results, especially at high velocities.

For motion at high velocities, there is an experimental method for evaluating the function Ψ [11, 167].

Figure 35 gives the experimental curves obtained by V. T. Sokolov (curve 1) and S. D. Chudinov (2), as well as a theoretical curve 3 and a curve 4 derived by plotting the N. Ye. Kochin solution. These graphs show that the experimental and the theoretical curves are in close agreement. A certain deviation from the Sokolov curve is

[378

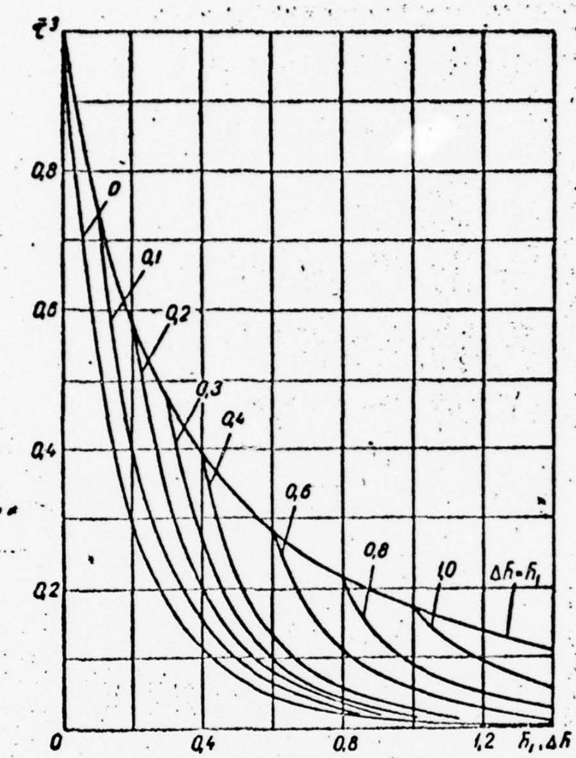

Fig. 34

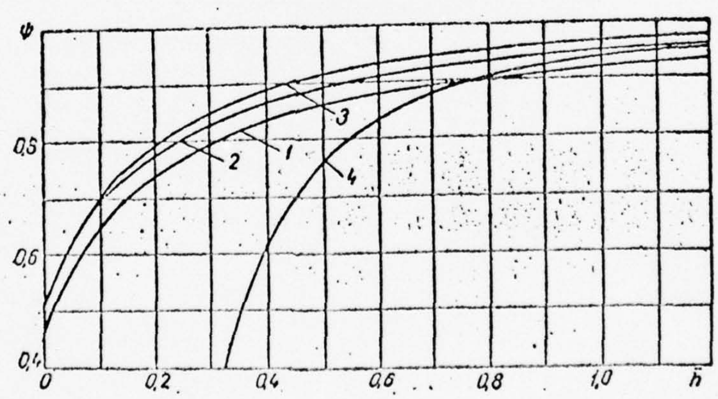

Fig. 35

attributed to the fact that the experimental curve $\Psi = f(h)$ [375
offered by him for a hydrofoil of an infinite span is, in reality, the curve $\bar{\psi} = \frac{\psi}{1+\frac{\psi a_{\infty}\zeta}{\pi\lambda}}$ for a hydrofoil of a finite

[379

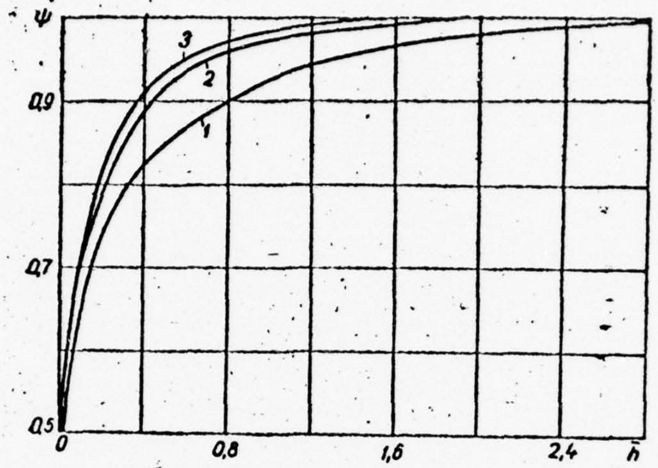

Fig. 36

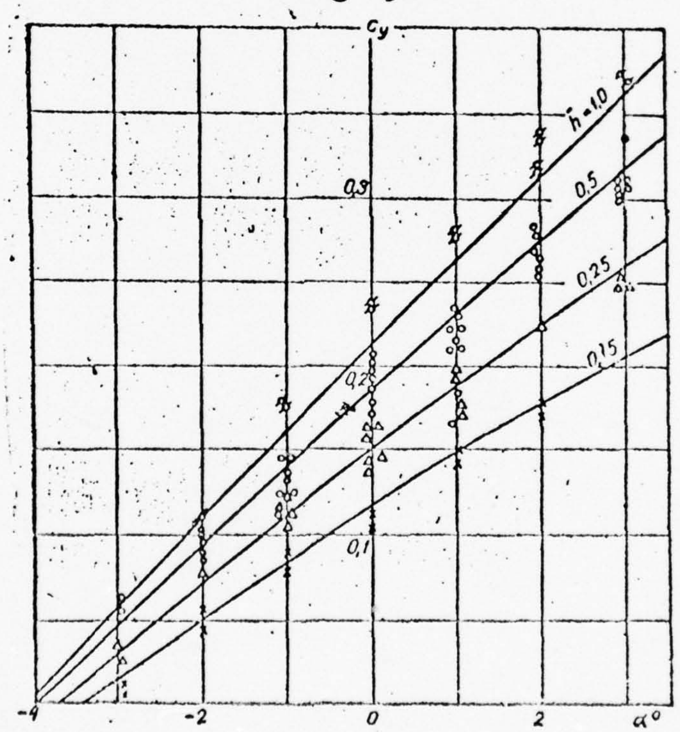

Fig. 37

span. This follows directly from the conditions of the [375
experiment which served as the basis for obtaining these [376
curves and from the methods used in processing the results.

V. T. Sokolov measured the distribution of pressure on the profile of a hydrofoil and the measurements (without recalculation) were used in the expression

$$\varphi = \frac{\Delta p_{b\bar{h}}}{\Delta p_{b_\infty}} \approx \frac{C_{yb\bar{h}}}{C_{yb_\infty}}$$

(where C_{yb} is the lifting force coefficient of the suction wall).

If we recalculate the experimental relationship $\Psi = f(\bar{h})$ for an infinite span we obtain a close agreement with the formula (VIII.218). This is illustrated in Fig. 36, where curve 1 is the Sokolov experimental curve, 2 is the corrected curve, and 3 is the theoretical curve.

In the recalculation the effective span/chord ratio $\lambda_э$ is chosen as 4.2.

The calculated data (the curves on the graph) and the experimental results obtained by M. B. Maseyev [108] (the points on the graph) for a foil with $\lambda = 3$, $\delta = 0.06$ and $k = 1$ are presented in Fig. 37.

The experimental results are also compared with the theoretical data in Figures 38-41. [377

In Fig. 38 the calculated data are represented by [380

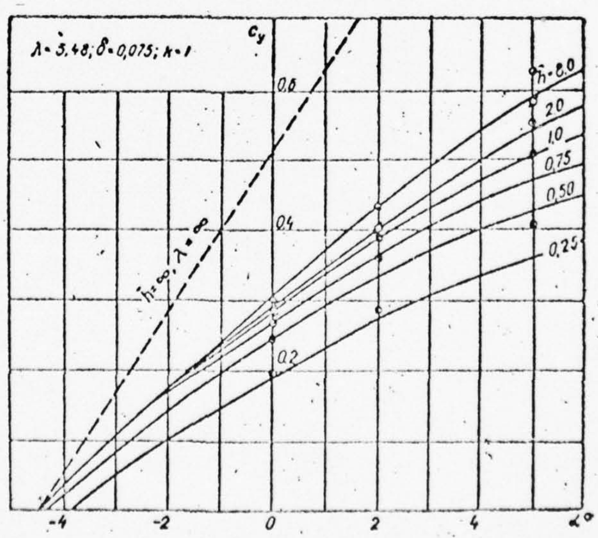

Fig. 38

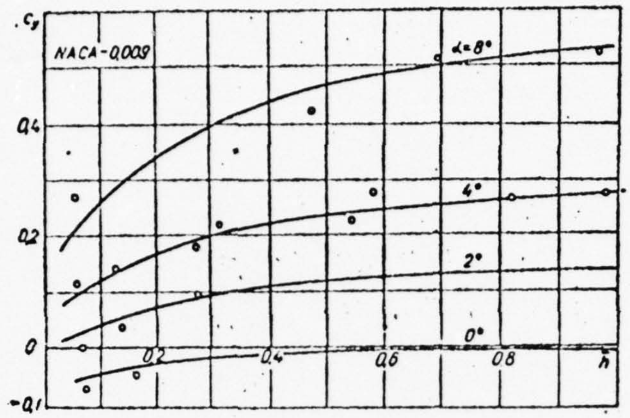

Fig. 39

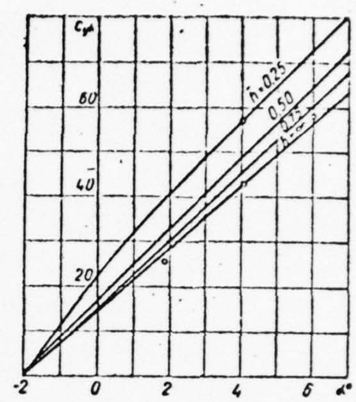

Fig. 40

[380
[381

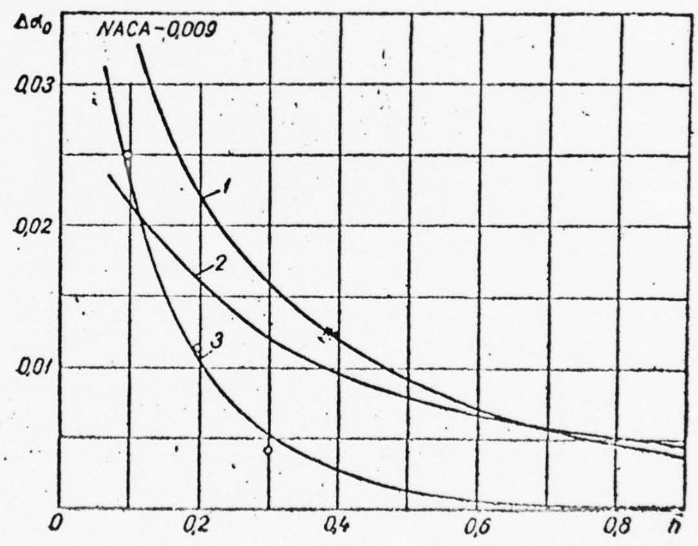

Fig. 41

points, while the experimental results are represented by curves; in Figures 39-40 the calculated results are the curves and the experimental ones are the points [14, 29]. In Fig. 41, curve 1 is based on the S. D. Chudinov formula, curve 2 on the formula by V. T. Sokolov, curve 3 was obtained from formula (VIII.220) and the points are based on the experiment by A. N. Vladimirov. [377

The experimental results are also compared with the theoretical data for arbitrary Froude numbers by T. Nishiyama [209-216].

The comparison shows that the theory being developed describes rather accurately the motion of lifting systems submerged under the heavy fluid.

8.15. Optimal Relationships for Submerged Hydrofoils* [382

*This section was written together with A. I. Yukhimenko.

The formulas (VIII.108) and (VIII.110) are used in solving many applied problems in hydrofoil dynamics. A simple solution for one such problem is given below [115]. Other simple problems are examined in studies [104] and [106].

As was already mentioned above, the coefficient for the total drag of a submerged hydrofoil can be expressed as follows:

$$C_x = C_{xp} + C_{xiB}.$$

If we determine C_{xiB} by formula (VIII.110), then for the inverse quantity $\varepsilon = \frac{C_x}{C_y}$ we will obtain

$$\varepsilon = \frac{C_y}{\pi\lambda}\zeta + \frac{C_{xp}}{C_y}. \qquad \text{(VIII.233)}$$

If we change the hydrodynamic angle of incidence only, then the optimal relationships for the submerged hydrofoil for a given relative span, submergence, shape of the hydrofoil in the plan view, and the Fr_b number are found from the condition of $\frac{d\varepsilon}{d\alpha} = 0$.

Taking the linearity of C_y as a function of α into account and disregarding the change in the profile drag with the change in the angle of incidence (the angles of incidence used in practice are sufficiently small, and hence this assumption is justified), we arrive at the condition $\frac{d\varepsilon}{dc_y} = 0$, from which, after differentiating, we will obtain

$$C^0_{xp} = \frac{C^2_{y\,\text{opt}}}{\pi\lambda}\zeta \qquad \text{(VIII.234)}$$

or

$$C^0_{xp} = C_{xi_B},$$

i.e., for optimal relationships of C_y and λ for the foil, the profile drag is equal to the sum of the induced and wave drags.

From the condition (VIII.234) the expression for C_y optimal for a flat foil can easily be found as follows:

$$C_{y\text{м opt}} = \sqrt{\frac{C_{xpm}}{\frac{1+\delta}{\pi\lambda_м}\zeta\left(\frac{\bar{h}}{\lambda}\right)}}; \qquad \text{(VIII.236)}$$

for a V-shaped foil

$$C_{y\text{v opt}} = \sqrt{\frac{C_{xpm}}{\frac{1+\delta}{\pi\lambda_v}\zeta\left(\frac{\operatorname{tg}\beta}{4}\right)}}, \qquad \text{(VIII.237)}$$

where β is the angle of the V-shaped foil.

The formulas (VIII.233), (VIII.236) and (VIII.237) make possible the qualitative comparison of foil systems using plane and V-shaped foils and having optimal C_{yv} and C_{ym} and the given λ_v and λ_m. [383

Let us examine a case of $\lambda_m = \lambda_v$ and $C_{xpv} = C_{xpm}$. The degree of effectiveness of such foil configurations will be determined by the quantity

$$\bar{k} = \frac{\varepsilon_v}{\varepsilon_m}.$$

With $\bar{k} > 1$, the V-shaped foil system is more advantageous, while with $\bar{k} < 1$, the plane foil system is better. Hence

$$\bar{k} = \frac{k_{1\text{ opt}}}{\cos\beta}, \quad \text{where} \quad k_{1\text{ opt}} = \frac{C_{y\text{м opt}}}{C_{y\text{v opt}}}.$$

Since

$$k_{1\text{ opt}} = \sqrt{\frac{\cos\beta\,\zeta\left(\frac{\operatorname{tg}\beta}{4}\right)}{\zeta\left(\frac{\bar{h}}{\lambda}\right)}}, \quad \bar{k} = \sqrt{\frac{\zeta(\operatorname{tg}\beta/4)}{\zeta\left(\frac{\bar{h}}{\lambda}\right)\cos\beta}}. \qquad \text{(VIII.238)}$$

The formula (VIII.238) makes the comparison of the hydrofoils with optimal C_y and λ relationships for $\lambda_v = \lambda_m$ possible. However, during the design, the consideration of strength, performance, and service flexibility does not always make the choice of hydrofoil configuration with optimal relationships possible. For this reason, below we will consider hydrofoils with arbitrary elements.

Let us approximate the function ξ:

$$\xi\left(\frac{\operatorname{tg}\beta}{4}\right) = \frac{0,92 + 4,13\cos\beta - 3\cos^2\beta}{1,10 + 3,65\cos\beta - 3,6\cos^2\beta}. \qquad \text{(VIII.239)}$$

Then the formula for ε_v will be expressed as follows:

$$\varepsilon_v = \frac{C_{xpv}}{\cos\beta} + \frac{C_{yv}^2}{\pi\lambda_v}(1+\delta)\frac{0,92 + 4,13\cos\beta - 3\cos^2\beta}{1,10 + 3,65\cos\beta - 3,6\cos^2\beta}. \qquad \text{(VIII.240)}$$

Let us determine the optimal angle for the V-shaped foil from the condition of $\frac{\partial\varepsilon_v}{\partial\beta} = 0$. If under this condition we will expand $\xi(\cos\beta)$ into the Taylor series in the neighborhood of $\cos\beta_0 = 0.866$, then we obtain a quadratic equation

$$\left[31,66C_{xpv} - 4,8\frac{C_{yv}^2}{\pi\lambda_v}(1+\delta)\right]\cos^2\beta + \left[73,2c_{xpv} + 0,02\frac{C_{yv}^2}{\pi\lambda_v}(1+\delta)\right]\cos\beta + \left[42,78c_{xpv} - 1,19\frac{C_{yv}^2}{\pi\lambda_v}(1+\delta)\right] = 0.$$

Solving this equation we get [384

$$\cos\beta_{\text{opt}} = \frac{1,157(1-0,425\sqrt{a})}{1-0,1515a}, \qquad \text{(VIII.241)}$$

where

$$a = \frac{C_{yv}^2}{\pi\lambda_v C_{xpv}}(1+\delta).$$

The formula (VIII.242) allows us to determine the optimum angle for the V-shaped foil. The graph of the expression $\beta_{opt} = f(a)$ is given in Fig. 42 (the value of $\beta_{opt}/10$ is plotted along the ordinate).

For actual hydrofoils $C_y = 0.25$-0.27; $\lambda = 4$-6; $C_{xp} = 0.006$-0.007, then $a = 0.4$-0.6 and $\beta_{opt} = 25$-36°.

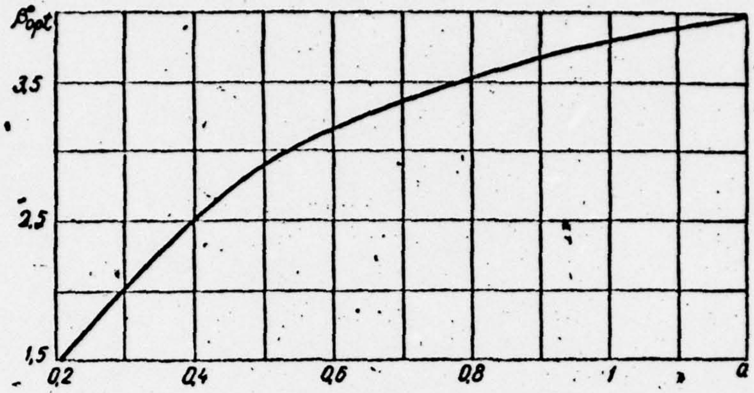

Fig. 42

The value of $\bar{k}$ for any two hydrofoil configurations with equal lifting forces can be written as follows:

$$\bar{k}=\frac{\frac{k_1}{\cos\beta}+a\zeta\left(\frac{\operatorname{tg}\beta}{4}\right)}{b}, \qquad \text{(VIII.242)}$$

where

$$k_1=\frac{C_{y\,\text{н}}}{C_{y0}};\quad k_2=\frac{\lambda_v}{\lambda_m};\quad b=1+k_1^2k_2a\zeta\left(\frac{\bar{h}}{\lambda}\right).$$

Let us determine the value of cos β for which the formula gives the required value, specifically that of $\bar{k}$, and the condition of equivalency for $\bar{k}$ = 1.

By approximating the function $\zeta\left(\frac{\operatorname{tg}\beta}{4}\right)$ [385

$$\zeta\left(\frac{\operatorname{tg}\beta}{4}\right)=\frac{6{,}855-4{,}52\cos\beta}{6{,}755-5{,}36\cos\beta},$$

we arrive at the quadratic equation

$$\cos^2\beta\,(4{,}52a-5{,}36\bar{k}b)+\cos\beta\,(6{,}755\bar{k}b+5{,}36k_1-6{,}855a)-$$
$$-6{,}755k_1=0,$$

from which

$$\cos\beta_{12}=\frac{6{,}755\bar{k}b+5{,}36k_1-6{,}855a}{2\,(4{,}52a-5{,}36\bar{k}b)}\pm$$

$$\pm\frac{\sqrt{(6{,}755\bar{k}b+5{,}36k_1-6{,}855a)^2-27{,}02\,(5{,}36\bar{k}b-4{,}52a)}}{2\,(4{,}52-5{,}36\bar{k}b)}. \qquad \text{(VIII.243)}$$

The formula (VIII.242) makes it possible to evaluate the advantages of one wing configuration over the other.

For example, with $\bar{k} = 1$, $\beta_1 < \beta < \beta_2$ ($\beta_1 < \beta_2$), and hence the V-shaped system is more advantageous, while with $\beta > \beta_2$ and $\beta < \beta_1$ the system using plane submerged hydrofoils is better. When $\beta_1 = \beta_2$ we will have only one equivalence point. The elements that determine this point are found from the condition in which the discriminant is zero:

$$\bar{k}b = 0{,}795k_1 + 1{,}015a + \sqrt{0{,}62ak_1}.$$

The relation b = f(a) as expressed by formula (VIII. 242) is presented in Fig. 43.

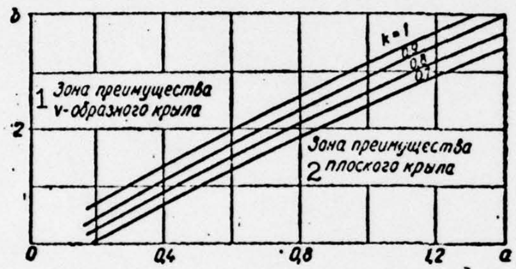

Fig. 43. [1 - area of advantage of V-shaped foil; 2 - area of advantage of plane foil]

Lines k_1 = const divide the graph into two zones. In the lower zone plane hydrofoil configurations are more advantageous, while in the upper the V-shaped foils are better. Obviously, the equivalence point is located in the region of β_{opt}. Therefore, the region where the V-shaped hydrofoils are more advantageous includes hydrofoils with the angle of β_{opt}.

CHAPTER IX. THE THEORY OF THE SUBMERGED HYDROFOIL IN A THREE-DIMENSIONAL FLUID FLOW OF FINITE DEPTH [386

9.1. The Velocity Potential for the Submerged Hydrofoil in a Fluid of Finite Depth

In studying the problem of hydrofoils moving in a fluid of finite depth [110] the velocity potential and the acceleration potential must satisfy the following additional condition on the bottom.

$$\varphi_z = 0, \quad Q_z = 0, \quad z = -h_0. \tag{IX.1}$$

Let us find the acceleration potential in the form

$$\Theta = -\frac{v_0}{4\pi} \iint_s \gamma(\theta) G(x, y, z)\, ds, \tag{IX.2}$$

where G(x, y, z) is the harmonic function in the area limited by surfaces x, y, 0 and x, y, $-h_0$ with the exception of point $Q(\xi, \eta, \zeta)$ on the surface x. For determining function G(x, y, z) we can use the boundary conditions of (VIII.1) and (IX.1).

The function G(x, y, z) is found in the form

$$G(x, y, z) = \frac{z - \zeta}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} - \frac{z + \zeta + 2h_0}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}} + G(x, y, z),$$

where $G_1(x, y, z)$ is the harmonic function within the entire area under consideration.

Let us note that

$$\frac{\partial}{\partial z} \cdot \frac{1}{r} = -\frac{z - \zeta}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{3/2}},$$

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

and using the integral expression [387

$$\frac{1}{r} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} e^{-\lambda[z-\zeta-i\omega]} d\theta\, d\lambda,$$

$$\frac{1}{r'} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} e^{-\lambda[z+\zeta+2h_0-i\omega]} d\theta\, d\lambda$$

under the condition that $z - \zeta > 0$ and $z + \zeta + 2h_0 > 0$.

Then we can evaluate the function G_1 using the equation

$$\frac{\partial^2 G_1}{\partial x^2} + \gamma \frac{\partial G_1}{\partial z} = \frac{1}{\pi} \int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{e^{-\lambda h_0} \operatorname{sh} \lambda (\zeta + h_0) \operatorname{ch} \lambda (z + h_0)}{\operatorname{ch} \lambda h_0} \times$$
$$\times (\lambda \cos^2\theta + \gamma) e^{i\lambda\omega} \lambda^2 d\lambda,$$

where

$$\omega = (x - \xi) \cos\theta + (y - \eta) \sin\theta, \tag{IX.3}$$

from where it follows

$$G = \frac{1}{\pi} \int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{e^{-\lambda h_0} \operatorname{sh} \lambda (\zeta + h_0) \operatorname{ch} \lambda (z + h_0) (\lambda \cos^2\theta + \gamma) e^{i\lambda\omega} \lambda \, d\lambda}{\operatorname{ch} \lambda h_0 [\gamma \operatorname{th} \lambda h_0 - \lambda \cos^2\theta]}. \tag{IX.4}$$

In order to satisfy the conditions at infinity the path for integration with respect to λ should bypass a special point $\lambda_0[\theta]$, where $\lambda_0[\theta]$ is the real and positive root of the following transcendental equation

$$\nu \operatorname{th} \lambda_0 h_0 = \lambda_0 \cos^2\theta. \tag{IX.5}$$

For $\dfrac{\nu h_0}{\cos^2\theta} > 1$ the equation (IX.5) has only one positive root for every θ, and for $\dfrac{\nu h}{\cos^2\theta} < 1$ it has only imaginary roots.

We can show that the conditions at infinity are satisfied if the integration path with respect to λ for the values of $|\theta| < \frac{1}{2}\pi$ bypasses the special point λ_0 from above and for $\pi > |\theta| > \frac{1}{2}\pi - c$ it does from below. Then

$$G = \frac{z - \zeta}{[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]^{3/2}} -$$
$$- \frac{z - \zeta + 2h_0}{[(x - \xi)^2 + (y - \eta)^2 + (z + \zeta + 2h_0)^2]^{3/2}} +$$
$$+ \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} d\theta \int_{0[L_1]}^{\infty} \frac{e^{-\lambda h_0} (\lambda \cos^2\theta + \gamma) \operatorname{sh} \lambda (\zeta + h_0) + \\ + \operatorname{ch} \lambda (z + h_0) \lambda e^{i\lambda\omega}}{\operatorname{ch} \lambda h_0 [\gamma \operatorname{th} \lambda h_0 - \lambda \cos^2\theta]} d\lambda +$$ [388

$$+\frac{1}{\pi}\int\limits_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} d\theta \int\limits_{0(L_2)}^{\infty} \frac{e^{-\lambda h}(\lambda\cos^2\theta+\gamma)\,\mathrm{sh}\,\lambda(\zeta+h_0)+ + \mathrm{ch}\,\lambda(z+h_0)\,\lambda e^{-i\lambda\omega}}{\mathrm{ch}\,\lambda h_0(\gamma\,\mathrm{th}\,\lambda h_0-\lambda\cos^2\theta)}\,d\lambda. \tag{IX.6}$$

Let L_1 bypass the specific point $\lambda_0(\theta)$ from above and let L_2 do it from below. Computing the remainders, the expression for G can be written as follows:

$$G=\frac{z-\zeta}{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{3/2}}-$$

$$-\frac{z+\zeta+2h_0}{[(x-\xi)^2+(y-\eta)^2+(z+\zeta+2h_0)^2]^{3/2}}+$$

$$+\mathrm{Re}\,2\nu i\int\limits_{-\pi/2}^{\pi/2}\frac{\mathrm{sh}\,\lambda_0(\zeta+h_0)\,\mathrm{ch}\,\lambda_0(z+h_0)\,e^{i\lambda_0\omega}}{\mathrm{ch}^2\lambda_0 h\cos^2\theta-\nu h_0}\,d\theta+$$

$$+\frac{1}{\pi}\int\limits_{-\pi}^{\pi}d\theta\int\limits_{0}^{\infty}\frac{e^{-\lambda h_0}(\lambda\cos^2\theta+\gamma)\,\mathrm{sh}\,\lambda(\zeta+h_0)\,\mathrm{ch}\,\lambda(z+h_0)\,e^{i\lambda\omega}}{\mathrm{ch}\,\lambda h_0(\nu\,\mathrm{th}\,\lambda h_0-\lambda\cos^2\theta)}\,d\lambda. \tag{IX.7}$$

where the integral with respect to λ is taken as the principal Cauchy's value. The velocity potential will be found from the expression (IX.2) as follows:

$$\varphi=+\frac{1}{4\pi}\iint\limits_{s}\gamma(\theta)\int\limits_{\infty}^{x}G(\tau,y,z:\xi,\eta,\zeta)\,ds\,dt.$$

Computing it, we obtain

$$\varphi=-\frac{1}{4\pi}\iint\limits_{s}\gamma(\theta)\left[\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{(x-\xi)}{r}-1\right)-\right.$$

$$-\frac{(z+\zeta+2h_0)}{(y-\eta)^2+(z+\zeta+2h_0)^2}\left(\frac{(x-\xi)}{r_1}-1\right)+$$

$$+\mathrm{Re}\,2\nu\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\mathrm{sh}\,\lambda_0(\zeta+h_0)\,\mathrm{ch}\,\lambda_0(z+h_0)\,e^{i\lambda_0\omega}}{(\mathrm{ch}^2\lambda_0 h_0\cos^2\theta-\gamma h_0)\cos\theta}\,d\theta- \quad [389$$

$$-\frac{i}{\pi}\int\limits_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\int\limits_{0}^{\infty}\frac{e^{-\lambda h_0}(\lambda\cos^2\theta+\gamma)\,\mathrm{sh}\,\lambda(\zeta+h_0)\,\mathrm{ch}\,\lambda(z+h_0)\,e^{i\lambda\omega}}{\mathrm{ch}\,\lambda h_0(\nu\,\mathrm{th}\,\lambda h_0-\lambda\cos^2\theta)}\,d\lambda-$$

$$-2\int_0^\infty \frac{e^{-\lambda h_0}\operatorname{sh}\lambda(\zeta+h_0)\operatorname{ch}\lambda(z+h_0)\cos\lambda(y-\eta)}{\operatorname{sh}\lambda h_0}d\lambda\Bigg]ds.$$

Using the Prandtl theory the velocity potential can be expressed as follows:

$$\varphi=\varphi_0+\frac{1}{4\pi}\int_{-b/2}^{b/2}\Gamma(\eta)\int_{\infty}^{\xi}G(\tau,y,z,\xi,\eta,\zeta)\,d\tau d\eta, \qquad \text{(IX.9) [sic]}$$

where φ_0 is the potential of the corresponding two-dimensional problem.

After computations, it follows from the expressions (IX.8) and (IX.9) that

$$\varphi=\varphi_0+\frac{1}{4\pi}\int_{-b/2}^{b/2}\Gamma(\eta)\left(\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}-\right.$$
$$-\frac{z+\zeta+2h_0}{(y-\eta)^2+(z+\zeta+2h_0)}-$$
$$-4\nu\int_0^{\frac{\pi}{2}}\frac{\operatorname{ch}\lambda_0(z+h_0)\operatorname{sh}\lambda_0(\zeta+h_0)\cos\lambda_0(y-\eta)\sin\theta}{\cos^3\theta\left(\operatorname{ch}^2\lambda_0h_0-\frac{\nu h}{\cos^2\theta}\right)}d\theta+$$
$$\left.+2\int_0^\infty\frac{e^{-\lambda h_0}\operatorname{sh}\lambda(\zeta+h_0)\operatorname{ch}\lambda(z+h_0)\cos\lambda(y-\eta)}{\operatorname{ch}\lambda h_0}d\lambda\right)d\eta. \qquad \text{(IX.10)}$$

With Fr $\to\infty$ and $\gamma\to 0$, the expression (IX.11) produces the value of the potential for the boundary condition on the free surface

$$\varphi=\varphi_0+\frac{1}{4\pi}\int_{-b/2}^{b/2}\Gamma(\eta)\left[\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}-\right.$$ [390
$$-\frac{z+\zeta+2h_0}{(y-\eta)^2+(z+\zeta+2h_0)^2}-$$
$$\left.-2\int_0^\infty\frac{e^{-\lambda h_0}\operatorname{sh}\lambda(\zeta+h_0)\operatorname{ch}\lambda(z+h_0)\cos\lambda(y-\eta)}{\operatorname{ch}\lambda h_0}d\lambda\right]d\eta. \qquad \text{(IX.11)}$$

For Fr → 0 and $\gamma \to \infty$, the formula (IX.11) gives the value for the potential for the hydrofoil moving between two solid walls.

$$\varphi = \varphi_0 + \frac{1}{4\pi} \int_{-b/2}^{b/2} \Gamma(\eta) \Bigg[\frac{z-\zeta}{(y-\eta)^2 + (z-\zeta)^2} - \frac{z+\zeta+2h_0}{(y-\eta)^2 + (z+\zeta+2h_0)^2} + 2\int_0^\infty \frac{e^{-\lambda h_0}\, \mathrm{sh}\, \lambda(\zeta+h_0)\, \mathrm{ch}\, \lambda(z+h_0) \cos\lambda(y-\eta)}{\mathrm{sh}\, \lambda h_0} d\lambda \Bigg] d\eta. \tag{IX.12}$$

The expressions (IX.11) and (IX.12) can also be obtained by analyzing the vortex models of corresponding motions directly.

The integration with respect to θ in the formula (IX. 10) extends only to values of θ for which the equation (IX.5) has a real positive root $\lambda_0(\theta)$. These values of θ will be determined by the inequality $\dfrac{\nu h_0}{\cos^2\theta} > 1$.

The formula (IX.10) can be expressed in a somewhat more convenient form. Let us choose a new variable

$$\lambda_0 = \frac{\nu}{\cos^2\theta}\, \mathrm{th}\, \lambda_0 h_0 .$$

Then

$$\frac{1}{\cos^2\theta\left(\mathrm{ch}^2\lambda_0 h - \dfrac{\nu h}{\cos^2\theta}\right)} d\theta = \frac{1}{2\nu\, \mathrm{ch}\, \lambda_0 h\, \mathrm{sh}\, \lambda_0 h \sqrt{1 - \dfrac{\gamma\, th_0 h}{\lambda_0}}} d\lambda,$$

$$\varphi = \varphi_0 + \frac{1}{4\pi} \int_{-b/2}^{b/2} \Gamma(\eta) \Bigg(\frac{z-\zeta}{(y-\eta)^2 + (z-\zeta)^2} - \frac{z+\zeta+2h_0}{(y-\eta)^2 + (z+\zeta+2h_0)^2} - 2\int_{\gamma_0}^{\infty} \frac{\mathrm{ch}\, \lambda(z+h_0)\, \mathrm{sh}\, \lambda(\zeta+h_0) \cos\left[\lambda(y-\eta)\sqrt{1 - \dfrac{\nu\, \mathrm{th}\, \lambda\eta_0}{\lambda}}\right]}{\mathrm{ch}\, \lambda h_0\, \mathrm{sh}\, \lambda h_0 \sqrt{1 - \dfrac{\nu\, \mathrm{th}\, \lambda h_0}{\lambda}}} d\lambda + $$ [391

$$+2\int_0^\infty \frac{e^{-\lambda h}\,\mathrm{sh}\,\lambda(\zeta+h_0)\,\mathrm{ch}\,\lambda(z+h_0)\cos\lambda(y-\eta)}{\mathrm{sh}\,\lambda h_0}\,d\lambda\Bigg)\,d\eta. \qquad \text{(IX.13)}$$

where $\gamma_0 = 0$ for $\nu h_0 < 1$; γ_0 is the real positive root of the equation (IX.5).

For $h_0 \to \infty$ the formula (IX.13) gives the following value for the potential in a fluid flow of infinite depth

$$\varphi = \varphi_0 + \frac{1}{4\pi}\int_{-b/2}^{b/2} \Gamma(\eta)\Bigg(\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2} + \int_0^\infty e^{\lambda(z+\zeta)}\cos(y-\eta)\,d\lambda -$$

$$-2\int_\gamma^\infty \frac{e^{\lambda(z+\zeta)}\cos\lambda(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1-\frac{\gamma}{\lambda}}}\,d\lambda\Bigg)\,d\eta. \qquad \text{(IX.14)}$$

9.2. <u>The Integral Equation for the Hydrofoil in a Fluid of Finite Depth</u>

Let us formulate the expression for the induced velocities φ_z in the following manner:

$$\varphi_z = \frac{1}{4\pi}\iint \gamma(\theta)\Bigg\{\frac{\partial}{\partial z}\Bigg[\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{(x-\xi)}{r_1}-1\right) -$$

$$-\frac{(z+\zeta+2h)}{(y-\eta)^2+(z+\zeta+2h)^2}\left(\frac{(x-\xi)}{r_1}-1\right)\Bigg] -$$

$$-\operatorname{Re} 2\gamma\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\mathrm{sh}\,\lambda_0(\zeta+h_0)\,\mathrm{sh}\,\lambda_0(z+h_0)}{\mathrm{ch}^2\lambda_0 h_0\cos^2\theta - \nu h_0}\cdot\frac{\lambda_0 e^{i\lambda_0\omega}}{\cos\theta}\,d\theta -$$

$$-\frac{i}{\pi}\int_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\int_0^\infty \frac{e^{-\lambda h_0}(\lambda\cos^2\theta+\gamma)\,\mathrm{sh}\,\lambda(\zeta+h_0)\,\mathrm{sh}\,\lambda(z+h_0)e^{i\lambda\omega}\lambda}{\mathrm{ch}\,\lambda h_0(\nu\,\mathrm{th}\,\lambda h_0-\lambda\cos^2\theta)}\,d\lambda +$$

$$+2\int_0^\infty \frac{e^{-\lambda h_0}\,\mathrm{sh}\,\lambda(\zeta+h_0)\,\mathrm{sh}\,\lambda(z+h_0)\cos\lambda(y-\eta)\,\lambda}{\mathrm{sh}\,\lambda h_0}\,d\lambda\Bigg\}. \qquad \text{(IX.15)} \quad [392$$

It follows, then, that the integral equation of the lifting surface with an arbitrary shape in the plan view will be as follows:

$$\frac{1}{4\pi}\iint\gamma(Q)\left[\frac{1}{(y-\eta)^2}\left(\frac{x-\zeta}{\sqrt{(x-\zeta)^2+(y-\eta)^2}}-1\right)-\right.$$

$$-\frac{\partial}{\partial z}\frac{z+\zeta+2h}{(y-\eta)^2+(z+\zeta+2h)^2}\left(\frac{x-\zeta}{r_1}\right)^{-1}-$$

$$-\operatorname{Re}2\nu\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\operatorname{sh}^2\lambda_0(h_0-h)\lambda_0 e^{i\lambda_0\omega}}{(\operatorname{ch}^2\lambda_0 h_0\cos^2\theta-\nu h_0)\cos\theta}d\theta-$$

$$-\frac{i}{\pi}\int\limits_{-\pi}^{+\pi}\frac{d_\theta}{\cos_\theta}\int\limits_0^\infty\frac{e^{-\lambda h_0}(\lambda\cos^2_\theta+\nu)\operatorname{sh}^2\lambda(h_0-h)e^{i\lambda\omega}}{\operatorname{ch}\lambda h_0(\nu\operatorname{th}\lambda h_0-\lambda\cos^2\theta)}d\lambda+$$

$$\left.+2\int\limits_0^\infty\frac{e^{-\lambda h_0}\operatorname{sh}^2\lambda(h_0-h)\lambda\cos\lambda(y-\eta)}{\operatorname{ch}\lambda h_0}d\lambda\right]ds=v_0\alpha. \qquad \text{(IX.16)}$$

From the lifting theory the resultant of the induced velocity along the axis z will be as follows:

$$\frac{\partial\varphi}{\partial z}=\frac{\partial\varphi_0}{\partial z}+\frac{1}{4\pi}\int\limits_{-\frac{b}{2}}^{\frac{b}{2}}\Gamma(\eta)\left(\frac{(y-\eta)^2-(z-\zeta)^2}{[(y-\eta)^2+(z-\zeta)^2]^2}-\right.$$

$$-\frac{(y-\eta)^2-(z+\zeta+2h)^2}{[(y-\eta)^2+(z+\zeta+2h)^2]^2}+$$

$$+2\int\limits_0^\infty\frac{\lambda e^{-\lambda h_0}\operatorname{sh}\lambda(\zeta+h_0)\operatorname{sh}\lambda(z+h_0)\cos\lambda(y-\eta)}{sh\,\lambda h_0}d\lambda-$$

$$\left.-2\int\limits_\gamma^\infty\frac{\lambda\operatorname{sh}\lambda(z+h_0)\operatorname{sh}\lambda(\zeta+h_0)\cos\lambda(y-\eta)\sqrt{1-\frac{\nu\operatorname{th}\lambda h_0}{\lambda}}}{\sqrt{1-\frac{\nu\operatorname{th}\lambda h_0}{\lambda}}\operatorname{ch}\lambda h_0\, sh\,\lambda h_0}d\lambda\right)d\eta. \qquad \text{(IX.17)}$$

The integral equation for the problem will be obtained [393
by taking $\frac{\partial\varphi}{\partial z}$ at the points on the surface of the hydrofoil $p=(0, y, -h)$ and by taking the boundary conditions of

$$-V_0\alpha = V_{0n} + V_{\lambda n},$$

where

$$V_{\lambda n}=\frac{1}{4\pi}\int\limits_{-\frac{b}{2}}^{\frac{b}{2}}\Gamma(\eta)\left(\frac{1}{(y-\eta)^2}-\frac{(y-\eta)^2-4(h_0-h)^2}{[(y-\eta)^2+4(h_0-h)^2]^2}+\right.$$

$$+\int_0^\infty \frac{\lambda e^{-\lambda h_0} \operatorname{sh}\lambda (h_0-h)\cos\lambda(y-\eta)}{\operatorname{sh}\lambda h_0}d\lambda -$$

$$-2\int_{\nu}^{\infty} \frac{\lambda \operatorname{sh}\lambda (h_0-h)\cos\left(\lambda(y-\eta)\sqrt{1-\frac{\nu\operatorname{th}\lambda h_0}{\lambda}}\right)}{\sqrt{1-\frac{\nu\operatorname{th}\lambda h_0}{\lambda}}\operatorname{ch}\lambda h_0\operatorname{sh}\lambda h_0}d\lambda\Bigg)d\eta. \qquad \text{(IX.18)}$$

Combining the values of V_{0n} with the values for circulation on the hydrofoil, we obtain the basic singular integro-differential equation for the proposed problem:

$$\Gamma(y)=\frac{V_0B(y)a_{hh_0}}{2}\Bigg\{\alpha(y)-\frac{1}{4\pi V_0}\Bigg[\int_{-\frac{b}{2}}^{\frac{b}{2}}\frac{\frac{d\Gamma}{d\eta}}{y-\eta}d\eta -$$

$$-\int_{-\frac{b}{2}}^{\frac{b}{2}}\Gamma(\eta)\Bigg(-\frac{(y-\eta)^2-4(h_0-h)^2}{[(y-\eta)^2+4(h_0-h)^2]^2}+$$

$$+2\int_0^\infty \frac{\lambda e^{-\lambda h_0}\operatorname{sh}^2\lambda(h_0-h)}{\operatorname{sh}\lambda h_0}\cos\lambda(y-\eta)\,d\lambda -$$

$$-2\int_{\nu_0}^{\infty}\frac{\lambda\operatorname{sh}^2\lambda(h_0-h)\cos\lambda(y-\eta)\sqrt{1-\frac{\nu\operatorname{th}\lambda h_0}{\lambda}}}{\operatorname{ch}\lambda h_0\operatorname{sh}\lambda h_0\sqrt{(1-\frac{\nu\operatorname{th}\lambda h_0}{\lambda}}}d\lambda\Bigg)d\eta\Bigg]\Bigg\}, \qquad \text{(IX.19)}$$

where $B(y)$ - chord of the hydrofoil; [394

a_{hh_0} - angular coefficient in the relationship $c_{yhh_0} = f(\alpha)$ for an infinite-span hydrofoil.

Another form of equation is obtained by integrating by parts the second integral with respect to η as follows:

$$\Gamma(y)=\frac{V_0B(y)a_{hh_0}}{2}\Bigg\{\alpha(y)-\frac{1}{4\pi V_0}\int_{-\frac{b}{2}}^{\frac{b}{2}}\frac{\frac{d\Gamma}{d\eta}}{y-\eta}d\eta\Bigg(\frac{1}{y-\eta}-$$

$$-\frac{y-\eta}{(y-\eta)+4(h_0-h)^2}-2\int_0^\infty \frac{e^{-\lambda h_0}\,\mathrm{sh}^2\lambda(h_0-h)}{\mathrm{sh}\,\lambda h_0}\sin(y-\eta)\,d\lambda+$$

$$+2\int_\nu^\infty \left(\frac{\mathrm{sh}^2\lambda(h_0-h)\sin\lambda(y-\eta)\sqrt{1-\frac{\nu\,\mathrm{th}\,\lambda h_0}{\lambda}}}{\left(1-\frac{\nu\,\mathrm{th}\,\lambda h_0}{\lambda}\right)\mathrm{ch}\,\lambda h_0\,\mathrm{ch}\,\lambda h_0}\right)d\eta\Bigg\}. \qquad \text{(IX.20)}$$

For the considerations that follow, it is more convenient to present the equation (IX.19) in the form of (VIII.15):

$$\Gamma(\bar{y})=\frac{a_{\bar{h}\bar{h}_0}}{2\lambda(\bar{y})}\left\{\alpha(\bar{y})-\frac{1}{2\pi}\int_{-1}^{1}\Gamma'(\eta)\left[\frac{1}{\bar{y}-\bar{\eta}}+G_{\bar{h}\bar{h}_0}(\bar{y}-\bar{\eta}]\,d\eta\right\},$$

$$G_{\bar{h}\bar{h}_0}(\bar{y}-\bar{\eta})=-\frac{\bar{y}-\bar{\eta}}{(\bar{y}-\bar{\eta})+16(\bar{h}_0-\bar{h})}-$$

$$-2\int_0^\infty \frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}^2 2\lambda(\bar{h}_0-\bar{h})}{\mathrm{sh}\,2\lambda h_0}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda+$$

$$+2\int_{\bar{\nu}}^\infty \frac{\mathrm{sh}^2 2\lambda(\bar{h}_0-\bar{h})\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\bar{\omega}\,\mathrm{th}\,2\lambda\bar{h}_0}{\lambda}}}{\left(1-\frac{\bar{\omega}\,\mathrm{th}\,2\lambda\bar{h}_0}{\lambda}\right)\mathrm{ch}\,2\lambda\bar{h}_0\,\mathrm{sh}\,2\lambda\bar{h}_0}\,d\lambda. \qquad \text{(IX.21)}$$

For $h_0\to\infty$, the function $G_{hh_0}(\bar{y}-\bar{\eta})$ will be as follows: [395

$$G_{\bar{h}\bar{h}_0}(\bar{y}-\bar{\eta})=-\int_0^\infty e^{-4\lambda\bar{h}}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda+$$

$$+2\int_\omega^\infty \frac{e^{-4\lambda\bar{h}}\sin(\bar{y}-\bar{\eta})\sqrt{1-\frac{\bar{\omega}}{\lambda}}}{1-\frac{\omega}{\lambda}}\,d\lambda,$$

while for $\omega\to\infty$ and $\omega\to 0$

$$G_{\bar{h}\bar{h}_0}(\bar{y}-\bar{\eta}) = \frac{(\bar{y}-\bar{\eta})}{(\bar{y}-\bar{\eta})^2 + 16(\bar{h}_0-\bar{h})^2} - $$

$$-2\int_0^{\infty} \frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}^2\,2\lambda(\bar{h}_0-\bar{h})}{\mathrm{sh}\,2\lambda\bar{h}_0}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda; \qquad \text{(IX.22)}$$

$$G_{\bar{h}\bar{h}_0}(\bar{y}-\bar{\eta}) = -\frac{\bar{y}-\bar{\eta}}{(\bar{y}-\bar{\eta})^2 + 16(\bar{h}_0-\bar{h})^2} +$$

$$+2\int_0^{\infty} \frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}^2\,2\lambda(\bar{h}-\bar{h}_0)}{\mathrm{ch}\,2\lambda\bar{h}_0}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda. \qquad \text{(IX.23)}$$

The formulas (IX.22) and (IX.23) correspond to the boundary conditions at the surface $\varphi_y = 0$ and $\varphi_x = 0$.

9.3. Determination of the Hydrofoil Velocity Potential in a Fluid of Finite Depth

In order to determine the velocity potential let us use the integral formulas (VII.43). First, let us determine the function $G(x, y, z, \xi, \eta, \zeta)$ using the integral expression (VIII.34). Let us find the function G in the following form:

$$G(x, y, z) = \frac{1}{r} + \frac{1}{r_1} + G(x, y, z).$$

We obtain the following:

$$\frac{1}{r} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty} e^{-\sqrt{k^2+\lambda^2}(z-\zeta)}\, e^{i(x-\xi)\lambda}\, e^{i(y-\eta)k}\,d\lambda,$$

$$\frac{1}{r_1} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty} e^{-\sqrt{k^2+\lambda^2}(z+\zeta+2h)}\, e^{i(x-\xi)\lambda}\, e^{i(y-\eta)k}\,d\lambda,$$ [396

$$\frac{1}{r} + \frac{1}{r_1} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+2} e^{-z\sqrt{k^2+\lambda^2}}\, e^{-h_0\sqrt{k^2+\lambda^2}}\,\mathrm{ch}\sqrt{k^2+\lambda^2}(\zeta+h_0)\times$$

$$\times e^{i(x-\xi)}\, e^{i(y-\eta)}\,d\lambda.$$

Let us define G_1 by the formula

$$G_1 = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}\times$$

$$\times e^{-h_0\sqrt{k^2+\lambda^2}} \frac{\operatorname{ch}\sqrt{k^2+\lambda^2}(\zeta+h_0)\operatorname{ch}\sqrt{k^2+\lambda^2}(z+h_0)A(k,\lambda)}{\operatorname{ch}\sqrt{k^2+\lambda^2}\,h_0} \times$$
$$\times e^{i(x-\xi)\lambda} e^{i(y-\eta)k}\,dk. \qquad \text{(IX.24)}$$

Let us select the function A(k, λ) using the condition (VIII.1) as follows:

$$A(\lambda, k) = -\frac{\dfrac{\lambda^2}{\sqrt{\lambda^2+k^2}}+\nu}{\left(\dfrac{\lambda^2}{\sqrt{\lambda^2+k^2}}+\dfrac{i\mu\lambda}{\sqrt{\lambda^2+k^2}}-\nu\operatorname{th}\sqrt{\lambda^2+k^2}\,h_0\right)}.$$

Then

$$G(x, y, z) = \frac{1}{r}+\frac{1}{r_1}-$$
$$-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}\frac{e^{-h_0\sqrt{k^2+\lambda^2}}\operatorname{ch}\sqrt{k^2+\lambda^2}(\zeta+h_0)}{\operatorname{ch}\sqrt{k^2+\lambda^2}\,h_0}\times$$
$$\times\frac{\operatorname{ch}\sqrt{k^2+\lambda^2}(z+h_0)\left(\dfrac{\lambda^2}{\sqrt{k^2+\lambda^2}}+\nu\right)e^{i(x-\xi)\lambda}e^{i(y-\eta)k}}{\dfrac{\lambda^2}{\sqrt{k^2+\lambda^2}}+\dfrac{i\mu\lambda}{\sqrt{k^2+\lambda^2}}-\nu\operatorname{th}\sqrt{k^2+\lambda^2}\,h_0}\,d\lambda, \qquad \text{(IX.25)}$$

or

$$G(x, y, z) = \frac{1}{r}+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}\times$$
$$\times\frac{e^{-h_0\sqrt{k^2+\lambda^2}}\left(\dfrac{k}{\sqrt{k^2+\lambda^2}}+\nu\right)}{\operatorname{ch}\sqrt{k^2+\lambda^2}\,h_0\left(\dfrac{\lambda^2}{\sqrt{k^2+\lambda^2}}+\dfrac{i\mu\lambda}{\sqrt{k^2+\lambda^2}}-\nu\operatorname{th}\sqrt{k^2+\lambda^2}\,h_0\right)}\times$$ [397
$$\times\left[e^{-\sqrt{k^2+\lambda^2}z}\operatorname{ch}\sqrt{k^2+\lambda^2}\,h_0\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}-\nu\operatorname{th}\sqrt{k^3+\lambda^2}\,h_0\right)-\right.$$
$$\left.-2\operatorname{ch}\sqrt{k^2+\lambda^2}(\xi+h_0)\operatorname{ch}\sqrt{k^2+\lambda^2}(z+h_0)\right]\times e^{i(x-\xi)}e^{i(y-\eta)}d\lambda. \qquad \text{(IX.26)}$$

Equating to formula (VII.43), we obtain the following:

$$N(\lambda, k) = \frac{1}{2}e^{i(y-\eta)k}e^{-h_0\sqrt{k^2+\lambda^2}}\left[e^{-\sqrt{k^2+\lambda^2}(z+\zeta)}\operatorname{ch}\sqrt{k^2+\lambda^2}\,h_0\times\right.$$
$$\times\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}-\nu\operatorname{th}\sqrt{k^2+\lambda^2}\,h_0\right)-2\operatorname{ch}\sqrt{k^2+\lambda^2}\times$$

$$\times \frac{(\zeta + h_0)\,\mathrm{ch}\sqrt{k^2+\lambda^2}(z+h_0)\left(\frac{\lambda^2}{\sqrt{k^2+\lambda^2}}+\nu\right)\Big]}{\sqrt{k^2+\lambda^2}\,\mathrm{ch}\sqrt{k^2+\lambda^2}\,h_0}$$

$$\left.\begin{aligned}
&Q(\lambda, k) = \frac{\lambda^2}{\sqrt{k^2+\lambda^2}} - \nu\,\mathrm{th}\sqrt{\lambda^2+k^2}\,h_0\\
&Q'(\lambda, k) = \frac{\lambda[(2k^2+\lambda^2)\,\mathrm{ch}^2\sqrt{k^2+\lambda^2}\,h_0 - \nu(k^2+\lambda^2)]}{(k^2+\lambda^2)^{3/2}\,\mathrm{ch}^2\sqrt{k^2+\lambda^2}\,h_0}\\
&\frac{N(0,k)}{Q(0,k)} = \frac{1}{2}\,\frac{e^{i(y-\eta)}e^{-|k|(z+\zeta+h_0)}}{|k|} +\\
&\quad + \frac{e^{-|k|h_0}e^{i(y-\eta)|k|}\,\mathrm{ch}|k|(\zeta+h_0)\,\mathrm{ch}|k|(z+h_0)}{|k|\,\mathrm{th}|k|h_0}
\end{aligned}\right\}. \quad \text{(IX.27)}$$

Let λ_0 be the real root of the transcendental equation

$$\frac{\lambda^2}{\sqrt{\lambda^2+k^2}} - \nu\,\mathrm{th}\sqrt{\lambda^2+k^2}\,h_0 = 0. \quad \text{(IX.28)}$$

Then, using formula (VII.43) we obtain

$$\varphi = -\frac{1}{4\pi}\iint_S \nu(\Theta)\left\{\int_{\infty}^{x}\frac{\partial}{\partial\zeta}\frac{1}{r}\,d\tau + \right.$$

$$+ \frac{\partial}{\partial\zeta}\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{e^{i(y-\eta)k}e^{-|k|(z+\zeta+h_0)}}{|k|}\,dk - \right.$$ [398

$$-\int_{-\infty}^{+\infty}\frac{e^{i(y-\eta)k}e^{-h_0|k|}\,\mathrm{ch}|k|(\zeta+h_0)\,\mathrm{ch}|k|(z+h_0)}{|k|\,\mathrm{sh}|k|h_0}\,dk +$$

$$+ 2\nu\int_{-\infty}^{+\infty}\frac{e^{-h_0\sqrt{\lambda^2+k^2}}e^{i(y-\eta)k}\,\mathrm{ch}\sqrt{\lambda^2+k^2}(\zeta+h_0)\,\mathrm{ch}\sqrt{\lambda^2+k^2}(z+h_0)(1+\mathrm{th}\sqrt{\lambda^2+k^2}\,h_0)}{\lambda_0^2[(2k^2+\lambda_0^2)\,\mathrm{ch}^2\sqrt{\lambda_0^2+k^2}\,h - \nu(k^2+\lambda_0^2)]}\times$$

$$\times\,\mathrm{ch}\sqrt{\lambda_0^2+k^2}\,h_0\,(k^2+\lambda_0^2)\cos\lambda_0(x-\xi)\,dk -$$

$$-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda_0^2}}\int_{-\infty}^{+\infty}\frac{e^{i(x-\xi)\lambda}e^{i(y-\eta)k}e^{-h_0\sqrt{k^2+\lambda^2}}}{\mathrm{ch}\sqrt{k^2+\lambda^2}\,h_0\left(\frac{\lambda^2}{\sqrt{\lambda^2+k^2}} - \nu\,\mathrm{th}\sqrt{k^2+\lambda^2}\,h_0\right)}\times$$

$$\times\left[e^{-\sqrt{k^2+\lambda^2}z}\,\mathrm{ch}\sqrt{k^2+\lambda^2}\,h_0\left(\frac{\lambda^2}{\sqrt{\lambda^2+k^2}} - \nu\,\mathrm{th}\sqrt{\lambda^2+k^2}\,h_0\right) - \right.$$

$$-2\,\mathrm{ch}\sqrt{k^2+\lambda^2}(\zeta+h_0)\times\mathrm{ch}\sqrt{k^2+\lambda^2}(z+$$
$$+h_0)\left(\frac{\lambda^2}{\sqrt{\lambda^2+k^2}}+\nu\right)\Bigg]\Bigg\}ds, \quad \text{(IX.29)}$$

The integration of the triple integral with infinite limits is carried out for the values of k to which the real roots λ_0 of equation (IX.28) correspond.

Behind the hydrofoil, at infinity, the potential will be determined by the formula

$$\varphi_{-\infty}=-\frac{1}{4\pi}\iint_s \gamma(\Theta)\Bigg\{\int_{+\infty}^{-\infty}\frac{\partial}{\partial\zeta}\cdot\frac{1}{r}\,d\tau+$$
$$+\frac{\partial}{\partial\zeta}\Bigg[-\int_{-\infty}^{+\infty}\frac{e^{i(y-\eta)k}e^{-|k|(z+\zeta+h_0)}}{|k|}dk-$$
$$-2\int_{-\infty}^{+\infty}\frac{e^{i(y-\eta)k}e^{-|k|h_0}\,\mathrm{ch}\,|k|(\zeta+h_0)\,\mathrm{ch}\,|k|(z+h_0)}{|k|\,\mathrm{sh}\,|k|h_0}dk+$$
$$+4\nu\int_{-\infty}^{+\infty}\frac{e^{-h_0\sqrt{k^2+\lambda_0^2}}e^{i(y-\eta)k}\,\mathrm{ch}\sqrt{k^2+\lambda_0^2}(\zeta+h_0)\,\mathrm{ch}\sqrt{k^2+\lambda_0^2}(z+h_0)}{\lambda_0^2[(2k^2+\lambda_0^2)\,\mathrm{ch}^2\sqrt{k^2+\lambda_0^2}h_0-\gamma(k^2+\lambda_0^2)}\times$$ [399

$$\times(1+\mathrm{th}\sqrt{k^2+\lambda_0^2}h_0)\,\mathrm{ch}\sqrt{k^2+\lambda_0^2}h_0\,(k^2+\lambda_0^2)\cos\lambda_0(x-\xi)\,dk\Bigg]\Bigg\}ds. \quad \text{(IX.30)}$$

For $\nu\to\infty$, the formula (IX.30) will be as follows:

$$\varphi_{-\infty}=-\frac{1}{4\pi}\iint_s \nu(\Theta)\Bigg\{-\int_{-\infty}^{+\infty}\frac{\partial}{\partial\zeta}\cdot\frac{1}{r}\,d\tau+$$
$$+\frac{\partial}{\partial\zeta}\Bigg[-\int_{-\infty}^{+\infty}\frac{e^{i(y-\eta)k}e^{-|k|(z-\zeta+h_0)}}{|k|}dk-$$
$$-2\int_{-\infty}^{+\infty}\frac{e^{i(y-\eta)k}e^{-|k|h_0}\,\mathrm{ch}\,|k|(\zeta+h_0)\,\mathrm{ch}\,|k|(z+h_0)}{|k|\,\mathrm{sh}\,|k|h_0}dk\Bigg]\Bigg\}ds. \quad \text{(IX.31)}$$

For $\nu\to 0$, $\lambda_0^2\to\nu|k|\,\mathrm{th}\,|k|h_0$ and then

$$\varphi_{-\infty}=-\frac{1}{4\pi}\iint_s V(0)\Bigg\{-\int_{-\infty}^{+\infty}\frac{\partial}{\partial\zeta}\frac{1}{r}\,d\tau+$$

$$+\frac{\partial}{\partial\zeta}\Bigg[-\int_{-\infty}^{+\infty}\frac{e^{i(y-\eta)k}e^{-|k|(s-\zeta+h_0)}}{|k|}\,dk+$$

$$+2\int_{-\infty}^{+\infty}\frac{e^{i(y-\eta)k}e^{-|k|h_0}\operatorname{ch}|k|(\zeta+h_0)\operatorname{ch}|k|(z+h_0)}{|k|\operatorname{ch}|k|h_0}\,dk\Bigg]\Bigg\}\,ds. \qquad \text{(IX.32)}$$

9.4. The Wave and the Induced Drag for the Simplest Hydrofoil Configurations in a Fluid of Finite Depth

The solution of the integro-differential equation and the problems connected with this equation (for example, the problem of optimum relations), can be obtained by using the methods given in Ch. VIII. The problems for $\omega \to \infty$ and $\omega \to 0$ are solved without much difficulty. In the special case when $\omega \to 0$, the function (IX.7) may be presented in the following form: [400

$$G_{\bar h\bar h_0}(\bar y)=\operatorname{Re}\sum_{k=0}^{\infty}\Bigg[\frac{1}{\bar y+4i\left(\frac{k+1}{2}\right)t}+\frac{1}{\bar y-4i\left(\bar h+\frac{k}{2}\bar t\right)}-$$

$$-\frac{1}{\bar y+4i\left(\bar h_1+\frac{k}{2}\bar t\right)}-\frac{1}{\bar y-4i\left(\frac{k+\frac{1}{2}}{2}\right)t}-$$

$$-\frac{1}{\bar y+4i\left(\frac{k+\frac{1}{2}}{2}\right)t}-\frac{1}{\bar y-4i\left(-\bar h_1+\frac{k+1}{2}\bar t\right)}+$$

$$+\frac{1}{\bar y+4i\left(-\bar h+\frac{k+1}{2}t\right)}+\frac{1}{\bar y-4i\left(\frac{k+1}{2}\right)\bar t}, \qquad \text{(IX.33)}$$

where $\bar h_1=\bar h_0-\bar h$; $t=4\bar h_0$.

For the function $\operatorname{Re}\frac{1}{z\pm 4i\bar x}$ the expansion in powers of parameter $\tau_x=\sqrt{4\bar x^2+1}-2\bar x$ is as follows:

$$\operatorname{Re}\frac{1}{z\pm 4ix}\sum_{n=2,4}^{\infty}\tau_x^n\sum_{p=0}^{\frac{n}{2}-1}\frac{(n-1-p)\ldots(p+1)}{(n-1-2p)!}(-1)^{\frac{n}{2}-p}z^{s-1-2p},$$

Then

$$G_{\bar{h}\bar{h}_0}(\bar{y}) = \sum_{k=0}^{\infty}\sum_{n=2,4}^{\infty}(2\tau_{k_1}^{n} + \tau_{k_2}^{n} - \tau_{k_3}^{n} - 2\tau_{k_4}^{n} - \tau_{k_5}^{n} + \tau_{k_6}^{n}) \times$$

$$\times \sum_{p=0}^{\frac{n}{2}-1} \frac{(n-1-p)\ldots(p+1)}{(n-1-2p)!}(-1)^{\frac{n}{2}-p} z^{s-1-2p}, \qquad \text{(IX.34)}$$

$$\tau_{k_1} = \sqrt{(k+1)^2\bar{t}^2+1} - (k+1)\bar{t},$$

$$\tau_{k_2} = \sqrt{4\left(\bar{h}+\frac{k}{2}\bar{t}\right)^2+1} - 2\left(\bar{h}+\frac{k}{2}\bar{t}\right),$$ [401

$$\tau_{k_3} = \sqrt{4\left(\bar{h}_1+\frac{k}{2}\bar{t}\right)^2+1} - 2\left(\bar{h}_1+\frac{k}{2}\bar{t}\right),$$

$$\tau_{k_4} = \sqrt{\left(k+\frac{1}{2}\right)^2\bar{t}^2+1} - \left(k+\frac{1}{2}\right)\bar{t},$$

$$\tau_{k_5} = \sqrt{4\left(\frac{k+1}{2}\bar{t}-\bar{h}_1\right)^2+1} - 2\left(\frac{k+1}{2}t-\bar{h}_1\right),$$

$$\tau_{k_6} = \sqrt{4\left(\frac{k+1}{2}\bar{t}-\bar{h}\right)^2+1} - 2\left(\frac{k+1}{2}\bar{t}-\bar{h}\right).$$

Now, utilizing the expansion (IX.34), it becomes not too difficult to obtain a solution which is similar to that presented in Chapter VIII.

A close first approximation for the case of an elliptical distribution of circulation $\Gamma(y)$ in an infinite flow can be obtained with the aid of the method similar to the first approximation method for solving regular equations [43].

Let us find the solution in the following form:

$$\Gamma(\bar{y}) = \Phi\sqrt{1-\bar{y}^2}. \qquad \text{(IX.35)}$$

Substituting the expression (IX.35) into the equation and integrating within the limits of 2 - 1 to +1 we obtain the following:

$$\Phi = \frac{a_{\bar{h}\bar{h}_0}}{2\lambda_0}\left[\alpha - \frac{\Phi}{2} + \frac{\Phi}{\Pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}d\bar{y}\int_{-1}^{+1}\times\right.$$

$$\times \frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}} G_{\bar{h}\bar{h}_0}(\bar{y}-\bar{\eta})\, d\bar{\eta}\Bigg]. \quad \text{(IX.36)}$$

$$\Phi = \frac{\dfrac{a_{\bar{h}\bar{h}_0}\alpha}{2\lambda_0}}{1+\dfrac{a_{\bar{h}\bar{h}_0}}{4\lambda_0}\left[1-\dfrac{2}{\pi^2}\displaystyle\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\frac{\xi\bar{\eta}}{\sqrt{1-\eta^2}}G_{\bar{h}\bar{h}_0}(\bar{y}-\bar{\eta})\,d\bar{\eta}\right]}.$$

For the case of the hydrofoil moving in an infinite fluid flow the formula (IX.36) produces exact values of Φ for the elliptical distribution of circulation along the span. However, for the motion in a restricted fluid flow, the optimal hydrofoil will be that which has the distribution of circulation different from the elliptical distribution. [402 The foil that has an elliptical distribution of circulation in an infinite fluid flow will not have an elliptical distribution when moving submerged under the surface. In this case, the formula (IX.36) will produce an approximate solution that corresponds, to a certain degree, to the approximate Prandtl solution, that is, to the analysis of the system with a minimum induced drag [39].

Going over to the coefficients of the lifting force and the induced wave drag for the hydrofoil under consideration, we can write

$$C_{\bar{y}\bar{h}\bar{h}_0} = \frac{a_{\bar{h}\bar{h}_0}}{1+\dfrac{a_{\bar{h}\bar{h}_0}}{\pi\lambda}\zeta}\alpha_0; \quad C_x = \frac{C^2_{\bar{y}\bar{h}\bar{h}}}{\pi\lambda}\zeta, \quad \text{(IX.37)}$$

$$\zeta = 1-\frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}}G_{\bar{h}\bar{h}_0}(\bar{y}-\bar{\eta})\,d\bar{\eta}. \quad \text{(IX.38)}$$

For the case of motion in which $\omega \to 0$, using the expression (IX.34) we obtain the following:

$$\zeta = 1 + 0{,}500\varepsilon_1 + 0{,}250\varepsilon_2 + 0{,}0625\varepsilon_3 + 0{,}0469\varepsilon_4 + 0{,}0237\varepsilon_5 + 0{,}0188\varepsilon_6, \quad \text{(IX.39)}$$

where

$$\varepsilon_1 = \sum_{k=0}^{\infty}(2\tau^2_{k_1} + \tau^2_{k_2} - \tau^2_{k_3} - 2\tau^2_{k_4} - \tau^2_{k_5} + \tau^2_{k_6}),$$

$$\varepsilon_2 = \sum_{k=0}^{\infty}(2\tau^4_{k_1} + \tau^4_{k_2} - \tau^4_{k_3} - 2\tau^4_{k_4} - \tau^4_{k_5} + \tau^4_{k_6}),$$

$$e_l = \sum_{k=0}^{\infty} (2\tau_{k_1}^{2l} + \tau_{k_2}^{2l} - \tau_{k_3}^{2l} - 2\tau_{k_4}^{2l} - \tau_{k_5}^{2l} + \tau_{k_6}^{2l}).$$

For analyzing the effect of shallow water, curves obtained by plotting formula (IX.39) and a computation Table 11 are given in Figures 44 and 45. It is clear that shallow water reduces the drag of the submerged hydrofoil considerably for the motion in which $\bar{\omega} \to 0$. The formulas (IX.37) and (IX.39) can be used in practical considerations of the effect of shallow water. [403

Table 11

$\bar{h}$	Values of function ζ for $\bar{h}_0$							
	0	0,05	0,1	0,2	0,3	0,4	0,6	∞
0	0,0000	0,4211	0,6347	1,0825	1,3385	1,5029	1,7977	2,0000
0,05	0,0000	0,3864	0,6247	0,9543	1,1091	1,3715	1,5530	1,6151
0,1	0,0000	0,3812	0,5927	0,8828	1,0620	1,1762	1,3048	1,4824
0,2	0,0000	0,3697	0,5624	0,8005	0,9154	1,0365	1,2105	1,2906
0,3	0,0000	0,3573	0,5340	0,7615	0,9039	0,9643	1,1356	1,1891
0,4	0,0000	0,3562	0,5288	0,7442	0,8570	0,9300	1,0570	1,1208
0,6	0,0000	0,3546	0,5225	0,7216	0,8301	0,8955	1,0014	1,0698
∞	0,0000	0,3546	0,5176	0,7094	0,8109	0,8702	0,9302	1,0000

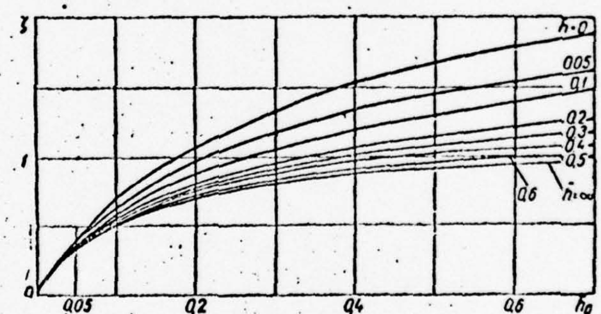

Fig. 44

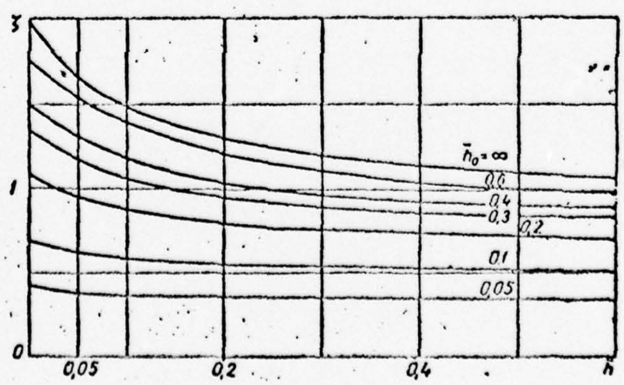

Fig. 45

The problem of the hydrofoil moving under the surface of a fluid of finite depth was also studied by I. Breslin [230]. For the hydrofoil with a constant circulation along the span and for the hydrofoil with an elliptical distribution of circulation he obtained formulas for $\frac{Cx}{Cy^2}$. Let us present the most important results of this investigation. [402

Breslin defines the potential of the hydrofoil in an infinite fluid flow as a potential of a layer of dipoles located within a semi-infinite strip whose width is equal to the hydrofoil span. [403

Breslin determines the wave potential φ_2 as follows: [404

$$\varphi_2 = \frac{2}{\pi}\Gamma_0 \int_{-\infty}^{0} d\tau \int_{-\frac{b}{2}}^{+\frac{b}{2}} \frac{\Gamma(\eta)}{\Gamma_0} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\lambda_0 \operatorname{sh} \lambda_0 (h_0 - h) \operatorname{ch} \lambda_0 (z + h_0)}{(1-\lambda) h \sec^2\theta \sec h^2 \lambda_0 h_0} \times$$

$$\times [\lambda_0 + \nu \sec^2\theta] \sin(x-\tau)\cos\theta] \cos[\lambda_0 (y-\eta)\sin\theta] e^{-\lambda_0 h_0}\, d\eta d\theta. \quad \text{(IX.40)}$$

From the general formula (IX.8) we can determine φ_2 as follows:

$$\varphi_2 = \frac{\operatorname{Re}\nu}{2\pi} \iint_s \gamma(\theta) \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\operatorname{sh}\lambda_0(\zeta + h_0)\operatorname{ch}\lambda_0(z+h_0) e^{i\lambda_0\omega}}{(\operatorname{ch}^2\lambda_0 h_0 \cos^2\theta - \nu h)\cos\theta} d\theta. \quad \text{(IX.41)}$$

If we integrate the expression (IX.40) with respect to τ then, after transformations, we can write φ_2 as follows:

$$\varphi_2 = + \frac{2\nu\Gamma_0}{\pi} \int_{-b}^{+b} \frac{\Gamma(\eta)}{\Gamma_0} \int_{\theta}^{\frac{\pi}{2}} \times$$

$$\times \frac{\operatorname{sh}\lambda_0(\zeta + h_0)\operatorname{ch}\lambda_0(z+h_0)\sin[\lambda_0 x \cos\theta \cos\lambda_0 (y-\eta)}{(\operatorname{ch}^2\lambda_0 h_0 \cos^2\theta - \nu h_0)\cos\theta} d\eta\lambda_0. \quad \text{(IX.42)}$$

This expression can also be obtained from (IX.41) if we introduce the lifting line approximation.

For determining the wave drag a formula is used which defines the wave drag on the basis of the free surface shape. The expression for the shape of the free surface may be written as follows:

$$\eta = \int_{\theta_0}^{\frac{\pi}{2}} Q_1 \sin \lambda_0 x \cos \theta \cos \lambda y \sin \theta d\theta, \qquad (IX.43)$$

where

$$Q_1 = -\frac{4V_0\Gamma_0}{\pi q} \frac{\lambda_0 \operatorname{sh} \lambda_0 (h_0 - h)(\lambda_0 + \nu \sec \theta) e^{-\lambda_0 h_0}}{1 - \nu h_0 \frac{\sec^2 \theta}{\operatorname{ch}^2 \lambda_0 h_0}} \times$$

$$\times \int_0^{\frac{b}{2}} \frac{\Gamma(\eta)}{\Gamma_0} \cos (\lambda_0 \eta \sin \theta) d\eta.$$

If the shape of the free surface is defined by the equation [405

$$\eta = \int_{\theta_0}^{\frac{\pi}{2}} (Q_1 \sin A \cos B + Q_2 \cos A \sin B + Q_3 \cos A \cos B + $$
$$+ Q_4 \sin A \sin B) d\theta, \qquad (IX.44)$$

then the wave drag is given by the formula

$$Q = \frac{\pi \varrho V_0^2}{4} \int_{\theta_0}^{\frac{\pi}{2}} (Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) \left(\operatorname{ct} h \lambda_0 h_0 - \frac{\lambda_0 h_0}{\operatorname{sh}^2 \lambda_0 h_0} \right) \cos^3 \theta d\theta, \qquad (IX.45)$$

where

$$A = \lambda_0 x \cos \theta; \quad B = \lambda_0 y \sin \theta.$$

For the hydrofoil with a constant load along the span after the transformations we obtain

$$\frac{C_{xb}}{C_{y^2}} = \frac{2}{\pi\lambda} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\operatorname{ct} h \lambda_0 h_0 \operatorname{sh}^2 \lambda_0 (h_0 - h) \sin \lambda_0 B \sin \theta d_\theta}{(\operatorname{ch}^2 \lambda h_0 - \nu h_0 \sec^2 \theta) \sin^2 \theta \cos \theta}. \qquad (IX.46)$$

For the case of a fluid of infinite depth this formula transforms into the following formula:

$$\frac{C_{xb}}{C_{y^2}} = \frac{2}{\pi\lambda} \int_0^{\frac{\pi}{2}} \frac{\sin^2 (\nu \sec^2 \theta) e^{-2\nu h \sec^2 \theta}}{\sin^2 \theta \cos \theta} d\theta. \qquad (IX.47)$$

For the hydrofoil with the elliptical distribution of load along the span, the wave amplitude will be

$$Q_1 = -\frac{4\nu\Gamma_0}{\Gamma q}\frac{\lambda_0 \operatorname{sh}\lambda_0 (h_0 - h)(\lambda_0 + \nu \sec^2\theta)\, e^{-\lambda_0 h_0}}{\left(1 - \nu h_0 \frac{\sec^2\theta}{\operatorname{ch}^2\lambda_0 h_0}\right)} \times$$

$$\times \int_0^{\frac{b}{2}} \sqrt{1-\bar{\eta}^2} \cos(\lambda_0 \eta \sin\theta)\, d\eta. \qquad \text{(IX.48)}$$

Then

$$\frac{C_{xb}}{C_{y^2}} = \frac{8}{\pi\lambda}\int_{\theta_0}^{\frac{\pi}{2}} \frac{J_1^2\left(\lambda_0 \frac{b}{2} \sin\theta\right) \operatorname{sh}^2\lambda_0 (h_0 - h)\, e^{-\lambda_0 h_0}}{(\operatorname{ch}^2\lambda_0 h_0 - \nu h_0 \sec^2\theta)\sin\theta\cos\theta}\, d\theta, \qquad \text{(IX.49)}$$

for $Fr \to \infty$ [406

$$\frac{C_{xb}}{C_{y^2}} = \frac{2}{\pi\lambda}\int_0^{\infty} \frac{J_1^2\left(\lambda \frac{b}{2}\right) \operatorname{sh}^2\lambda (h_0 - h)}{\lambda \operatorname{sh} 2\lambda_0 h_0}\, d\lambda. \qquad \text{(IX.50)}$$

In solving for the induced drag I. Breslin takes the velocity potential as follows:

$$\varphi = \varphi_0 + \varphi_1 + \varphi_b + \varphi_{2b},$$

where φ_0 - velocity potential for the wing in an infinite fluid;

φ_1 - velocity potential that satisfies the condition of $(\varphi_0 + \varphi_1)_z = 0, \quad z = -h_0;$

$\varphi_b + \varphi_{2b}$ - wave components of the potential

$$\varphi_b + \varphi_{2b} = \varphi_{01} + \varphi_{11} + \varphi_b',$$

where φ_{01} satisfies the following conditions:

$$\varphi_{01}z = 0, \quad z = -h_0, \quad (\varphi_0 + \varphi_{01})_z = 0, \quad z = 0;$$
$$\varphi_{11}z = 0, \quad z = -h_0, \quad (\varphi_1 + \varphi_{11})_z = 0, \quad z = 0.$$

For $Fr \to 0$, $\varphi_b' \to 0$, and for $Fr \to \infty$ Breslin defines φ_b' as $\varphi_b' = -2(\varphi_{01} + \varphi_{11})$. Obviously, this type of definition is incorrect, because with $Fr \to \infty$ we have the condition of $\varphi_x = 0$ on the free surface. We can define φ_b' in this way only for a fluid of infinite depth, when φ_b' will have the particular points only on the surface, which represents the mirror reflection of the surface $s + \Sigma$ from the plane xy. For a fluid of finite depth φ_b defines the grid of imaginary surfaces. On the basis of the expression for φ_b'

with Fr → ∼, Breslin determines the induced drag as one-half of the wave drag with $Fr \to \infty$, taken with the opposite sign. For the same reason, this type of determination is incorrect, a fact which is quite evident from the comparison of the nuclei $G_{m0}(\bar{y} - \bar{\eta})$ with $\omega \begin{smallmatrix}\to \infty \\ \to 0\end{smallmatrix}$ using formulas (IX.22) and (IX.23).

9.5. Motion of a Submerged Hydrofoil in a Channel of Rectangular Cross Section

Let us consider a hydrofoil moving through a channel with a rectangular cross section and width b.

The potentials θ and φ in this problem must satisfy also the additional conditions at the walls of the channel

$$\theta_y = 0, \quad \varphi_y = 0, \quad y = \pm \frac{b}{2}. \tag{IX.51}$$

The function $G(x, y, z, \zeta, \eta, \xi)$, in this case, can be found by the simple mirror reflection method. Using the expression (IX.8) we obtain [407

$$G(x, y, z, \xi, \eta, \zeta) = \sum_{-\infty}^{+\infty} \left\{ \frac{z-\zeta}{[(x-\zeta)^2 + (y-\eta-2nb)^2 + (z-\zeta)^2]^{\frac{3}{2}}} + \right.$$

$$+ \frac{z-\zeta}{[(x-\xi) + [y-\eta+b(1-2n)]^2 + (z-\zeta)^2]^{\frac{3}{2}}} -$$

$$- \frac{z+\zeta+2h_0}{[(x-\xi)^2 + (y-\eta-2nb)^2 + (z+\zeta+2h_0)^2]^{\frac{3}{2}}} -$$

$$- \frac{z+\zeta+2h_0}{\{(x-\xi)^2 + [y+\eta+b(1-2n)]^2 + (z+\zeta+2h_0)]\}^{\frac{3}{2}}} +$$

$$+ \operatorname{Re} 2\gamma \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\operatorname{ch} \lambda_0 (\zeta + h_0) \operatorname{ch} \lambda (z + h_0) e^{i\lambda_0 (x-\xi)\cos\theta}}{\operatorname{ch}^2 \lambda_0 h_0 \cos^2\theta - \nu h_0} \times$$

$$\times [e^{i\lambda_0 (y-\eta+2nb)\sin\theta} + e^{i\lambda_0 [y-\eta+b(1-2n)]\cos\theta}] \, d\theta -$$

$$- \frac{i}{2\pi} \int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{e^{-\lambda h_0} (\lambda \cos^2\theta + \nu) \operatorname{sh} \lambda (\zeta + h_0) \operatorname{ch} \lambda (z + h_0) e^{i\lambda (x-\xi)\sin\theta}}{\operatorname{ch} \lambda h_0 (\nu \operatorname{th} \lambda h_0 - \lambda \cos^2\theta)} \times$$

$$\times [e^{i\lambda(y-\eta+2nb)\sin\theta} + e^{i\lambda[y+\eta+b(1-2n)\cos\theta]}]d\theta\Bigg\}. \quad \text{(IX.52)}$$

Then the velocity potential can be written as follows:

$$\varphi = -\frac{1}{4\pi}\iint_s \gamma(\theta)\Bigg\{\sum_{-\infty}^{+\infty}\frac{(z-\zeta)}{(y-\eta-2nb)^2+(z-\zeta)^2}\times$$

$$\times\left[\frac{x-\xi}{\sqrt{(x-\xi)^2+(y-\eta-2nb)^2+(z-\zeta)^2}}-1\right]+$$

$$+\frac{(z-\zeta)}{[y-\eta+b(1-2n)]^2+(z-\zeta)^2}\times$$

$$\times\left[\frac{x-\xi}{\sqrt{(x-\xi)^2+[y+\eta+b(1-2n)]^2+(z-\zeta)^2}}-1\right]-$$

$$-\frac{z+\zeta+2h_0}{(y-\eta-2nb)^2+(z+\zeta+2h_0)^2}\times$$

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$$\times\left[\frac{(x-\xi)}{\sqrt{(x-\xi)^2+(y-\eta-2nb)^2+(z+\zeta+2h_0)}}-1\right]-$$

$$-\frac{(z+\zeta+2h_0)}{[y+\eta+b(1-2n)]^2+(z+\zeta+2h_0)}\times$$

$$\times\left[\frac{(x-\xi)}{\sqrt{(x-\xi)^3+[y+\eta+b(1-2n)]-(z+\zeta+2h_0)^2}}-1\right]+$$

$$+\operatorname{Re}2\gamma\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\operatorname{sh}\lambda_0(\zeta+h_0)\operatorname{ch}\lambda_0(z+h_0)e^{i\lambda_0(x-\xi)\cos\theta}}{\operatorname{ch}^2\lambda_0 h_0\cos^2\theta-\nu h_0}\times$$

$$\times[e^{i\lambda_0(y-\eta+2nb)\sin\theta}+e^{i\lambda_0[y+\eta+y(1-2n)\cos\theta}]d\theta+$$

$$+\frac{i}{\pi}\int_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\int_0^\infty\frac{e^{-\lambda h_0}(\lambda\cos^2\theta+\nu)\operatorname{sh}\lambda(\zeta+h_0)\operatorname{ch}(z+h_0)e^{i\lambda(x-\xi)\cos\theta}}{\operatorname{ch}\lambda h_0[\nu\operatorname{th}\lambda h_0-\lambda\cos^2\theta]}\times$$

$$\times[e^{i\lambda(y-\eta-2nb)\sin\theta}+e^{i\lambda[y+\eta+b(1-2n)]\cos\theta}]d\lambda-$$

$$-2\int_0^\infty\frac{e^{-\lambda h_0}\operatorname{sh}\lambda(\zeta+h_0)\operatorname{ch}\lambda(z+h_0)}{\operatorname{ch}\lambda h_0}\times$$

$$\times[\cos\lambda(y-\eta-2nb)+\cos\lambda(y+\eta+b)(1-2n)]d\lambda\Bigg\}ds. \quad \text{(IX.53)}$$

If we use the integro-differential equation (VIII.15) then the nucleus $G(y - \eta)$ will be determined by the formula

$$G(\bar{y}-\bar{\eta}) = \sum_{-\infty}^{+\infty}{}' \left(\frac{1}{(\bar{y}-\bar{\eta}-2n\bar{b})} + \frac{1}{[\bar{y}+\bar{\eta}+\bar{b}(1-2n)]} \right) +$$

$$+ \sum_{-\infty}^{+\infty} \Bigg\{ - \frac{(\bar{y}-\eta-2n\bar{b})}{(\bar{y}-\eta-2n\bar{b})^2+16(\bar{h}_0-\bar{h})^2} -$$

$$- \frac{[\bar{y}+\bar{\eta}+\bar{b}(1-2n)]}{[\bar{y}+\bar{\eta}+\bar{b}(1-2h)]^2+16(\bar{h}_0-\bar{h})^2} -$$

$$-2\int_0^{\infty} \frac{e^{-2\lambda h_0} \operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h})}{\operatorname{sh} 2\lambda h_0} \times$$ [409

$$\times [\sin\lambda(\bar{y}-\bar{\eta}-2n\bar{b}) + \sin(\bar{y}+\bar{\eta}+\bar{b}(1-2n)] +$$

$$+2\int_{\bar{\nu}}^{\infty} \frac{\operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h})}{\left(1-\frac{\omega \operatorname{th} 2\lambda\bar{h}_0}{\lambda}\right) \operatorname{ch} 2\lambda\bar{h}_0 \operatorname{sh} 2\lambda\bar{h}_0} d\lambda \times$$

$$\times \left[\sin\lambda \sqrt{1-\frac{\omega \operatorname{th} 2\lambda\bar{h}_0}{\lambda}}(\bar{y}-\bar{\eta}-2n\bar{b})\right] +$$

$$+ \sin\lambda \sqrt{1-\frac{\omega \operatorname{th} 2\lambda\bar{h}_0}{\lambda}}[\bar{y}+\bar{\eta}+\bar{b}(1-2n)]\Bigg\}. \qquad \text{(IX.54)}$$

where symbol $\sum'$ denotes that the sum does not contain $\frac{1}{\bar{y}-\bar{\eta}}$.

For an infinite channel depth the function $G(\bar{y} - \bar{\eta})$ will be as follows:

$$G(\bar{y}-\bar{\eta}) = \sum_{-\infty}^{+\infty}{}' \left(\frac{1}{(\bar{y}-\bar{\eta}-2n\bar{b})} + \frac{1}{[\bar{y}+\bar{\eta}+\bar{b}(1-2h)]} \right) +$$

$$+ \sum_{-\infty}^{+\infty} \Bigg\{ \int_0^{\infty} e^{-4\lambda\bar{h}_0} [\sin\lambda(\bar{y}-\bar{\eta}-2n\bar{b}) + \sin\lambda(\bar{y}+\bar{\eta}+\bar{b}(1-2n)] d\lambda +$$

$$+2\int_{\omega}^{\infty} \frac{e^{-4\lambda\bar{h}}}{1-\frac{\omega}{\lambda}} \left[\sin\lambda\sqrt{1-\frac{\omega}{\lambda}}(\bar{y}-\bar{\eta}-2n\bar{b}) + \right.$$

$$\left. + \sin\lambda\sqrt{1-\frac{\omega}{\lambda}}[\bar{y}+\bar{\eta}+\bar{b}(1-2n)]\right] d\lambda \Bigg\}. \qquad \text{(IX.55)}$$

The integration in the integro-differential equation (VIII.15) is performed within the limits of -1 + a to +1 - -a, where a is the relative displacement of the axes of the hydrofoil and the channel along the horizontal.

With $\bar{a} = 0$, the coefficients C_y and C_x for a hydrofoil with the elliptical distribution of circulation in an infinite fluid will be determined by the formulas (IX.37), where function ζ is also given by the formula (IX.38), in which $G(y - \eta)$ is determined by formula (IX.54).

CHAPTER X. INTERACTION OF SUBMERGED HYDROFOILS IN A THREE-DIMENSIONAL FLOW [410

10.1. The Velocity Potential and the Integral Equations for Hydrofoil Systems

The interaction theory for the submerged hydrofoils can be developed on the basis of the results obtained in Chapters VIII-IX.

The problem of motion of a system of submerged hydrofoils is analyzed also with the boundary conditions on the free surface (VIII.1), (33).

On the surface of the i-th hydrofoil, the condition (31) must be satisfied:

$$\varphi_z = v_{z_{i_{z=z_i}}} \text{ on } s_i.$$

Let us examine the steady-state motion of a system of n arbitrarily positioned submerged hydrofoils.

When crossing the surface s_1 the acceleration potential undergoes a jump $\theta_- + \theta_+ = -\gamma_i(\theta)$. Taking this fact into account, the expression for the acceleration potential can be written as follows:

$$\theta = \frac{v_0}{4\pi}\sum_{i=1}^{n}\iint_{s_i}\gamma_i(\theta)\frac{\partial}{\partial\zeta}\left[\frac{1}{r_i}+k_i(x, y, z)\right]ds_i. \qquad (X.1)$$

Let us obtain the expression for the velocity potential which satisfies the condition (33) through the integral (X.1):

$$\varphi = -\sum_{i=1}^{n}\frac{1}{4\pi}\iint_{s_i}\gamma_i(\theta)\int_{\infty}^{x}\frac{\partial}{\partial\zeta}\left[\frac{1}{r_i}+k_i(x, y, z)\right]ds\cdot d\tau. \qquad (X.2)$$

Using the value φ_z and the boundary conditions (31), [411 we arrive at the system of integral equations for determining $\gamma_i(\theta)$ as follows:

$$\sum_{i=1}^{n}\left(\frac{1}{4\pi}\frac{\partial}{\partial z}\iint_{s_i}\gamma_i(\theta)\int_{\infty}^{x_i}\frac{\partial}{\partial\zeta_i}\left[\frac{1}{r_i}+k_i(x, y, z)\right]ds\,dx\right)_{z=z_i} = -v_{ni}. \qquad (X.3)$$

The harmonic function k(x, y, z) is determined from the boundary conditions (VIII.1). For a fluid of infinite depth the function k(x, y, z) is given by formula (VIII.6).

This problem will fall within the limits of the lifting line theory approximation if we select for the upper limit of the integral with respect to x, the values that are equal to the points along the abscissa for the i-th foil, i.e., the points located on the lifting line. The system (X.3) will then acquire the following form:

$$\sum_{i=1}^{n}\left(\frac{1}{4\pi}\frac{\partial}{\partial z}\iint_{s_i}\gamma_i(\theta)\int_{\infty}^{x_{0i}}\frac{\partial}{\partial Q_i}\left[\frac{1}{r_i}+k_i(x,y,z)\right]ds_i\,dx\right)_{z=z_j}=-v_{nj}+v_{0j}, \quad (X.4)$$

$$(j=1,\ 2,\dots n)$$

where v_{0i} - normal velocity on the i-th foil that corresponds to the two-dimensional problem;

x_{0i} - abscissa of the i-th lifting vortex.

Introducing the circulation $\Gamma_j(\eta)$ around the i-th foil we arrive at a system of integro-differential equations of the problem:

$$\Gamma_j(y)=\frac{vb_j(y)\,a_{nj}}{2}\left\{\alpha(y)-\frac{1}{4\pi v_0}\int_{-\frac{1}{2}b_j}^{\frac{1}{2}b_j}\frac{\frac{d\Gamma_j}{d\eta}d\eta}{y-\eta}-\right.$$

$$-\frac{1}{4\pi v_0}\left|\frac{\partial}{\partial z}\int_{-\frac{1}{2}b_j}^{\frac{1}{2}b_j}\Gamma_j(\eta)\int_{\infty}^{x_{0j}}\frac{\partial}{\partial\zeta}k_j(x,y,z)\,dx\,d\eta\right|_{z=z_j}-$$

$$\left.-\sum_{i=1}^{n}\left|\frac{1}{4\pi v_0}\frac{\partial}{\partial z}\int_{-\frac{1}{2}b_i}^{+\frac{1}{2}b_i}\Gamma_i(\eta)\,d\eta\int_{\infty}^{x_{0i}}\frac{\partial}{\zeta}\left[\frac{1}{r_i}+k_i(x,y,z)dx\right|_{z=z_j}\right\}. \quad (X.5)$$

$$(j=1,2,\dots n)$$

10.2. The Motion of a Biplanar Hydrofoil System Submerged Under the Free Surface of a Fluid of Infinite Depth [412

Let us examine the motion of two biplanar foils submerged under the free fluid surface.

Performing computations for the potential φ_λ and the induced velocity, we obtain

$$\varphi_\lambda = \frac{1}{4\pi}\left\{ \int\limits_{-\frac{1}{2}b_1}^{+\frac{1}{2}b_1} \Gamma_1(\eta)\left[\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2} + 2\int\limits_0^\infty e^{\lambda(z+\zeta_1)}\cos\lambda(y-\eta)\,d\lambda - \right.\right.$$

$$\left. - 2\int\limits_\nu^\infty e^{\lambda(z+\zeta_1)}\frac{\cos(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1-\frac{\nu}{\lambda}}}\,d\lambda\right]d\eta +$$

$$+ \int\limits_{-\frac{1}{2}b_2}^{+\frac{1}{2}b_2} \Gamma_2(\eta)\left[\frac{(z-\zeta_2)}{(y-\eta)^2+(z-\zeta_2)^2} + 2\int\limits_0^\infty e^{\lambda(z+\zeta_2)}\cos\lambda(y-\eta)\,d\lambda - \right.$$

$$\left.\left. - 2\int\limits_\nu^\infty e^{\lambda(z+\zeta_2)}\frac{\cos\lambda(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1-\frac{\nu}{\lambda}}}\right]d\eta\right\}, \tag{X.6}$$

$$\varphi_{z\lambda} = \frac{1}{4\pi}\left\{ \int\limits_{-\frac{1}{2}b_1}^{+\frac{1}{2}b_1} \Gamma_1(\eta)\left[\frac{(y-\eta)^2-(z-\zeta_1)^2}{[(y-\eta)^2+(z-\zeta_1)^2]^2} + 2\int\limits_0^\infty e^{\lambda(z+\zeta_1)}\lambda\cos\lambda(y-\eta)\times\right.\right.$$

$$\left.\times\,d\lambda - 2\int\limits_\nu^\infty \lambda e^{\lambda(z-\zeta_1)}\frac{\cos\lambda(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1-\frac{\nu}{\lambda}}}\,d\lambda\right]d\eta +$$

$$+ \int\limits_{-\frac{b_2}{2}}^{+\frac{b_2}{2}} \Gamma_2(\eta)\left[\frac{(y-\eta)^2-(z-\zeta_2)^3}{[(y-\eta)^2+(z-\zeta_1)^2]^2} + 2\int\limits_0^\infty e^{\lambda(z+\zeta_2)}\lambda\cos(y-\eta)\,d\lambda - \right.$$ [413

$$\left.\left. - 2\int\limits_\nu^\infty \lambda e^{\lambda(z+\zeta_2)}\frac{\cos\lambda(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1-\frac{\nu}{\lambda}}}\,d\lambda\right]d\eta\right\}. \tag{X.7}$$

Then the relations (X.5)-(X.7) produce a system of two integro-differential equations.

$$\Gamma_1(y) = \frac{v_0 b_1(y) a h_1}{2}\left\{a_1(y) - \frac{1}{4\pi v_0}\left[\int_{-\frac{b_1}{2}}^{+\frac{b_1}{2}} \frac{\frac{d\Gamma_1}{d\eta}}{y-\eta} d\eta + \right.\right.$$

$$+ \int_{-\frac{b_1}{2}}^{+\frac{b_1}{2}} \Gamma_1(\eta)\left(\frac{(y-\eta)^2-(2h_1)^2}{[(y-\eta)^2+(2h_1)^2]^2} - 2\int_{\nu}^{\infty} \frac{\lambda}{\sqrt{1-\frac{\nu}{\lambda}}} e^{-2\lambda h_1} \times\right.$$

$$\left.\times \cos\lambda(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}\, d\lambda\right) d\eta + \int_{-\frac{b_2}{2}}^{+\frac{b_2}{2}} \Gamma_2(y)\left(\frac{(y-\eta)^2-(h_1-h_2)^2}{[(y-\eta)^2+(h_1+h_2)^2]^2} + \right.$$

$$+ \int_0^{\infty} e^{-\lambda(h_1+h_2)} \lambda \cos\lambda(y-\eta)\, d\lambda -$$

$$\left.\left.\left. - 2\int_{\nu}^{\infty} e^{-\lambda(h_1+h_2)} \frac{\lambda \cos(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1+\frac{\nu}{\lambda}}} d\lambda\right)\right]\right\},$$

$$\Gamma_2(y) = \frac{v_0 b_2(y) a h_2}{2}\left\{a_2(y) - \frac{1}{4\pi v_0}\left[\int_{-\frac{b_2}{2}}^{+\frac{b_2}{2}} \frac{\frac{d\Gamma_2}{d\eta}}{y-\eta} d\eta + \right.\right.$$

$$+ \int_{-\frac{b_2}{2}}^{+\frac{b_2}{2}} \Gamma_2(\eta)\left(+\frac{(y-\eta)^2-(2h_2)^2}{[(y-\eta)^2+(2h_2)^2]^2} - 2\int_{\nu}^{\infty} \lambda e^{-2\lambda h_2} \times\right.$$ [414

$$\left.\times \frac{\cos(y-\eta_1)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1-\frac{\nu}{\lambda}}} d\lambda\right) + \int_{-\frac{b_1}{2}}^{+\frac{b_1}{2}} \Gamma_1(\eta)\left(\frac{(y-\eta)^2-(h_1-h_2)^2}{[(y-\eta)^2+(h_1-h_2)^2]^2} + \right.$$

$$+ \int_0^{\infty} e^{-\lambda(h_1+h_2)} \lambda \cos\lambda(y-\eta)\, d\lambda -$$

$$\left.\left.\left. - 2\int_{\nu}^{\infty} \lambda e^{-\lambda(h_1+h_2)} \frac{\cos\lambda(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1-\frac{\nu}{\lambda}}} d\lambda\right)\right]\right\}, \qquad (X.8)$$

or in the dimensionless form

$$\Gamma_1(\bar{y}) = \frac{a_{h_1}}{2\lambda_1(\bar{y})}\left\{\alpha_1(\bar{y}) - \frac{1}{2\pi}\left[\int_{-1}^{+1}\frac{\Gamma_2'(\bar{\eta})\,d\bar{\eta}}{\bar{y}-\bar{\eta}} + \int_{-1}^{+1}\Gamma_1(\bar{\eta})\,G_1(\bar{y}-\bar{\eta})\,d\bar{\eta} + \right.\right.$$

$$\left.\left. + \int_{-1}^{+1}\Gamma_2(\eta)\,G_2(\bar{y}-\bar{\eta})\,d\bar{\eta}\right]\right\},$$

$$\Gamma_2(\bar{y}) = \frac{a_{h_2}}{2\lambda_2(\bar{y})}\left\{\alpha_2(y) - \frac{1}{2\pi}\left[\int_{-1}^{+1}\frac{\Gamma_2'(\bar{\eta})}{\bar{y}-\bar{\eta}}\,d\bar{\eta} + \int_{-1}^{+1}\Gamma_2(\bar{\eta})\,G_3(\bar{y}-\bar{\eta})\,d\bar{\eta} + \right.\right.$$

$$\left.\left. + \int_{-1}^{+1}\Gamma_1(\eta)\,G_4(\bar{y}-\bar{\eta})\,d\eta\right]\right\}, \qquad \text{(X.9)}$$

where

$$\left.\begin{aligned}
G_1(\bar{y}-\eta) &= \frac{(\bar{y}-\bar{h})^2-(4\bar{h}_1)^2}{[y-\eta)^2+(4\bar{h}_1)^2]^2} - \\
&- 2\int_{\omega}^{\infty}\lambda e^{-4\lambda\bar{h}_1}\frac{\cos\lambda(y-\eta)\sqrt{1-\frac{\omega}{\lambda}}}{\sqrt{1-\frac{\omega}{\lambda}}}\,d\lambda \\
G_2(\bar{y}-\bar{\eta}) &= \frac{(y-\eta)^2-4k^2(\bar{h}_1-\bar{h}_2)^2}{[(y-\eta)^2+4k^2(h_1-\bar{h}_2)^2]^2} + \\
&+ 2\int_0^{\infty}e^{-2\lambda k(\bar{h}_1+h_2)}\lambda\cos\lambda(y-\eta)\,d\lambda - \\
&- 2\int_{\omega}^{\infty}e^{-4\lambda k(\bar{h}_1+\bar{h}_2)}\frac{\lambda\cos(\bar{y}-\eta)\sqrt{1-\frac{\omega}{\lambda}}}{\sqrt{1-\frac{\omega}{\lambda}}}\,d\lambda \\
G_3(\bar{y}-\bar{\eta}) &= \frac{(\bar{y}-\bar{\eta})^2-k^2(4\bar{h}_2)^2}{[(y-\eta)^2+k^2(4\bar{h}_2)^2]^2} - \\
&- 2\int_{\omega}^{\infty}e^{-4\lambda k\bar{h}_2}\frac{\cos\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\omega}{\lambda}}}{\sqrt{1-\frac{\omega}{\lambda}}}\,d\lambda \\
G_4(\bar{y}-\bar{\eta}) &= \frac{(\bar{y}-\bar{\eta})^2-4(\bar{h}_1-\bar{h}_2)^2}{[(y-\eta)^2+4(\bar{h}_1+\bar{h}_2)^2]^2} +
\end{aligned}\right|; \qquad \text{(X.10)}$$

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$$+\,2\int_{0}^{\infty} e^{-2\lambda(\bar{h}_1+\bar{h}_2)}\lambda\cos\lambda\,(\bar{y}-\bar{\eta})\,d\lambda\,-$$

$$-\,2\int_{\omega}^{\infty} e^{-4\lambda(\bar{h}_1+\bar{h}_2)}\,\frac{\cos\lambda\,(y-\eta)\sqrt{1-\frac{\omega}{\lambda}}}{\sqrt{1-\frac{\omega}{\lambda}}}\,d\lambda \;\Bigg|$$

$$\Gamma_1(\bar{y})=\frac{\Gamma_1(y)}{l_1 v_0},\qquad \Gamma_2(\bar{y})=\frac{\Gamma_2(y)}{l_2 v_0},\qquad \lambda_i=\frac{l_i}{b_i(y)},$$

$$\bar{y}=\frac{2y}{l_1},\qquad \bar{h}_i=\frac{h_i}{l_i},\qquad k=\frac{l_1}{l_2},\qquad a_{h_i}=\frac{dc_{y\,i\infty}}{d\alpha}.$$

With $\bar{h}\to\infty$, the system (X.9) will transform into a system of equations describing the motion of a biplane in an infinite fluid flow with the foils separated by 2H: [416

$$\Gamma_1(y)=\frac{a_{\infty 1}}{2\lambda_1(y)}\left[\alpha_1(\bar{y})-\frac{1}{2\pi}\left(\int_{-1}^{+1}\frac{\Gamma_1'(\bar{\eta})}{\bar{y}-\eta}\,d\bar{\eta}\,+\right.\right.$$

$$\left.\left.+\int_{-1}^{+1}\Gamma_2(\bar{\eta})\,\frac{(\bar{y}-\eta)^2-16k^2\bar{H}^2}{[(y-\eta)^2+16k^2\bar{H}^2]^2}\,d\eta\right)\right],$$

$$\Gamma_2(\bar{y})=\frac{a_{\infty 2}}{2\lambda_2(y)}\left[\alpha_2(\bar{y})-\frac{1}{2\pi}\left(\int_{-1}^{+1}\frac{\Gamma_2'(\bar{\eta})}{\bar{y}-\eta}\,d\bar{\eta}\,+\right.\right.$$

$$\left.\left.+\int_{-1}^{+1}\Gamma_1(\bar{h})\,\frac{(\bar{y}-\bar{\eta})^2-16\bar{H}^2}{[(\bar{y}-\bar{\eta})^2+16\bar{H}^2]^2}\,d\eta\right)\right]. \qquad \text{(X.11)}$$

With $\omega\to 0$ the nuclei $G_n(y-\eta)$ will be as follows:

$$\left.\begin{aligned}
G_1(y-\eta)&=\frac{(\bar{y}-\bar{\eta})^2-(4\bar{h}_1)^2}{[(\bar{y}-\bar{\eta})^2+(4\bar{h}_1)^2]^2}\\
G_2(\bar{y}-\bar{\eta})&=\frac{(\bar{y}-\bar{\eta})^2-4k^2(\bar{h}_1-\bar{h}_2)^2}{[(\bar{y}-\bar{\eta})^2+4k^2(\bar{h}_1-\bar{h}_2)^2]^2}+\frac{(\bar{y}-\bar{\eta})^2-4k^2(\bar{h}_1+\bar{h}_2)^2}{[(\bar{y}-\bar{\eta})^2+4k^2(\bar{h}_1+\bar{h}_2)^2]^2}\\
G_3(\bar{y}-\bar{\eta})&=\frac{(\bar{y}-\bar{\eta})^2-k^2(4\bar{h}_2)^2}{[(\bar{y}-\bar{h})^2+k^2(4\bar{h}_2)^2]^2}
\end{aligned}\right| \qquad \text{(X.12)}$$

$$G_4(\bar{y}-\bar{h}) = \frac{(\bar{y}-\bar{\eta})^2 - 4(\bar{h}_1-\bar{h}_2)^2}{[(\bar{y}-\bar{\eta})^2+4(\bar{h}_1-\bar{h}_2)^2]^2} + \frac{(y-\eta) - 4(\bar{h}_1+\bar{h}_2)^2}{[(\bar{y}-\bar{\eta})^2+4(\bar{h}_1+\bar{h}_2)^2]^2}\Bigg|$$

With $\omega \to \infty$, nuclei G_1 and G_3 will have the opposite signs, while in nuclei G_2 and G_4 the sign of the second term will change.

The coefficients of the lifting force and the induced drag for the hydrofoil system will be determined by the formula

$$C = \lambda_i \int_{-1}^{+1} \Gamma_i(\bar{y})\, d\bar{y},$$

$$C_{xi1} = \frac{\lambda_1}{2\pi}\int_{-1}^{+1} \Gamma_1(\bar{y})\, d\bar{y}\left[\int_{-1}^{+1} \frac{\Gamma_1'(\bar{\eta})}{\bar{y}-\bar{\eta}}\, d\bar{\eta} + \int_{-1}^{+1} \Gamma_1(\bar{\eta})\, G_1(\bar{y}-\bar{\eta})\, d\bar{\eta} + \right.$$
$$\left. + \int_{-1}^{+1} \Gamma_2(\bar{\eta})\, G_2(\bar{y}-\bar{\eta})\, d\bar{\eta}\right]. \qquad \text{(X.13)}$$ [417

$$C_{xi2} = \frac{\lambda_2}{2\pi}\int_{-1}^{+1} \Gamma_2(\bar{y})\left[\int_{-1}^{+1} \frac{\Gamma_2(\bar{\eta})}{\bar{y}-\bar{\eta}}\, d\bar{\eta} + \int_{-1}^{+1} \Gamma_2(\bar{\eta})\, G_3(\bar{y}-\bar{\eta})\, d\bar{\eta} + \right.$$
$$\left. + \int_{-1}^{+1} \Gamma_1(\bar{\eta})\, G_1(\bar{y}-\bar{\eta})\, d\bar{\eta}\right]. \qquad \text{(X.14)}$$

where $\lambda_i = \frac{l_i^2}{s_i}$; and s_i is the area of the hydrofoil.

The system of equations (X.11) can be solved with the aid of methods discussed in Ch. VIII.

Let us obtain an approximate solution for a hydrofoil with the elliptical distribution of circulation in an infinite flow. This solution permits studying the free surface effect on the hydromechanical characteristics of biplanar foils and is satisfactory for the majority of practical cases.

Let us take the function $\frac{1}{\lambda_i(y)}$ in the form of

$$\frac{1}{\lambda_i(y)} = \frac{4}{4\lambda_i}\sqrt{1-\bar{y}^2}.$$

Let us find the solution of the system (X.9) in the form $\Gamma_i(y) = \Phi_i\sqrt{1-\bar{y}^2}$ and integrate the equation with respect

to y from -1 to +1. Then

$$\Phi_1 = \frac{2a_{h_1}}{\pi\lambda_1}\left\{\alpha_1 - \frac{\Phi_1}{2}\left[1 + \frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_1(\bar{y}-\bar{\eta})d\eta dy\right] - \right.$$

$$\left. - \frac{\Phi_2}{2}\frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_2(\bar{y}-\bar{\eta})\,d\bar{\eta}d\bar{y}\right\}, \qquad (X.15)$$

$$\Phi_2 = \frac{2a_{h_2}}{\pi\lambda_2}\left\{\alpha_2 - \frac{\Phi_2}{2}\left[1 + \frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_3(\bar{y}-\bar{\eta})d\eta\right] - \right.$$

$$\left. - \frac{\Phi_1}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_4(\bar{y}-\bar{\eta})\,d\bar{\eta}d\bar{y}\right\},$$

or

$$\Phi_1\left(1 + \frac{a_{\bar{h}_1}}{\pi\lambda_1}\zeta_1\right) + \frac{a_{\bar{h}_1}}{\pi\lambda_1}\zeta_{12}\Phi_2 = \frac{2a_{\bar{h}_1}}{\lambda\pi_1}\alpha_1,$$

$$\Phi_2\left(1 + \frac{a_{\bar{h}_2}}{\pi\lambda_2}\zeta_2\right) + \frac{a_{h_2}}{\pi\lambda_2}\zeta_{21}\Phi_1 = \frac{2a_{h_2}}{\pi\lambda_2}\alpha_2, \qquad (X.16)$$

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Hence it follows

$$\Phi_1 = \frac{\frac{2a_{h_1}}{\pi\lambda_1}\alpha_1\left(1 + \frac{a_{h_2}}{\pi\lambda_2}\zeta_2\right) - \frac{2a_{h_2}}{\pi\lambda_2}\alpha_2\frac{a_{h_1}}{\pi\lambda_1}\zeta_{12}}{\left(1 + \frac{a_{h_1}}{\pi\lambda_1}\zeta_1\right)\left(1 + \frac{a_{\bar{h}_2}}{\pi\lambda_2}\zeta_2\right) - \frac{a_{h_2}}{\pi\lambda_2}\zeta_{21}\frac{a_{h_1}}{\pi\lambda_1}\zeta_{12}}, \qquad (X.17)$$

$$\Phi_2 = \frac{\frac{2a_{h_2}}{\pi\lambda_2}\alpha_2\left(1 + \frac{a_{h_1}}{\pi\lambda_1}\zeta_1\right) - \frac{2a_{h_1}}{\pi\lambda_1}\alpha_1\frac{a_{h_2}}{\pi\lambda_2}\zeta_{21}}{\left(1 + \frac{a_{h_1}}{\pi\lambda_1}\zeta_1\right)\left(1 + \frac{a_{h_2}}{\pi\lambda_2}\zeta_2\right) - \frac{a_{h_2}}{\pi\lambda_2}\zeta_{21}\frac{a_{h_1}}{\pi\lambda_1}\zeta_{12}}, \qquad (X.18)$$

where

$$\zeta_1 = 1 + \frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_1(\bar{y}-\bar{\eta})\,d\bar{\eta}d\bar{y}, \qquad (X.19)$$

$$\zeta_2 = 1 + \frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_3(\bar{y}-\bar{\eta})\,d\bar{\eta}d\bar{y}, \qquad (X.20)$$

$$\zeta'_{12} = \frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_2(\bar{y}-\bar{\eta})\,d\bar{\eta}d\bar{y}, \qquad (X.21)$$

$$\zeta'_{21} = \frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_4(\bar{y}-\bar{\eta})\,d\bar{\eta}d\bar{y}.$$

Using formulas (X.13), (X.14), (X.19) and (X.21), the

coefficients C_{yj} and C_{xij} will be determined from the expressions

$$C_{yl}=\frac{a_{\bar{h}l}}{1+\frac{a_{\bar{h}l}}{\pi\lambda_l}\zeta_{ll}}\alpha_0, \qquad (X.22)$$

$$C_{xll}=\frac{C_{yl}^2}{\pi\lambda_l}\zeta_{ll'} \qquad (X.23)$$

where

$$\left.\begin{aligned}\zeta_{12}&=\zeta_1+\frac{\zeta_{12}'}{\overline{\Phi}}\\ \zeta_{21}&=\zeta_2+\overline{\Phi}\zeta_{21}'\\ \overline{\Phi}&=\frac{\Phi_1}{\Phi_2}\end{aligned}\right\}. \qquad (X.24)$$

The determination of functions ζ_1, ζ_2, ζ_{12} and ζ_{21} [419
by formulas (X.19), (X.20) and (X.21) involves very cumbersome calculations and, in addition, it is impossible to obtain their values in a closed form. Interesting results are obtained when determining them in the form of a series in powers of the $\tau=\sqrt{4a^2+1}-2a$ parameter. The expansion of nuclei $G_n(y-\eta)$ will be as follows:

$$G_l(\bar{y}-\bar{\eta})=\sum_{n=2,4\ldots}^{\infty}\tau^n\sum_{k=0,2\ldots}^{\frac{n}{2}-1}\frac{(n-1-k)\ldots(k+1)(n-1-2k)}{(n-1-2k)!}\times$$
$$\times(-1)^{\frac{n}{2}-k+1}(\bar{y}-\bar{\eta})^{n-2-2k}G_{nk}\left(\frac{\bar{\omega}}{\tau_l}\right) \qquad l=1,3,\ldots$$

$$G_l(\bar{y}-\bar{\eta})=\sum_{n=2,4\ldots}^{\infty}\tau_{l_1}^n\sum_{k=0,2\ldots}^{\frac{n}{2}-1}\frac{(n-1-k)\ldots(k+1)(n-1-2k)}{(n-1-2k)!}\times$$
$$\times(-1)^{\frac{n}{2}-k+1}(\bar{y}-\bar{\eta})^{n-2-2k}+\sum_{n=2,4\ldots}^{\infty}\tau_{l_2}\sum_{k=0,2\ldots}^{\frac{n}{2}-1}\frac{(n-1-k)\ldots(k+1)(n-1-2k)}{(n-1-2k)!}\times$$
$$\times(-1)^{\frac{n}{2}-k+1}(\bar{y}-\bar{\eta})^{n-2-2k}G_{nk}\left(\frac{\bar{\omega}}{\tau_{l_2}}\right), \qquad l=2,4\ldots \qquad (X.25)$$

where

$$\tau_1=\sqrt{4\bar{h}_1^2+1}-2\bar{h}_1; \qquad \tau_3=\sqrt{4k^2\bar{h}_2^2+1}-2k\bar{h}_2;$$

$$\tau_{21} = \sqrt{k^2(\bar{h}_1 - \bar{h}_2)^2 + 1} - k(\bar{h}_1 - \bar{h}_2);$$
$$\tau_{22} = \sqrt{k^2(\bar{h}_1 + \bar{h}_2)^2 + 1} - k(\bar{h}_1 + \bar{h}_2);$$
$$\tau_{41} = \sqrt{(\bar{h}_1 - \bar{h}_2)^2 + 1} - (\bar{h}_1 - \bar{h}_2);$$
$$\tau_{42} = \sqrt{(\bar{h}_1 + \bar{h}_2)^2 + 1} - (\bar{h}_1 + \bar{h}_2).$$

After performing a large volume of computations we obtain:

$$\begin{aligned}
\zeta_1 = {} & 1 + \frac{1}{2} G_{2,0}\left(\frac{\bar{\omega}}{\tau_1}\right)\tau_1^2 + \left[G_{4,1}\left(\frac{\bar{\omega}}{\tau_1}\right) - 0{,}7G_{4,0}\left(\frac{\bar{\omega}}{\tau_1}\right)\right]\tau_1^4 + \\
& + \left[1{,}5G_{6,2}\left(\frac{\bar{\omega}}{\tau_1}\right) - 3G_{6,1}\left(\frac{\bar{\omega}}{\tau_1}\right) + 1{,}5625G_{6,0}\left(\frac{\bar{\omega}}{\tau_1}\right)\right]\tau_1^6 + \\
& + [2G_{8,3} - 7{,}5G_{8,2} + 9{,}375G_{8,1} - 3{,}82809G_{8,0}]\tau_1^8 + \\
& + [2{,}5G_{10,4} - 15G_{10,3} + 32{,}8125G_{10,2} - 30{,}62472G_{10,1} + \\
& + 10{,}33587G_{10,0}]\tau_1^{10} + [3G_{12,5} - 26{,}25G_{12,4} + 87{,}5G_{12,3} - \\
& - 137{,}81124G_{12,2} + 103{,}3587G_{12,1} - 29{,}777336G_{12,0}]\tau_1^{12},
\end{aligned} \tag{X.26}$$ [420

$$\begin{aligned}
\zeta_2 = {} & 1 + \frac{1}{2} G_{2,0}\left(\frac{\bar{\omega}}{\tau_3}\right)\tau_3^2 + \left[G_{4,1}\left(\frac{\bar{\omega}}{\tau_3}\right) - 0{,}75G_{4,0}\left(\frac{\bar{\omega}}{\tau_3}\right)\right]\tau_3^4 + \\
& + [1{,}5G_{6,2} - 3G_{6,2} + 1{,}5625G_{6,0}]\tau_3^6 + [2G_{8,3} - 7{,}5G_{8,2} + \\
& + 9{,}375G_{8,1} - 3{,}82809G_{8,0}]\tau_3^8 + [2{,}5G_{10,4} - 15G_{10,3} + 32{,}8125G_{10,2} - \\
& - 30{,}62472G_{10,1} + 10{,}33587G_{10,0}]\tau_3^{10} + [3G_{12,5} - 26{,}25G_{12,4} + \\
& + 87{,}5G_{12,3} - 137{,}81124G_{12,2} + 103{,}3587G_{12,1} - 29{,}77733G_{12,0}]\tau_3^{12} + \ldots,
\end{aligned} \tag{X.27}$$

$$\begin{aligned}
\zeta_{12}^1 = {} & 0{,}5\tau_{12}^2 + 0{,}25\tau_{21}^4 + 0{,}0625\tau_{21}^6 + 0{,}00469\tau_{21}^8 + 0{,}0188\tau_{12}^{12} + \\
& + 0{,}5G_{2,0}\left(\frac{\bar{\omega}}{\tau_{22}}\right)\tau_{22}^2 + \left[G_{4,1}\left(\frac{\bar{\omega}}{\tau_{22}}\right) - 0{,}75G_{4,0}\left(\frac{\bar{\omega}}{\tau_{22}}\right)\right]\tau_{22}^4 + \\
& + [1{,}5G_{6,2} - 3G_{6,1} + 1{,}5625G_{6,0}]\tau_{22}^6 + [2G_{8,3} - 7{,}5_{8,2} + \\
& + 9{,}375G_{8,1} - 3{,}82809G_{8,0}]\tau_{22}^8 + [3G_{12,8} - 26{,}25G_{12,4} + 87{,}5G_{12,3} - \ldots - \\
& - 29{,}77733G_{12,0}]\tau_{22}^{12},
\end{aligned} \tag{X.28}$$

$$\begin{aligned}
\zeta_{21}^1 = {} & 0{,}5\tau_{41}^2 + 0{,}25\tau_{41}^4 + 0{,}0625\tau_{41}^6 + 0{,}0469\tau_{4,1}^8 + 0{,}0188\tau_{41}^{12} + \\
& + 0{,}5G_{2,0}\left(\frac{\bar{\omega}}{\tau_{42}}\right)\tau_{42}^2 + \left[G_{4,2}\left(\frac{\bar{\omega}}{\tau_{4,2}}\right) - 0{,}75G_{4,0}\left(\frac{\bar{\omega}}{\tau_{42}}\right)\right]\tau_{42}^4 + \\
& + [1{,}5G_{62} - 3G_{6,1} + 1{,}5625G_{6,2}]\tau_{42}^6 + [2G_{8,3} - 7{,}5G_{8,2} + 9{,}375G_{8,1} - \\
& - 3{,}82809G_{8,0}]\tau_{42}^8 + [2{,}5G_{10,4} - \ldots]\tau_{42}^{10} + [3G_{12,5} - \ldots]\tau_{42}^{12}.
\end{aligned} \tag{X.29}$$

When $\frac{\omega}{\tau} \to 0$, $G_{nm}(\frac{\omega}{\tau}) \to 1$, while for $\frac{\omega}{\tau} \to \infty$, $G_{nm} \to -1$.

In these instances the formulas will be as follows:

$$\zeta_1 = 1 \pm (0{,}5\tau_1^2 + 0{,}25\tau_1^4 + 0{,}0625\tau_1^6 + 0{,}0469\tau_1^8 + 0{,}0237\tau_1^{10} + 0{,}0188\tau_1^{12} + \ldots) \quad (X.30)$$

$$\zeta_2 = 1 \pm (0{,}5\tau_3^2 + 0{,}25\tau_3^4 + 0{,}0625\tau_3^6 + 0{,}0469\tau_3^8 + 0{,}0237\tau_3^{10} + 0{,}0188\tau_3^{12} + \ldots), \quad (X.31)$$

$$\zeta_{12}^1 = 0{,}5(\tau_{21}^2 \pm \tau_{22}^2) + 0{,}25(\tau_{21}^4 \pm \tau_{22}^4) + 0{,}0625(\tau_{21}^6 \pm \tau_{22}^6) + 0{,}0469(\tau_{21}^8 \pm \tau_{22}^8) + 0{,}0237(\tau_{21}^{10} \pm \tau_{22}^{10}) + 0{,}0188(\tau_{21}^{12} \pm \tau_{22}^{12}), \quad (X.32)$$ [422

$$\zeta_{21} = 0{,}5(\tau_{41}^2 \pm \tau_{42}^2) + 0{,}25(\tau_{41}^4 \pm \tau_{42}^4) + 0{,}0625(\tau_{41}^6 \pm \tau_{42}^6) + 0{,}0469(\tau_{41}^8 \pm \tau_{42}^8) + 0{,}0237(\tau_{41}^{10} \pm \tau_{42}^{10}) + 0{,}0188(\tau_{41}^{12} \pm \tau_{42}^{12}). \quad (X.33)$$

The + sign corresponds to $\omega \to 0$, while the - sign to $\omega \to \infty$. The values of the function ζ_{ij} for a number of b, $\bar{\varphi}$, $\bar{h}$ values are given in Tables 12-21. Figures 45-50 show curves which illustrate the effect of $\bar{h}$, Fr, b and $\bar{\Phi}$ on ζ_{ij} for the biplanar foils. [421

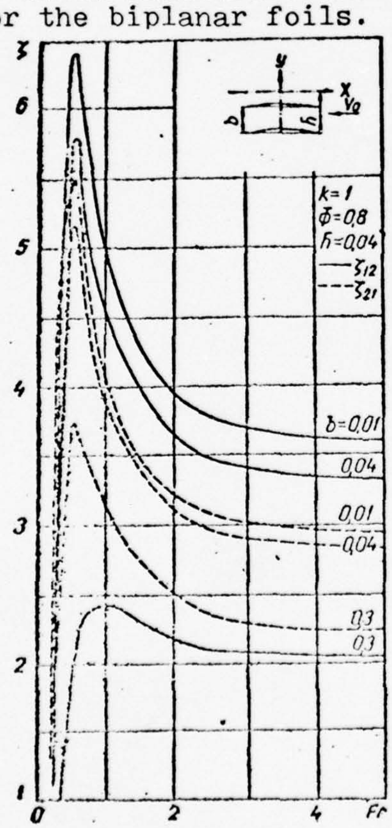

Fig. 46

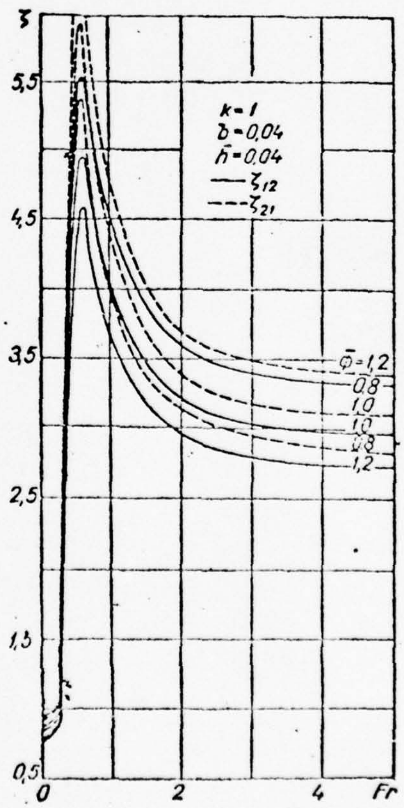

Fig. 47

[421

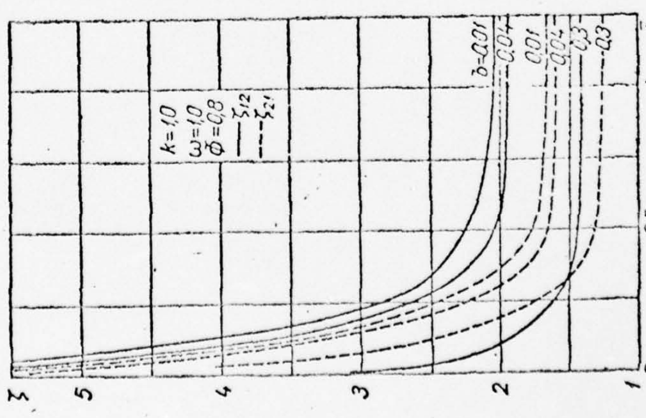

Fig. 48

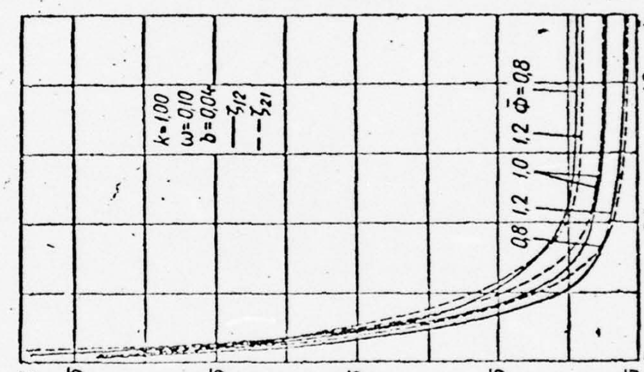

Fig. 49

[422

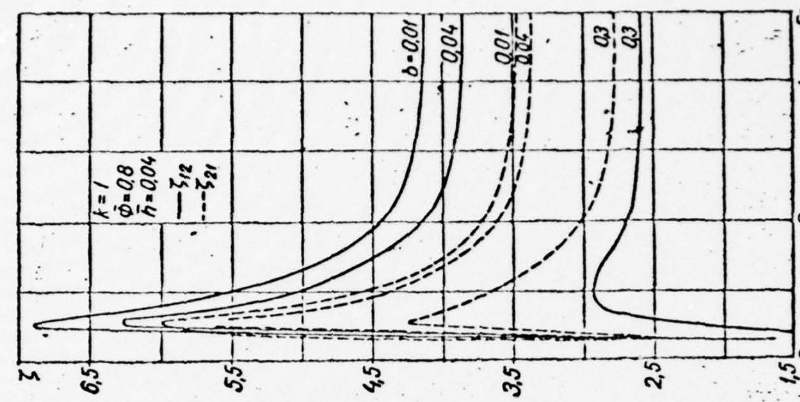

Fig. 50

Table 12

$k=1,\ \bar{\Phi}=1{,}2$

$\bar{h}$	b	Values of ζ_{21} for ω							
		0,02	0,1	0,6	1,0	2,0	3,0	4,0	7,0
0,04	0,01	3,5516	3,7896	5,1595	5,9491	6,4782	5,4475	3,8904	1,0744
	0,04	3,3844	3,6096	4,8989	5,6133	5,9987	4,9556	3,4879	0,9612
	0,3	2,5217	2,6929	3,5609	3,9336	3,8831	3,0707	2,1347	0,6538
0,1	0,01	3,1221	3,2938	4,1529	4,4820	4,2127	3,2236	2,2737	1,1252
	0,04	2,9758	3,1409	3,9488	4,2360	3,9126	2,9528	2,0677	1,0428
	0,3	2,2321	2,3605	2,8965	3,0003	2,5828	1,8990	1,3472	0,7724
0,3	0,01	2,4747	2,5491	2,6971	2,5581	2,0679	1,7852	1,6778	1,7035
	0,04	2,3591	2,4317	2,5671	2,4244	1,9467	1,6810	1,5831	—
	0,3	1,7804	1,8403	1,9068	1,7597	1,3833	1,2066	1,1469	—
0,5	0,01	2,2605	2,3012	2,2689	2,1234	1,9036	1,8509	1,8423	—
	0,04	2,1532	2,1930	2,1575	2,0137	1,8018	1,7526	1,7449	—
	0,3	1,6216	1,6557	1,6027	1,4762	1,3152	1,2828	1,2789	—
1,0	0,01	2,1131	2,1266	2,0477	2,0025	1,9826	1,9839	—	—
	0,04	2,0098	2,0231	1,9446	1,9004	1,8812	1,8829	—	—
	0,3	1,5046	1,5163	1,4411	1,4039	1,3894	1,3975	—	—

Table 13

$k=1,\ \bar{\Phi}=1$

$\bar{h}$	b	Values of ζ_{12} for ω							
		0,02	0,1	0,6	1,0	2,0	3,0	4,0	7,0
0,04	0,01	2,9472	3,1377	4,2359	4,8529	5,2127	4,3426	3,0912	0,9049
	0,04	2,7258	2,8944	3,8144	4,2674	4,3274	3,4501	2,4040	0,8333
	0,3	1,7418	1,8152	2,0192	1,9504	1,5339	1,1971	1,0172	1,1006
0,1	0,01	2,5997	2,7393	3,4277	3,6790	3,4231	2,6126	1,8567	0,9643
	0,04	2,4306	2,5556	3,1305	3,2907	2,9262	2,1893	1,5817	0,9594
	0,3	1,6471	1,7061	1,8155	1,7063	1,3383	1,1204	1,0298	1,8613
0,3	0,01	2,0767	2,1378	2,2544	2,1365	1,7342	1,5077	1,4235	—
	0,04	1,9760	2,0325	2,1212	1,9959	1,6232	1,4345	1,3795	—
	0,3	1,4793	1,5116	1,4812	1,3668	1,1980	1,1563	1,1496	—
0,5	0,01	1,9026	1,9361	1,9075	1,7871	1,6081	1,5660	1,5595	—
	0,04	1,8200	1,8516	1,8156	1,6996	1,5386	1,5041	1,4998	—
	0,3	1,4091	1,4291	1,3648	1,2881	1,2224	1,2153	1,2229	—
1,0	0,01	1,7819	1,7931	1,7275	1,6904	1,6743	1,6754	1,9241	—
	0,04	1,7087	1,7194	1,6516	1,6197	1,6052	1,6060	—	—
	0,3	1,3482	1,3554	1,2989	1,2785	1,2727	—	—	—

Table 14 [424

$k=1,\ \bar{\Phi}=1$

$\bar{h}$	b	Values of ζ_{21} for ω							
		0,02	0,1	0,6	1,0	2,0	3,0	4,0	7,0
0,04	0,01	3,2451	3,4590	4,7535	5,4265	5,9115	4,9753	3,5575	0,9891
	0,04	3,1033	3,3090	4,4893	5,1466	5,5120	4,5054	3,2220	0,8948
	0,3	2,3844	2,5451	3,3744	3,7468	3,7491	2,9946	2,0944	0,6386
0,1	0,01	2,8512	3,0075	3,7899	4,0903	3,8470	2,9472	2,0819	1,0335
	0,04	2,7293	2,8801	3,6198	3,8852	3,5969	2,7213	1,9118	0,9648
	0,3	2,1096	2,2297	2,7429	2,8555	2,4888	1,8433	1,3098	0,7395
0,3	0,01	2,2617	2,3296	2,4644	2,3381	1,8922	1,6348	1,5369	1,56085
	0,04	2,1655	2,2317	2,3561	2,2268	1,7912	1,5480	1,4580	—
	0,3	1,6833	1,7389	1,8059	1,6728	1,3217	1,1526	1,0945	—
0,5	0,01	2,0669	2,1039	2,0747	1,9424	1,7423	1,6943	1,6865	—
	0,04	1,9775	2,0138	1,9818	1,8510	1,6575	1,6124	1,6053	—
	0,3	1,5345	1,5660	1,5195	1,4030	1,2519	1,2209	1,2170	—
1,0	0,01	1,9328	1,9450	1,8733	1,8322	1,8141	—	—	—
	0,04	1,8467	1,8588	1,7874	1,7471	1,7297	—	—	—
	0,3	1,4257	1,4365	1,3678	1,3334	1,3198	—	—	—

Table 15

Biplane $k=1,\ \bar{\Phi}=1$

h	b	Values of ζ_{12} for ω							
		0,02	0,1	0,6	1,0	2,0	3,0	4,0	7,0
0,04	0,01	3,2052	3,4133	4,6134	5,2885	5,6848	4,7361	5,3687	0,9758
	0,04	2,9601	3,1449	4,1556	4,6563	4,7330	3,7753	2,6256	0,8888
	0,3	1,8563	1,9384	2,1747	2,1060	1,6457	1,2605	1,0508	1
0,1	0,01	2,8254	2,9779	3,7302	4,0055	3,7278	2,8429	2,0166	1,0402
	0,04	2,6360	2,7729	3,4047	3,5831	3,1892	2,3820	1,7133	1,0241
	0,3	1,7492	1,8150	1,9435	1,8270	1,4166	1,1667	1,0610	0,7395
0,3	0,01	2,2541	2,3208	2,4483	2,3191	1,8896	1,6331	1,5410	1,6170
	0,04	2,1374	2,1992	2,2971	2,1607	1,7528	1,5454	1,4748	1,6753
	0,3	1,5602	1,5961	1,5654	1,4392	1,2493	1,2013	1,1933	11,4261
0,5	0,01	2,0639	2,1006	2,0694	1,9380	1,7425	1,6965	1,6894	—
	0,04	1,9665	2,0009	1,9620	1,8353	1,6588	1,6210	1,6162	—
	0,3	1,4817	1,5030	1,4342	1,3491	1,2751	1,2669	1,2747	—
1,0	0,01	1,9322	1,9444	1,8728	1,8323	1,8146	1,8159	—	—
	0,04	1,8146	1,8563	1,7856	1,7474	1,7316	1,7326	—	—
	0,3	1,4140	1,4220	1,3600	1,3373	1,3307	—	—	—

Table 16

Biplane $k=1,\ \bar{\Phi}=0{,}8$

h	b	Values of ζ_{12} for ω							
		$\omega=0{,}02$	$\omega=0{,}1$	$\omega=0{,}6$	$\omega=1{,}0$	$\omega=2{,}0$	$\omega=3{,}0$	$\omega=4{,}0$	$\omega=7{,}0$
0,04	0,01	3,5921	3,8266	5,1796	5,9418	6,3931	5,3264	3,7849	1,0822
	0,04	3,3116	3,5207	4,6676	5,2397	5,3413	4,2631	2,9579	0,9721
	0,3	2,0280	2,1233	2,4079	2,3394	1,8133	1,3556	1,1012	1,1346
0,1	0,01	3,1640	3,3358	4,1840	4,4952	4,1849	3,1884	2,2564	1,1539
	0,04	2,9141	3,0990	3,8159	4,0215	3,5838	2,6711	1,9105	1,1212
	0,3	1,9024	1,9785	2,1355	2,0080	1,5341	1,2362	1,1077	—
0,3	0,01	2,5202	2,5953	2,7392	2,5946	2,1002	1,8212	1,7172	—
	0,04	2,3794	2,4492	2,5609	2,4077	1,9471	1,7118	1,6313	—
	0,3	1,6817	1,7229	1,6916	1,5478	1,3263	1,2688	1,2589	—
0,5	0,01	2,3060	2,3472	2,3122	2,1843	1,9441	1,7923	1,8842	—
	0,04	2,1861	2,2250	2,1816	2,0387	1,8393	1,7962	1,7907	—
	0,3	1,5906	1,6160	1,5382	1,4406	1,3541	1,3443	1,3522	—
1,0	0,01	2,1576	2,1713	2,0909	2,0451	2,0252	2,0267	2,3752	—
	0,04	2,0485	2,0616	1,9821	1,9390	1,9211	1,9225	2,3516	—
	0,3	1,5126	1,5217	1,4516	1,4254	1,4178	1,4718	—	—

Table 17

$k=1,\ \bar{\Phi}=0{,}8$

$\bar{h}$	b	Values of ζ_{21} for ω							
		0,02	0,1	0,6	1,0	2,0	3,0	4,0	7,0
0,04	0,01	2,9355	3,1283	4,2536	4,9038	5,3451	4,5031	3,2215	0,9038
	0,04	2,8221	3,0083	4,0798	4,6799	5,0253	4,1751	2,9561	0,8284
	0,3	2,2470	2,3972	3,1879	3,5601	3,6150	2,9185	2,0540	0,6234
0,1	0,01	2,5804	2,7212	3,4268	3,6985	3,4814	2,6709	1,8900	0,9418
	0,04	2,4828	2,6193	3,2908	3,5345	3,2813	2,4903	1,7540	0,8869
	0,3	1,9872	2,0990	2,5893	2,7107	2,3948	1,7878	1,2724	—
0,3	0,01	2,0489	2,1100	2,2317	2,1182	1,7166	1,4843	1,3959	—
	0,04	1,9719	2,0317	2,1451	2,0291	1,6358	1,4149	1,3328	—
	0,3	1,5861	1,6375	1,7049	1,5859	1,2601	1,0986	1,0420	—
0,5	0,01	1,8733	1,9067	1,8804	1,7613	1,5810	1,5377	1,5306	—
	0,04	1,8017	1,8346	1,8061	1,6882	1,5132	1,4722	1,4657	—
	0,3	1,4474	1,4763	1,4363	1,3298	1,1887	1,1596	1,1551	—
1,0	0,01	1,7524	1,7635	1,6970	1,6619	1,6456	1,6467	—	—
	0,04	1,6836	1,6916	1,6302	1,5938	1,5789	1,5793	1,9176	—
	0,3	1,3468	1,3567	1,2916	1,2629	1,2502	1,2558	—	—

Table 18

$h_2 = 0.02,\ h_1 = h_2 + b$

b	$\bar{\omega}$	Values of ζ_{12} for ω															
		0,02	0,05	0,10	0,20	0,40	0,60	0,80	0,90	1,00	1,50	2,00	2,50	3,00	4,00	5,00	6,00
0,01	0,6	4,5606	4,7126	4,9708	5,4894	6,4815	7,3588	8,0826	8,3793	8,6302	9,1957	8,7635	7,6911	6,3517	3,8519	2,1801	1,2741
	0,8	3,8556	3,9802	4,1919	4,6172	5,4318	6,1547	6,7546	7,0024	7,2134	7,7144	7,4054	6,5638	5,4856	3,4177	1,9849	1,1781
	1,0	3,4326	3,5408	3,7245	4,0938	4,8020	5,4322	5,9578	6,1762	6,3633	6,8256	6,5904	5,8875	4,9659	3,1572	1,8677	1,1205
	1,2	3,1506	3,2478	3,4130	3,7449	4,3821	4,9505	5,4266	5,6254	5,7966	6,2331	6,0472	5,4366	4,6195	2,9835	1,7896	1,0820
0,015	0,6	4,5062	4,6560	4,9102	5,4200	6,3916	7,2457	7,9446	8,2289	8,4656	8,9800	8,5171	7,4391	6,1150	3,6790	2,0729	1,2329
	0,8	3,8093	3,9320	4,1401	4,5578	5,3545	6,0570	6,6352	6,8720	7,0723	7,5260	7,1882	6,3397	5,2728	3,2591	1,8855	1,1375
	1,0	3,3911	3,4975	3,6781	4,0404	4,7322	5,3438	5,8496	6,0579	6,2352	6,6535	6,3909	5,6800	4,7675	3,0072	1,7730	1,0802
	1,2	3,1124	3,2079	3,3701	3,6955	4,3174	4,8683	5,3258	5,5152	5,6771	6,0719	5,8593	5,2402	4,4306	2,8392	1,6981	1,0420
0,02	0,6	4,4531	4,6007	4,8511	5,3524	6,3040	7,1353	7,8103	8,0826	8,3094	8,7713	8,2798	7,1979	5,8899	3,5167	1,9813	1,1745
	0,8	3,7642	3,8850	4,0898	4,4999	5,2793	5,9619	6,5192	6,7455	6,9354	7,3439	6,9796	6,1256	5,0710	3,1107	1,7997	1,0851
	1,0	3,3508	3,4555	3,6330	3,9884	4,6644	5,2579	5,7445	5,9432	6,1109	6,4875	6,1994	5,4822	4,5796	2,8671	1,6907	1,0315
	1,2	3,0753	3,1692	3,3285	3,6474	4,2545	4,7885	5,2280	5,4084	5,5613	5,9166	5,6793	5,0532	4,2520	2,7046	1,6180	0,9957
0,025	0,6	4,4013	4,5468	4,7934	5,2864	6,2186	7,0278	7,6795	7,9401	8,1556	8,5692	8,0513	6,9670	5,6758	3,3646	1,8931	1,1331
	0,8	3,7203	3,8392	4,0407	4,4436	5,2060	5,8694	6,4064	6,6225	0,8024	7,1680	6,7790	5,9211	4,8795	2,9720	1,7181	1,0468
	1,0	3,3116	3,4146	3,5891	3,9379	4,5984	5,1744	5,6425	5,8319	5,9905	6,3274	6,0157	5,2935	4,4017	2,7365	1,6130	0,9950
	1,2	3,0392	3,1315	3,2880	3,6008	4,1934	4,7110	5,1332	5,3048	5,4492	5,7669	5,5068	4,8752	4,0832	2,5794	1,5430	0,9604
0,03	0,6	4,3508	4,4942	4,7371	5,2220	6,1352	6,9229	7,5521	7,8035	8,0060	8,3734	7,8311	6,7458	5,4720	3,2223	1,8129	1,1042
	0,8	3,6775	3,7946	3,9929	4,3887	5,1346	5,7793	6,2967	6,5030	6,6733	6,9980	6,5863	5,7257	4,6977	2,8427	1,6440	1,0187
	1,0	3,2735	3,3748	3,5463	3,8887	4,5342	5,0932	5,5434	5,7239	5,8737	6,1728	5,8394	5,1136	4,2332	2,6149	1,5426	0,9673
	1,2	3,0042	3,0949	3,2486	3,5554	4,1339	4,6358	5,0412	5,2045	5,3406	5,6226	5,3415	4,7056	3,9235	2,4630	1,4751	0,9331

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Table 19

$h_2 = 0.02,\ h_1 = h_2 + b,\ k = 0.7$

b	$\bar{\Phi}$	Values of ζ_{21} for ω															
		0,02	0,05	0,1	0,2	0,4	0,6	0,8	0,9	1,0	1,5	2,0	2,5	3,0	4,0	5,0	6,0
0,01	0,6	2,8104	2,9036	3,0622	3,3814	3,9956	4,5438	5,0010	5,1905	5,3523	5,7393	5,5054	4,8629	4,0413	2,4792	1,4146	0,8301
	0,8	3,1372	3,2392	3,4125	3,7617	4,4345	5,0370	5,5425	5,7535	5,9350	6,3899	6,1729	5,5034	4,6230	2,9014	1,6853	0,9910
	1,0	3,4640	3,5747	3,7629	4,1421	4,8734	5,5302	6,0840	6,3166	6,5177	7,0404	6,8405	6,1438	5,2047	3,3236	1,9560	1,1519
	1,2	3,7907	3,9102	4,1133	4,5224	5,3125	6,0234	6,6255	6,8796	7,1003	7,6910	7,5080	6,7842	5,7864	3,7459	2,2266	1,3128
0,015	0,6	2,7944	2,8873	3,0452	3,3631	3,9741	4,5186	4,9719	5,1593	5,3191	5,6963	5,4549	4,8084	3,9863	2,4317	1,3798	0,8074
	0,8	3,1159	3,2174	3,3899	3,7373	4,4058	5,0034	5,5037	5,7120	5,8907	6,3326	6,1056	5,4306	4,5497	2,8381	1,6389	0,9608
	1,0	3,4373	3,5475	3,7346	4,1115	4,8376	5,4882	6,0354	6,2646	6,4623	6,9688	6,7563	6,0529	5,1131	3,2445	1,8980	1,1141
	1,2	3,7588	3,8776	4,0793	4,4857	5,2693	5,9730	6,5672	6,8173	7,0339	7,6050	7,4070	6,6752	5,6764	5,6510	2,1571	1,2675
0,02	0,6	2,7788	2,8713	3,0286	3,3451	3,9530	4,4939	4,9434	5,1288	5,2866	5,6543	5,4057	4,7554	3,9330	2,3860	1,3465	0,7858
	0,8	3,0951	3,1961	3,3678	3,7133	4,3777	4,9705	5,4656	5,6713	5,8474	6,2766	6,0400	5,3600	4,4786	2,7772	1,5944	0,9319
	1,0	3,4113	3,5209	3,7070	4,0816	4,8024	5,4471	5,9879	6,2138	6,4081	6,8988	6,6744	5,9646	5,0243	3,1684	1,8424	1,0181
	1,2	3,7276	3,8456	4,0462	4,4498	5,2271	5,9237	6,5102	6,7562	6,9689	7,5211	7,3087	6,5691	5,5699	3,5595	2,0904	1,2242
0,025	0,6	2,7635	2,8557	3,0124	3,3275	3,9324	4,4698	4,9154	5,0990	5,2548	5,6133	5,3578	4,7039	3,8814	2,3420	1,3145	0,7650
	0,8	3,0747	3,1753	3,3462	3,6809	4,3502	4,9383	5,4284	5,6315	5,8050	6,2219	5,9761	5,2913	4,4098	2,7184	1,5518	0,9042
	1,0	3,3859	3,4948	3,6799	4,0523	4,7680	5,4069	5,9414	6,1640	6,3551	6,8305	6,5945	5,8787	4,9382	3,0949	1,7892	1,0434
	1,2	3,6970	3,8144	4,0137	4,4147	5,1859	5,8755	6,4544	6,6965	6,9053	7,4391	7,2128	6,4661	5,4666	3,4714	2,0265	1,1826
0,03	0,6	2,7486	2,8404	2,9965	5,3104	3,9122	4,4462	4,8881	5,0698	5,2237	5,5733	5,3111	4,6538	3,8314	2,2995	1,2839	0,7454
	0,8	3,0548	3,1549	3,3250	3,6670	4,3233	4,9068	5,3920	5,5925	5,7635	6,1685	5,9139	5,2245	4,3431	2,6619	1,5111	0,8781
	1,0	3,3610	3,4694	3,6535	4,0237	4,7344	5,3675	5,8959	6,1153	6,3033	6,7638	6,5166	5,7953	4,8548	3,0242	1,7382	1,0107
	1,2	3,6672	3,7838	3,9820	4,3804	5,1455	5,8282	6,3997	6,6381	6,8430	7,3590	7,1194	6,3660	6,3665	3,3866	1,9653	1,1434

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Table 20

$h_2 = 0.02,\ h_1 = h_2 + b,\ k = 0.8$

b	$\bar{\Theta}$	Values of ξ_{12} for ω														
		0,02	0,05	0,10	0,20	0,40	0,60	0,80	0.90	1.00	1.5	2.00	2.50	3.00	4.00	5.00	6.00
0,01	0,6	4,5259	4,6625	4,8946	5,3619	6,2639	7,0788	7,7750	8,0723	8,3333	9,0753	8,9422	8,1558	7,0032	4,5351	[illegible]	[illegible]
	0,8	3,8295	3,9427	4,1347	4,5215	5,2686	5,9446	6,5239	6,7721	6,9908	7,6245	7,5393	6,9124	5,9743	3,9301	2,5367	1,3514
	1,0	3,4117	3,5107	3,6788	4,0173	4,6715	5,2641	5,7733	5,9920	6,1852	6,7537	6,6976	6,1664	5,3569	3,5671	2,1492	1,2671
	1,2	3,1332	3,2228	3,3749	3,6812	4,2734	4,8105	5,2728	5,4719	5,6482	6,1732	6,1365	5,6690	4,9453	3,3251	2,0242	1,2042
0,015	0,6	4,4674	4,6019	4,8300	5,2887	6,1709	6,9630	7,6349	7,9197	8,1685	8,8554	8,6850	7,8860	6,7430	4,3355	2,5242	1,4528
	0,8	3,7802	3,8914	4,0800	4,4593	5,1890	5,8450	6,4029	6,6402	6,8480	7,4325	7,3141	6,6748	5,7438	3,7515	2,2240	1,3023
	1,0	3,3679	3,4651	3,6300	3,9616	4,5998	5,1742	5,6637	5,8724	6,0557	6,5787	6,4916	5,9481	5,1443	3,4011	2,0438	1,2121
	1,2	3,0030	3,1809	3,3300	3,6298	4,2070	4,7270	5,1709	5,3606	5,5275	6,0096	5,9432	5,4637	4,7446	3,1674	1,9237	1,1519
0,02	0,6	4,4104	4,5427	4,7671	5,2175	6,0802	6,8502	7,4985	7,7714	8,0083	8,6420	8,4373	7,6276	6,4952	4,1478	2,4086	1,3979
	0,8	3,7322	3,8415	4,0268	4,3987	5,1114	5,7481	6,2853	6,5121	6,7095	7,2470	7,0976	6,4478	5,5249	3,5840	2,1201	1,2526
	1,0	3,3252	3,4207	3,5826	3,9075	4,5301	5,0868	5,5574	5,7565	5,9303	6,4100	6,2939	5,7399	4,9428	3,2457	1,9470	1,1655
	1,2	3,0539	3,1402	3,2865	3,5800	4,1426	4,6460	5,0721	5,2528	5,4107	5,8520	5,7580	5,2680	4,5547	3,0202	1,8317	1,1074
0,025	0,6	4,3548	4,4851	4,7058	5,1480	5,9919	6,7404	7,3658	7,6272	7,8527	8,4356	8,1987	7,3800	6,2591	3,9713	2,3014	1,3439
	0,8	3,6854	3,7929	3,9750	4,3398	5,0360	5,6539	6,1711	6,3878	6,5752	7,0678	6,8896	6,2308	5,3170	3,4270	2,0243	1,2048
	1,0	3,2837	3,3776	3,5365	3,8548	4,4624	5,0019	5,4543	5,6442	5,8087	6,2472	6,1042	5,5413	4,7517	3,1005	1,8580	1,1214
	1,2	3,0160	3,1007	3,2442	3,5316	4,0800	4,5673	4,9764	5,1484	5,2977	5,7001	5,5805	5,0817	4,3748	2,8828	1,7472	1,0658
0,03	0,6	4,3007	4,4288	4,6460	5,0802	5,9058	6,6334	7,2367	7,4869	7,7014	8,2357	7,9689	7,1427	6,0341	3,8052	2,2027	1,2918
	0,8	3,6399	3,7456	3,9245	4,2824	4,9625	5,5622	6,0601	6,2671	6,4448	6,8947	6,6897	6,0234	5,1193	3,2798	1,9363	1,1593
	1,0	3,2434	3,3356	3,4916	3,8036	4,3965	4,9195	5,3542	5,5352	5,6909	6,0902	5,9221	5,3518	4,5705	2,9646	1,7765	1,0799
	1,2	2,9791	3,0623	3,2030	3,4845	4,0192	4,4910	4,8835	5,0472	5,1883	5,5538	5,4104	4,9041	4,2046	2,7545	1,6699	1,0269

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Table 21

$h_2 = 0.02,\ h_1 = h_2 + b,\ k = 0.8$

b	$\overline{\Phi}$	Values of ξ_{21} for ω															
		0,02	0,05	0,10	0,20	0,40	0,60	0,8	0,90	1,00	1,50	2,00	2,50	3,00	4,00	5,00	6,00
0,01	0,6	2,7947	2,8787	3,0215	3,3097	3,8693	4,3792	4,8194	5,0092	5,1773	5,6738	5,6263	5,1643	4,4620	2,9226	1,7225	0,9884
	0,8	3,1215	3,2142	3,3719	3,6900	4,3082	4,8724	5,3609	5,5723	5,7599	6,3244	6,2939	5,8048	5,0437	3,3448	1,9932	1,1493
	1,0	3,4483	3,5408	3,7222	4,0704	4,7471	5,3656	5,9024	6,1353	6,3426	6,9749	6,9614	6,4452	5,6254	3,7670	2,2639	1,3102
	1,2	3,7751	3,8853	4,0726	4,4507	5,1860	5,8588	6,4439	6,6983	6,9253	7,6255	7,6290	7,0856	6,2071	4,1892	2,5346	1,4711
0,015	0,6	2,7788	2,8624	3,0045	3,2913	3,8477	4,3540	4,7903	4,9781	5,1440	5,6308	5,5759	5,1098	4,4070	2,8751	1,6877	0,9657
	0,8	3,1002	3,1925	3,3493	3,6656	4,2795	4,8388	5,3221	5,5307	5,7156	6,2671	6,2266	5,7320	4,9704	3,2815	1,9468	1,1190
	1,0	3,4217	3,5226	3,6940	4,0398	4,7112	5,3236	5,8539	6,0833	6,2872	6,9033	6,8773	6,3543	5,5337	3,6879	2,2059	1,2724
	1,2	3,7431	3,8526	4,0387	4,4140	5,1429	5,8084	6,3857	6,6360	6,8588	7,5395	7,5280	6,9766	6,0971	4,0943	2,4650	1,4257
0,02	0,6	2,7631	2,8164	2,9880	3,2734	3,8266	4,3293	4,7618	4,9476	5,1115	5,5888	5,5267	5,0568	4,3537	3,8294	1,6544	0,9441
	0,8	3,0794	3,1712	3,3271	3,6416	4,2514	4,8059	2,2841	5,4900	5,6723	6,2111	6,1610	5,6614	4,8993	3,2206	1,9024	1,0902
	1,0	3,3956	3,4959	3,6663	4,0099	4,6761	5,2825	5,8063	6,0325	6,2331	6,8333	6,7953	6,2659	5,4449	3,6117	2,1503	1,2364
	1,2	3,7119	3,8207	4,0055	4,3781	5,1008	5,7591	6,3286	6,5750	6,7938	7,4556	7,4296	6,8705	5,9905	4,0029	2,3083	1,3825
0,025	0,6	2,7479	2,8308	2,9717	3,2558	3,8060	4,3052	4,7339	4,9177	5,0797	5,5478	5,4788	5,0052	4,3021	2,7853	1,6224	0,9233
	0,8	3,0590	3,1504	3,3055	3,6182	4,2238	4,7737	5,2469	5,4502	5,6299	6,1564	6,0971	5,5927	4,8305	[illegible]	[illegible]	[illegible]
	1,0	3,3702	3,4699	3,6393	3,9806	4,6417	5,2423	5,7598	5,9828	6,1801	6,7650	6,7154	6,1801	5,3589	[illegible]	[illegible]	[illegible]
	1,2	3,6814	3,7895	3,9731	4,3430	5,0595	5,7108	6,2728	6,5153	6,7302	7,3736	7,3338	6,7675	5,8873	3,9148	[illegible]	[illegible]
0,03	0,6	2,7329	2,8155	2,9559	3,2387	3,7858	4,2816	4,7066	4,8885	5,0486	5,5078	5,4321	4,9552	4,2520	2,7420	1,5918	[illegible]
	0,8	3,0391	3,1300	3,2843	3,5953	4,1969	4,7422	5,2104	5,4113	5,5884	6,1030	6,0348	5,5259	4,7638	3,1053	1,8190	[illegible]
	1,0	3,3453	3,4445	3,6128	3,9520	4,6080	5,2029	5,7143	5,9341	6,1282	6,6982	6,6376	[illegible]	5,2755	3,4676	2,0464	1,1600
	1,2	3,6515	3,7589	3,9413	4,3086	5,0191	5,6636	6,2182	6,4568	6,6680	7,2935	7,2404	6,6674	5,7872	3,8300	2,2732	1,3017

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10.3. Motion of a Tandem Hydrofoil System Under the Free Surface of a Fluid of Infinite Depth [426

The tandem system with arbitrary distances between the hydrofoils can be analyzed with the aid of the general system of equations [X.5]. However, with the separation of approximately four chords between the hydrofoils, the effect of the stern hydrofoil on the bow hydrofoil becomes small. The effect of the bow hydrofoil on the stern hydrofoil can be described by asymptotic values when $x \to \infty$. Most of the units have considerably larger separations between the foils. Because of this, it is necessary to perform a more comprehensive analysis of the case with a considerable separation between the foils. In this problem the characteristics of the bow hydrofoil will be determined as characteristics of an isolated hydrofoil and the motion of this system will be described not by a system of equations, but simply by two equations. At large distances behind the hydrofoil the expression for φ_{1z} will be

$$\varphi_{1z} = -\frac{1}{2\pi}\int_{-1}^{+1}\Gamma(\eta)\left(\frac{[(y-\eta)^2-(z-\zeta)^2}{[(y-\eta)^2+(z-\zeta)^2]^2} + \int_0^\infty \lambda e^{\lambda(z+\zeta)}\cos(y-\eta)\lambda d\lambda - \right.$$
$$\left. - 2\int_\nu^\infty \frac{e^{\lambda(z+\zeta)}\cos\lambda\left[(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}\right.}{\sqrt{1-\frac{\nu}{\lambda}}}\cos(x-a)\sqrt{\lambda\nu}d\lambda\right)d\eta, \qquad (X.34)$$ [427

and then the integro-differential equation for the stern hydrofoil will be as follows:

$$\Gamma_2(y) = v_0 a(y) a_{h_2}^{-}\left\{a_2(y) - \frac{1}{4\pi v_0}\left[\int_{-b_2}^{+b_2}\frac{\Gamma_2'(\eta)}{y-\eta}d\eta + \right.\right.$$
$$+ 2\int_{-b_1}^{+b_1}\Gamma_1(\eta)\left(\frac{(y-\eta)^2+(h_1-h_2)^2}{[(y-\eta)^2+(h_1-h_2)^2]^2} + \int_0^\infty e^{\lambda(h_1+h_2)}\lambda\cos\lambda(y-\eta)\,d\lambda - \right.$$ [428
$$\left.\left.\left. - \int_\nu^\infty \frac{\lambda e^{-\lambda(h_1+h_2)}\cos\lambda(y-\eta)\sqrt{1-\frac{\nu}{\lambda}}}{\sqrt{1-\frac{\nu}{\lambda}}}\cos L\sqrt{\lambda\nu}d\lambda\right)d\eta\right]\right\}, \qquad (X.35)$$

where L is the separation between the hydrofoils of the system.

Analyzing the approximate solution for the hydrofoil with the elliptical distribution of circulation in an infinite fluid for the function ζ_{21L} in formula (X.24) we have the following:

$$\zeta'_{21L} = \zeta'_{21\infty} + \Delta\zeta'_{21} = -\frac{4\omega^2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,d\bar{\eta}\,\times$$

$$\times\int_0^\infty e^{-4\omega h_{cp}(\lambda+1)}\frac{(\lambda+1)^{\frac{3}{2}}\cos\omega\sqrt{(\lambda+1)\lambda}(y-\eta)(\cos\bar{\omega}L\sqrt{\lambda+1}-1)}{\sqrt{\lambda}}\,d\lambda, \quad \text{(X.36)}$$ [429

where $\zeta'_{21\infty} = 2\zeta'_{21}$, and ζ_{21} is defined by formula (X.29).

Performing the integration in formula (X.36) with respect to y and η we obtain

$$\Delta\zeta_{21} = -\frac{16}{\pi^2}\Gamma^2\left(\frac{3}{2}\right)\Gamma^2\left(\frac{1}{2}\right)\int_0^\infty e^{-4\omega\bar{h}_{cp}(\lambda+1)}\frac{J_1^2(\omega\sqrt{(\lambda+1)\lambda})}{\lambda^{\frac{5}{2}}}\,\times$$

$$+(\cos\bar{\omega}\bar{L}\sqrt{\lambda+1}-1)\,d\lambda. \quad \text{(X.37)}$$

With C_{y2} = const, the extreme distance between the hydrofoils will be determined by the condition

$$\frac{\partial\zeta_{21}}{\partial L} = 0, \quad \text{(X.38)}$$

which leads us to the functional equation [432

$$\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,d\bar{\eta}\int_0^\infty e^{-4\omega\bar{h}_{cp}(\lambda+1)}\,\times$$

$$\times\int_0^\infty\frac{(\lambda+1)^2}{\sqrt{\lambda}}\cos\bar{\omega}\sqrt{(\lambda+1)\lambda}(\bar{y}-\bar{\eta})\sin\bar{\omega}\bar{L}\sqrt{\lambda+1}\,\Big]\,d\lambda = 0. \quad \text{(X.39)}$$

Integrating, we obtain

$$\int_0^\infty e^{-4\omega\bar{h}_{cp}(\lambda+1)}\frac{(\lambda+1)}{\lambda^{\frac{3}{2}}}J_1^2(\omega\sqrt{(\lambda+1)\lambda})\sin(\bar{\omega}\bar{L}\sqrt{\lambda+1})\,d\lambda = 0. \quad \text{(X.40)}$$

The equation (X.40) has one trivial solution: $\bar{L} = 0$. We can look for the approximate solution of equation (X.40)

if we express it in the form of an algebraic equation of infinite order

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{g=0}^{\infty}\frac{(-1)^{k+p}\Gamma(3+2k)\tau^{2g+2p+2k}\left(\frac{\omega}{\tau}\right)^{g+2p+2k}}{\Gamma(3+k)\,\Gamma^2(2+k)\,g!\,(2p+1)!2^{2k}}\,\overline{L}^{2p}\times$$

$$\times\, k_{p,k,g}\left(\frac{\overline{\omega}}{\tau}\right)=0, \qquad (X.41)$$

$$k_{p,k,g}\left(\frac{\overline{\omega}}{\tau}\right)=\int_0^{\infty}e^{-\frac{\omega}{\tau}\lambda}(\lambda+1)^{\frac{5}{2}+p+k}\lambda^{-\frac{1}{2}+k+g}\,d\lambda. \qquad (X.42)$$

CHAPTER XI. THE THEORY OF THE SUBMERGED HYDROFOIL IN A THREE-DIMENSIONAL UNSTEADY STATE [433

11.1. General Considerations. Boundary Conditions

The theory of lifting surfaces in an unsteady infinite fluid flow has been developed rather well.

The lifting line theory in an unsteady flow has been developed in [102, 116-118] and others. In the studies by S. M. Belotserkovskiy [2, 3] the solutions are given which take into account the distribution of vortices along the chord. The theory for short-span hydrofoils in an unsteady fluid flow has been developed in studies by Lawrence and Herbert [76]. For the solution of the latter problem Küssner [186, 235] has adopted the acceleration potential method. However, the results dealing with the unsteady motion of submerged hydrofoils in the three-dimensional fluid flow have not yet been published in literature.

This chapter develops the theory for the submerged hydrofoil in an unsteady infinite flow. The problems of the unsteady hydrofoil motion in fluids of different densities will be analyzed in the next chapter.

It has already been established in Chapter VIII that the boundary conditions on the free surface for the velocity and acceleration potentials are similar. For the case of an ideal incompressible fluid the acceleration-potential boundary conditions on the free surface have the form (in a fixed coordinate system)

$$\theta_u + g\theta_z = 0. \qquad \text{(XI.1)}$$

$$z = 0.$$

Let us study only such unsteady motions which are characterized by the finite values of the average forward velocity v_0 and small unsteady velocity increments. For this type of motion the acceleration potential can be found in the form:

$$\Theta(x, y, z, t) = \Theta_1(x, y, z) + \Theta_2(x, y, z, t), \qquad \text{(XI.2)}$$

where $\theta_1(x, y, z)$ is the acceleration potential for the steady forward motion; $\theta_2(x, y, z, t)$ is the acceleration potential for the unsteady motion. [434

Motions which are characterized by the potential θ_1

have been studied rather fully in Chapters VIII-X. In this chapter we will study the motion which is described by the potential θ_2. Without limiting the scope of the general considerations, one may study only the unsteady motions which periodically change with time. Let us assume that for the case of the periodic motion we have

$$\Theta_2(x, y, z, t) = \Theta(x, y, z)\, e^{i\omega t}.$$

Then, the velocity potential will be defined by the formula (VII.5)

$$\varphi = -\frac{e^{ipx}}{v_0}\int_{\infty}^{x}\Theta(x, y, z)\, e^{-ip\tau}\, d\tau, \qquad p = \frac{\omega}{v_0}.$$

Now we can state the boundary conditions for the potentials θ and φ as follows:

$$\Theta_z - 2i\tau_0(1 - i\beta)\Theta_x - \nu_1(1 - 2i\beta)\Theta + \frac{\tau_0^2}{\nu_1}\Theta_{xx} = 0, \quad z = 0 \qquad \text{(XI.3)}$$

$$\beta = \frac{\mu}{2\omega}; \qquad \tau_0 = \frac{v_0\omega}{g}; \qquad \nu_1 = \frac{\omega^2}{g}, \qquad \lim_{\substack{x\to\infty \\ z\to\pm\infty}}\Theta = 0, \qquad \text{(XI.4)}$$

$$\varphi_z = v_z \text{ on } s, \qquad \text{(XI.5)}$$

$\nabla\varphi$ - finite at the trailing edge of the surface;

$\nabla\varphi \sim \frac{1}{\delta^\alpha}$ - on the leading edge of the surface $s(\alpha < 1)$.

When crossing the surface s the acceleration potential will experience a jump

$$\Theta_- - \Theta_+ = -\gamma(Q)\, v_0.$$

The acceleration potential for the horizontal hydrofoil will be determined from formula (VII.33):

$$\Theta(x, y, z) = \frac{v_0}{4\pi}\iint_s \gamma(Q)\frac{\partial}{\partial\xi}G(x, y, z, \xi, \eta, \zeta)\, ds. \qquad \text{(XI.6)}$$

The basic integral equation is defined by the expression (XI.5) as follows:

$$v_z(x, y, z) = -\frac{1}{v_0}\left\{\frac{\partial}{\partial z}\left[e^{+ipx}\int_{+\infty}^{x}\Theta(\tau, y, z)\, e^{-ip\tau}\, d\tau\right]\right\}. \qquad \text{(XI.7)}$$

11.2. Velocity and Acceleration Potentials for a Submerged Hydrofoil in a Fluid of Infinite Depth [435

The function $G(x, y, z, \xi, \eta, \zeta)$ represents the velocity potential of a pulsating source under the free surface and moving with the velocity v_0. The solution of the problem of motion of a pulsating source was obtained by L. N. Sretenskiy and also by M. D. Khaskind [162]. Let us examine the solution by M. D. Khaskind. Let us find the function G in the following form:

$$G(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} - \frac{1}{r_1} + G_1(x, y, z, \xi, \eta, \zeta), \tag{XI.8}$$

where G_1 is the function which is harmonic throughout the entire lower half-space.

From the condition (XI.2) for determining function G_1 we obtain an equation which holds true for the entire lower half-space:

$$\nu G_{1z} - 2i\tau_0\nu_1(1 - i\beta)G_{1x} - \nu_1^2(1 - 2i\beta)G_1 + \tau_0^2 G_{1xx} = 2\nu_1\left(\frac{1}{r_1}\right)_z. \tag{XI.9}$$

To solve this equation the following integral expression is used:

$$\frac{1}{r_1} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\int_0^\infty e^{\lambda(z+\zeta+i\omega)}\,d\Theta\,d\lambda,$$

$$(z + \zeta < 0).$$

The function G_1 is obtained in the form

$$G_1 = -\frac{\nu_1}{\pi}\int_{-\pi}^{+\pi}\int_0^\infty \frac{\lambda e^{\lambda(z+\zeta+i\omega)}\,d\theta\,d\lambda}{\tau_0^2\lambda^2\cos^2\theta - 2\tau_0\nu_1(1 - i\beta)\lambda\cos\theta - \nu_1\lambda + \nu_1^2(1 - 2i\beta)}. \tag{XI.10}$$

The integrand expression is determined in terms of the roots of the following equation:

$$\tau_0^2\lambda^2\cos^2\theta - [2\tau_0(1 - i\beta)\cos\theta + 1]\nu_1\lambda + \nu_1^2(1 - 2i\beta) = 0. \tag{XI.11}$$

Solving this equation, we find:

$$\lambda_1' = \nu_1 \frac{1 + 2\tau_0(1 - i\beta)\cos\theta + \sqrt{1 + 4\tau_0\cos\theta - 4\tau_0 i\beta\cos\theta - 4\tau_0^2\beta^2\cos^2\theta}}{2\tau_0^2\cos^2\theta},$$

$$\lambda_2' = \nu_1 \frac{1 + 2\tau_0(1 - i\beta)\cos\theta - \sqrt{1 + 4\tau_0\cos\theta - 4\tau_0 i\beta\cos\theta - 4\tau_0^2\beta^2\cos^2\theta}}{2\tau_0^2\cos^2\theta}. \quad \text{(XI.12)}$$

When $\beta = 0$ the roots are in a simple form: [436

$$\lambda_{1,2} = \frac{1 + 2\tau_0\cos\theta \pm \sqrt{1 + 4\tau_0\cos\theta}}{2\tau_0^2\cos\theta}. \quad \text{(XI.13)}$$

Let us consider the angle θ_0 in the $(0, \frac{\pi}{2})$ interval

$$\theta_0 = \begin{cases} 0 & \text{with } |\tau_0| < \frac{1}{4}, \\ \arccos\frac{1}{4|\tau_0|} & \text{with } |\tau_0| > \frac{1}{4}. \end{cases} \quad \text{(XI.14)}$$

Then the roots λ_1 and λ_2 are real and positive for the following values of θ:

$$|\theta| < \pi - \theta_0, \quad \text{if} \quad \tau_0 > 0,$$
$$\pi > |\theta| > \theta_0, \quad \text{if} \quad \tau_0 < 0.$$

For the remaining values of θ in the $(-\pi, +\pi)$ interval these roots are complex.

For small values of μ

$$\lambda_1' = \lambda_1 - i\frac{\mu\sqrt{1 + 4\tau_0\cos\theta} + 1}{2v_0\cos\theta\sqrt{1 + 4\tau_0\cos\theta}} + \Theta(\mu^2),$$

$$\lambda_2' = \lambda_2 - i\frac{\mu\sqrt{1 + 4\tau_0\cos\theta} - 1}{2v_0\cos\theta\sqrt{1 + 4\tau_0\cos\theta}} + \theta(\mu^2), \quad \text{(XI.15)}$$

from where

$$\operatorname{Sign}\operatorname{Im}\lambda_1' = -\operatorname{Sign}\cos\theta; \quad \operatorname{Sign}\operatorname{Im}\lambda_2' = -\operatorname{Sign}\tau_0.$$

For $\tau_0 > 0$, let us write the function G_1 in the following form:

$$G = F_1(x, y, z) + F_2(x, y, z) + F_3(x, y, z),$$

where

$$F_1(x, y, z) = -\frac{\nu_1}{\pi\tau_0^2}\int_{-\pi+\theta_0}^{+\pi-\theta_0} d\theta \int_0^\infty \frac{\lambda e^{\lambda(z+\zeta+i\omega)}}{\cos^2\theta(\lambda_1' - \lambda_2')(\lambda - \lambda_1')}\, d\lambda, \quad \text{(XI.16)}$$

$$F_3(x, y, z) = \frac{\nu_1}{\pi\tau_0^2} \int_{-\pi+\theta_0}^{+\pi-\theta_0} d\theta \int_0^\infty \frac{\lambda e^{\lambda(z+\zeta+i\omega)}}{\cos^2\theta(\lambda_1' - \lambda_2')(\lambda - \lambda_2')} d\lambda, \quad \text{(XI.17)}$$

$$F_3(x, y, z) = \frac{\nu_1}{\pi\tau_0^2} \int_{-\theta_0}^{+\theta_0} d\theta \int_0^\infty \frac{\lambda e^{\lambda(z+\zeta+i\omega)}}{\cos^2(\lambda - \bar{\lambda}_1')(1 - \bar{\lambda}_2)} d\lambda, \quad \text{(XI.18)}$$

$$\bar{\lambda}_{1,2}'(\theta) = \lambda_{1,2}'(\pi + \theta).$$

In the limit, when $\mu \to 0$, the integration path in the formulas for F_1 and F_2 must be curvilinear, bypassing in a certain way the specific points λ_1 and λ_2 located on the real axis: [437

$$F_1 = -\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}{}' \int_{0(L)_1}^{\infty} \frac{\lambda e^{\lambda(z+\zeta+i\omega)}}{(\lambda - \lambda_1)\sqrt{1+4\tau_0\cos\theta}} d\theta d\lambda -$$

$$-\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}{}' \int_{0(\bar{L}_1)}^{\infty} \frac{\lambda e^{\lambda(z+\zeta-i\omega)}}{(\lambda - \bar{\lambda}_2)\sqrt{1-4\tau_0\cos\theta}} d\theta d\lambda. \quad \text{(XI.19)}$$

$$F_2 = \frac{1}{\pi} \int_{-\pi+\theta_0}^{+\pi-\theta_0} \int_{0(L_1)}^{\infty} \frac{\lambda e^{\lambda(z+\zeta+i\omega)}}{(\lambda - \lambda_2)\sqrt{1+4\tau_0\cos\theta}} d\theta d\lambda. \quad \text{(XI.20)}$$

$$F_3 = -\frac{\nu_1}{\pi\tau_0} \int_{-\theta_0}^{+\theta_0} \int_0^\infty \frac{\lambda e^{\lambda(z+\zeta-i\omega)}}{\cos^2\theta\,(\lambda - \bar{\lambda}_1)(\lambda - \bar{\lambda}_2)} d\theta d\lambda. \quad \text{(XI.21)}$$

The contour L_1 bypasses the specific point from above, while the contour $\bar{L}_1$ from below. The integral prime sign denotes integration within specified limits with the exception of the $(-\theta_0, +\theta_0)$ interval.

$$\int_{-a}^{+a}{}' f(\theta)\,d\theta = \int_{-a}^{-\theta_0} f(\theta)\,d\theta + \int_{+\theta_0}^{+a} f(\theta)\,d\theta.$$

The functions F_j with $\tau_0 < 0$ are determined in a similar manner. We will consider the case of $\tau_0 > 0$ only. The formulas (XI.12)-(XI.21) were derived by M. D. Khaskind.

For further discussion it is convenient to transform the expressions (XI.19) and (XI.20).

Selecting the remainders, we can write the formula for G(x, y, z) as follows:

$$G(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} - \frac{1}{r_1} - \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{\lambda e^{\lambda(z+\zeta+i\omega)}}{(\lambda-\lambda_1)\sqrt{1+4\tau_0\cos\theta}}\, d\lambda +$$

$$+ \frac{i}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{\lambda e^{\lambda(z+\zeta+i\omega)}}{(\lambda-\lambda_2)\sqrt{1+4\tau_0\cos\theta}}\, d\lambda +$$

$$+ i\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\lambda_1 e^{\lambda_1(z+\zeta+i\omega)}}{\sqrt{1+4\tau_0\cos\theta}}\, d\theta - \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\bar{\lambda}_1 e^{\bar{\lambda}(z+\zeta-i\omega)}}{\sqrt{1-4\tau_0\cos\theta}}\, d\theta -$$

$$- i\int_{-\pi+\theta_0}^{+\pi-\theta_0} \frac{\lambda_2 e^{\lambda_2(z+\zeta+i\omega)}}{\sqrt{1+4\tau_0\cos\theta}}\, d\theta. \qquad \text{(XI.22)}$$

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The velocity potential φ can be written as follows:

$$\varphi = -\frac{1}{4\pi}\iint_s \gamma(Q)\, e^{-i(\xi-x)} \int_{\infty}^{x} \frac{\partial}{\partial\zeta} G(\tau, y, z, \xi, \eta, \zeta)\, e^{-ip(\tau-e)}\, d\tau ds. \qquad \text{(XI.23)}$$

Performing calculations, we can write

$$\varphi = -\frac{1}{4\pi}\iint_s \gamma(Q)\, e^{-ip(\xi-x)} \left[\int_{\infty}^{x} \frac{\partial}{\partial\zeta}\frac{1}{r} e^{-ip(\tau-\xi)} + \right.$$

$$+ \frac{\partial}{\partial\zeta}\left(\frac{i}{2\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{e^{\lambda[z+\zeta+i(y-\eta)\sin\theta]} e^{i(x-\xi)(\lambda\cos\theta-p)}}{\lambda\cos\theta - p}\, d\lambda + \right.$$

$$+ \frac{i}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{\lambda e^{\lambda[z+\zeta+i(y-\eta)\sin\theta]} e^{i(x-\xi)(\lambda\cos\theta-p)}}{(\lambda\cos\theta-p)(\lambda-\lambda_1)\sqrt{1+4\tau_0\cos\theta}}\, d\lambda -$$

$$- \frac{i}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{\lambda e^{\lambda[z+\zeta+i(y-\eta)\sin\theta]} e^{i(x-\xi)(\lambda\cos\theta-p)}}{(\lambda\cos\theta-p)(\lambda-\lambda_0)\sqrt{1+4\tau_0\cos\theta}}\, d\lambda +$$

$$+ \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\lambda_1 e^{\lambda_1[z+\zeta+i(y-\eta)\sin\theta]} e^{i(x-\xi)(\lambda_1\cos\theta-p)}}{(\lambda\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}}\, d\theta +$$

$$+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\bar{\lambda}_1 e^{\bar{\lambda}_1[z+\zeta+i(y-\eta)\sin\theta]}e^{-i(x-\xi)(\bar{\lambda}_1\cos\theta+p)}}{(\bar{\lambda}_1\cos\theta+p)\sqrt{1-4\tau_0\cos\theta}}\,d\theta-$$

$$-\int_{-\pi+\theta_0}^{+\pi-\theta_0}\frac{\lambda_2 e^{\lambda_2[z+\zeta+i(y-\eta)\sin\theta]}e^{i(x-\xi)(\lambda_2\cos\theta-p)}}{(\lambda_2\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}}\,d\theta\Bigg)ds\Bigg]. \qquad \text{(XI.24)}$$

In order to obtain the final result it remains necessary to transform the term which describes the lifting surface motion in an infinite fluid and to separate the remainders at the point $\cos\theta_0 = \frac{p}{\lambda}$. Determining the first integral by using the Küssner method [92] and separating the remainders at the point $\theta_0 = \frac{p}{\lambda}$, we obtain: [439

$$\varphi=-\frac{1}{4\pi}\iint_s \gamma(Q)\,e^{-ip(\xi-x)}\Bigg[-\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\times$$

$$\times\int_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z-\zeta)^2}}}\frac{e^{-ipu\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{\frac{3}{2}}}\,du+$$

$$+\frac{\partial}{\partial\zeta}\Bigg(\frac{i}{2\pi}\int_{-\pi}^{+\pi}d\theta\int_0^\infty\frac{e^{\lambda[z+\zeta+i(y-\eta)\sin\theta]}(\lambda^2\cos^2\theta-\lambda\nu(1-2\tau_0\cos\theta)+\nu^2\tau_0^2)\,e^{i(x-\xi)(\lambda\cos\theta-p)}}{(\lambda\cos\theta-p)[\lambda^2\cos^2\theta-\nu\lambda(1+2\tau_0\cos\theta)+\nu^2\tau_0^2]}\times$$

$$\times d\lambda+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\lambda_1 e^{\lambda_1[z+\zeta+i(y-\eta)\sin\theta]}e^{i(x-\xi)(\lambda_1\cos\theta-p)}}{(\lambda_1\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}}\,d\theta+$$

$$+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\bar{\lambda}_1 e^{\bar{\lambda}_1[z+\zeta+i(y-\eta)\sin\theta]}e^{-i(x-\xi)(\bar{\lambda}_1\cos\theta+p)}}{(\bar{\lambda}_1\cos\theta+p)\sqrt{1-4\tau_0\cos\theta}}\,d\theta-$$

$$-\int_{-\pi+\theta_0}^{+\pi-\theta_0}\frac{\lambda_2 e^{\lambda_2[z+\zeta+i(y-\eta)\sin\theta]}e^{i(x-\xi)(\lambda_2\cos\theta-p)}}{(\lambda_2\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}}\,d\theta-$$

$$-\operatorname{Re}\int_p^\infty\frac{e^{\lambda\left[z+\zeta+i(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}\right]}\left[p^2+\lambda\nu\left(1-\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right]}{\lambda\sqrt{1-\frac{p^2}{\lambda^2}}\left[p^2-\lambda\nu\left(1+\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right]}\,d\lambda\Bigg)\Bigg]ds, \qquad \text{(XI.25)}$$

Here $\nu = \frac{\nu_1}{\tau_0^2} = \frac{g}{v_0^2}$.

In the limiting case of steady motion [440

$$\tau_0 = 0, \quad \lambda_1 = \nu \sec^2 \theta, \quad \lambda_2 = 0, \quad p = 0.$$

Then, for the potential we obtain the formula

$$\varphi = -\frac{1}{4\pi} \iint_s \gamma(Q) \left[-\frac{(z-\zeta)}{(y-\eta)^2 + (z-\zeta)^2} \int\limits_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z-\zeta)^2}}}^{\infty} \times \right.$$

$$\times \frac{d\lambda}{(\lambda^2+1)^{\frac{3}{2}}} + \frac{\partial}{\partial \zeta} \left(\frac{i}{2\pi} \int\limits_{-\pi}^{+\pi} \frac{d\theta}{\cos\theta} \int\limits_0^{\infty} \frac{e^{\lambda[z+\zeta+i\omega]}(\lambda \cos^2\theta + \nu)}{\lambda(\lambda\cos^2\theta - \nu)} d\lambda + \right.$$

$$\left. \left. + 2\mathrm{Re} \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\nu}{\cos^2\theta}[z+\zeta+i\omega]} \sec\theta d\theta - \mathrm{Re} \int\limits_0^{\infty} \frac{e^{\lambda[z+\zeta+i(y-\eta)]}}{\lambda} d\lambda \right) \right] ds, \qquad \text{(XI.26)}$$

which can be easily written in the form of (VIII.12).

We will obtain the integral equation of the problem by the usual method using the condition (XI.5), determining φ_z as follows:

$$\varphi_z = -\frac{1}{4\pi} \iint_s \gamma(Q) e^{-ip(x-\xi)} \left[\frac{\partial}{\partial z} \left(-\frac{(z-\zeta)}{(y-\eta)^2 + (z-\zeta)^2} \times \right. \right.$$

$$\left. \times \int\limits_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z-\zeta)^2}}}^{\infty} \frac{e^{-ip\lambda\sqrt{(y-\eta)+(z-\zeta)}}}{(\lambda^2+1)^{\frac{3}{2}}} d\lambda \right) +$$

$$+ \int\limits_{-\pi}^{+\pi} d\theta \int\limits_0^{\infty} \frac{e^{\lambda[z+\zeta+i(y-\eta)\sin\theta]} \lambda^2 [\lambda^2\cos^2\theta + \lambda\nu(1 - 2\tau_0\cos\theta) + \nu^2\tau_0^2] e^{i(x-\xi)(\lambda\cos\theta - p)}}{(\lambda\cos\theta - p)[\lambda^2\cos^2\theta - \lambda\nu(1+2\tau_0\cos\theta) + \nu^2\tau_0^2]} d\lambda +$$

$$+ \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\lambda_1^3 e^{\lambda_1(z+\zeta+i(y-\eta)\sin\theta]} e^{i(x-\xi)(\lambda_1\cos\theta - p)}}{(\lambda_1\cos\theta - p)\sqrt{1+4\tau_0\cos\theta}} d\theta +$$

$$+\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\bar{\lambda}_1^3 e^{\bar{\lambda}_1[z+\zeta+i(y-\eta)\sin\theta]} e^{i(x-\xi)(\bar{\lambda}_1\cos\theta+p)}}{(\bar{\lambda}_1\cos\theta+p)\sqrt{1-4\tau_0\cos\theta}}\, d\theta -$$ [441

$$-\int\limits_{-\pi+\theta_0}^{+\pi-\theta_0} \frac{\lambda_2^3 e^{\lambda_2[z+\zeta+i(y-\eta)\sin\theta]} e^{i(x-\xi)(\lambda_2\cos\theta+p)}}{(\lambda_2\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}}\, d\theta -$$

$$-\operatorname{Re}\int\limits_{p}^{\infty} \frac{\lambda e^{\lambda\left[z+\zeta+i(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}\right]}\left[p^2+\lambda\nu\left(1-\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right]}{\sqrt{1-\frac{p^2}{\lambda^2}}\left[p^2-\lambda\nu\left(1+\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right]}\, d\lambda\Bigg] ds. \qquad \text{(XI.27)}$$

11.3. Integro-Differential Equation for the Submerged Hydrofoil Moving at High Velocities

Before deriving the integro-differential equation let us examine a case of motion when $Fr_b \to 0$, i.e., when the problem can be solved by another method that generalizes the method used in aerodynamics. For this case the boundary conditions on the free surface will be as follows:

$$\theta_x = 0, \qquad \text{(XI.28)}$$

and then the acceleration potential will be defined by the formula

$$\Theta(x, y, z) = \frac{v_0}{4\pi}\iint\limits_{S} \gamma(\xi, \eta)\frac{\partial}{\partial\zeta}\left[\frac{1}{r}-\frac{1}{r_1}\right] ds, \qquad \text{(XI.29)}$$

where $r=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$; $r_1=\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}$.

Substituting $\tau-\zeta=u\sqrt{(y-\eta)^2+z^2}$, we may write the expression for the potential φ as follows:

$$\varphi=\frac{1}{4\pi}\iint\limits_{S}\gamma(\xi,\eta)\, e^{-ip(\xi-x)}du\left[\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\int\limits_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z-\zeta)^2}}}^{\infty}\times\right.$$

$$\times\frac{e^{-ipu\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{3/2}}du+\frac{z+\zeta}{(y-\eta)^2+(z-\zeta)^2}\int\limits_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z-\zeta)^2}}}^{\infty}\times$$

$$\left.\times\frac{e^{-ipu\sqrt{(y-\eta)^2+(z+\zeta)^2}}}{(u^2+1)^{3/2}}du\right]d\xi d\eta. \qquad \text{(XI.30)}$$

Then the equation (XI.7) will be in the form: [442

$$v_{z_0} = \frac{1}{4\pi v_0} \iint_s \gamma(\xi, \eta)\, e^{-ip(\xi - x)} \times$$

$$\times \frac{\partial}{\partial z}\left(\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2} \int\limits_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z+\zeta)^2}}}^{\infty} \frac{e^{-ipu\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{3/2}}\, du + \right.$$

$$\left. + \frac{z+\zeta}{(y-\eta)^2+(z+\zeta)^2} \int\limits_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z+\zeta)^2}}}^{\infty} \frac{e^{-ipu\sqrt{(y-\eta)^2+(z+\zeta)^2}}}{(u^2+1)^{3/2}}\, du \right) d\xi\, d\eta \Bigg]_{(z=\zeta=-h)}. \qquad \text{(XI.31)}$$

Let us introduce into the equation (XI.31) the approximations used in the lifting line theory. Let us assume that the lower limit of the integral with respect to u is equal to zero. For the second term this assumption gives a better approximation, since the denominator of this limit is always greater than both the denominator of the limit of the first integral and zero, so that the lower limit is finite at all points.

Assuming $\Gamma(\eta) = \int\limits_{-a}^{+a} \gamma(\xi, \eta)\, e^{ip\xi} d\xi,$

we obtain

$$v_z(x, y, z) = v_{z_0}(x, y, \zeta) + \frac{1}{4\pi v_0}\left[e^{ipx} \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \Gamma(\eta)\, d\eta \frac{\partial}{\partial z} \times \right.$$

$$\times \left(\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2} \times \int\limits_0^{\infty} \frac{e^{-ipu\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{3/2}}\, du \right) + e^{ipx} \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} \times$$

$$\left. \times \bar{\Gamma}(\eta)\, d\eta \frac{\partial}{\partial z}\left(\frac{z+\zeta}{(y-\eta)^2+(z-\zeta)^2} \times \int\limits_0^{\infty} \frac{e^{ipu\sqrt{(y-\eta)^2+(z+\zeta)^2}}}{(u^2+1)^{3/2}}\, du \right) \right]_{(z=\zeta=-h)}, \qquad \text{(XI.32)}$$

where $v_{z0}(x, y, \zeta)$ is the induced velocity with $l = \infty$.

Let us introduce the function $F_{-1}(iy) = \int\limits_0^{\infty} \frac{e^{-iyu}}{(u^2+1)^{3/2}}\, du.$ [443

This function can be written as $F_{-1}(y) = \beta iy Z_{-1}(iy),$ where

$Z_{-1}(iy)$ is a linear combination of the Bessel functions. Then the equation (XI.32) can be transformed into the following form:

$$v_z(x, y-h, \zeta) = v_{z_0}(x, y-h, \zeta) + \frac{1}{4\pi v_0}\left[e^{ipx}\int_{-b}^{b}\frac{\Gamma(\eta)}{(y-\eta)^2}\times\right.$$
$$\times F_{-1}(p(y-\eta))\,d\eta + e^{ipx}\int_{-b}^{+b}\Gamma(\eta)\left(\frac{(y-\eta)^2-(2h)^2}{[(y-\eta)^2+(2h)^2]^2}\times\right.$$
$$\times F_{-1}(p\sqrt{(y-\eta)^2+(2h)^2}) - \frac{(2h)^2}{[(y-\eta)^2+(2h)^2]^2}\times$$
$$\left.\left.\times F_0(p\sqrt{(y-\eta)^2+(2h)^2}\right)d\eta\right],$$

where $F_0(y) = \beta y^2 Z_0(iy)$.

Let us assume that $v(x, y-h, t) = -B(y)e^{ipx}v_0$ and define $v_{z_0}(x, y - h, t)$ by the formula

$$v_{z_0}(x, y-h) = -\frac{\Gamma(y)}{4\pi a(y)}f_1(k), \qquad \text{(XI.33)}$$

where $a(y)$ - a half-chord;
$f_1(k)$ - some function of the Strouhal number $k = \frac{pa(0)}{2}$ which can be determined from the data given in Ch. VII.

For an isolated hydrofoil the value of the function $f_1(k)$ is given by Küssner. Introducing these values into the equation we obtain:

$$\Gamma(y) = 2v_0 a(y) f_1(k)\left\{B(y) + \frac{1}{4\pi v_0}\left[\int_{-b}^{+b}\frac{\Gamma(\eta)}{(y-\eta)^2}F_{-1}(p(y-\eta))\,d\eta + \right.\right.$$
$$+\int_{-\frac{l}{2}}^{+\frac{l}{2}}\Gamma(\eta)\left[\frac{(y-\eta)^2-(2h)^2}{[(y-\eta)^2+(2h)^2]^2}F_{-1}(p\sqrt{(y-\eta)^2+(2h)^2}) - \right.$$
$$\left.\left.\left. - \frac{(2h)^2}{[(y-\eta)^2+(2h)^2]^2}F_0(p\sqrt{(y-\eta)^2+(2h)^2})\right]d\eta\right\}. \qquad \text{(XI.34)}$$

Calculations produce the following [444

$$F_{-1}(y)=\frac{\pi}{2}iy[N_{-1}(iy)-H_{-1}(iy)],$$

$$F_0(y)=-\frac{\pi}{2}y^2[N_0(iy)-H_0(iy)], \tag{XI.35}$$

where $N_0(iy)$, $N_{-1}(iy)$ - the Neumann function;
$H_0(iy)$, $H_{-1}(iy)$ - the Struve function.

By determining the real and imaginary parts we obtain

$$\operatorname{Re}F_{-1}(y)=yK_1(y),\quad \operatorname{Im}F_1(y)=-\frac{\pi}{2}y[I_1(y)+L_{-1}(y)],$$

$$\operatorname{Re}F_0(y)=y^2K_0(y),\quad \operatorname{Im}F_0(y)=-\frac{\pi}{2}y^2[I_0(y)+L_0(y)],$$

where $K_n(y)$, $I_n(y)$ - the MacDonald functions;
$I_n(y)$ - the Struve function.

The equation (XI.34) can be expressed in a different way by performing integration by parts:

$$\Gamma(y)=2v_0\pi a(y)f_1(k)\left\{B(y)-\frac{1}{4\pi v_0}\left[\int_{-b}^{+b}\frac{d\Gamma}{d\eta}\frac{N_{-1}(p(y-\eta))}{(y-\eta)}d\eta+\right.\right.$$
$$+\int_{-b}^{+b}\frac{d\Gamma}{d\eta}\left(\frac{(y-\eta)}{(y-\eta)^2+(2h)^2}N_{-1}(p\sqrt{(y-\eta)^2+(2h)^2})-\right.$$
$$\left.\left.\left.-\frac{2h}{(y-\eta)^2+(2h)^2}N_0(p\sqrt{(y-\eta)^2+(2h)^2})\right)d\eta\right]\right\}, \tag{XI.36}$$

where

$$N_{-1}(\sqrt{\sigma^2+(2hp)^2})=\frac{\sigma^2+(2hp)^2}{\sigma}\int_{\sigma}^{\infty}\frac{[\nu^2-(2hp)^2]}{[\nu^2+(2hp)^2]^2}\times$$
$$\times F_{-1}(\sqrt{\nu^2+(2hp)^2})\,d\nu,$$

$$N_0=\frac{[\sigma^2+(2hp)^2]}{2hp}\int_{0}^{\infty}\frac{(2hp)^2}{[\nu^2+(2hp)^2]^2}F_0(\sqrt{\nu^2+(2hp)^2})\,d\nu. \tag{XI.37}$$

The function $N_{-1}(\sigma)$ represents the general form of the Küssner function. It is not difficult to show that with $2hp = 0$, $N(\sigma) = s(\sigma)$

$$N_{-1}(\sigma)=\sigma\int_{\sigma}^{\infty}\frac{1}{\nu^2}F_{-1}(\nu)\,d\nu. \tag{XI.38}$$

[92] The Küssner function is represented by the formula [445

$$s(\sigma) = -\sigma \int_{\sigma}^{\infty} \frac{d\nu}{\nu} \frac{d}{d\nu} \int_{0}^{\infty} \frac{e^{-i\lambda\nu}}{\sqrt{\lambda^2+1}} d\lambda.$$

To prove it, it is enough to show that

$$-\frac{d}{d\nu} \int_{0}^{\infty} \frac{e^{-i\lambda\nu}}{\sqrt{\lambda^2+1}} d\lambda = \frac{1}{\nu} \int_{0}^{\infty} \frac{e^{-i\lambda\nu}}{(\lambda^2+1)^{3/2}} d\lambda.$$

Substituting the variables $x = \lambda\nu$, we obtain

$$J = \int_{0}^{\infty} \frac{e^{-i\lambda\nu}}{\sqrt{\lambda^2+1}} d\lambda = \int_{0}^{\infty} \frac{e^{-ix}}{\sqrt{x^2+\nu^2}} dx,$$

$$\frac{dJ}{dt} = -\nu \int_{0}^{\infty} \frac{e^{-ix}}{(\sqrt{x^2+\nu^2})^3} dx = -\frac{1}{\nu} \int_{0}^{\infty} \frac{e^{-i\lambda\nu}}{(\sqrt{\lambda^2+1})^3} d\lambda. \quad \text{(XI.39)}$$

Let us analyze the boundary value of $N_{-1}(\sqrt{\sigma^2+(2hp)^2})$ with $\sigma \to 0$ ($p = 0$):

$$\lim_{\sigma\to 0} N_{-1} = \lim_{\sigma\to 0} \frac{\dfrac{d}{d\sigma} \displaystyle\int_{0}^{\infty} \frac{\nu^2-(2hp)^2}{[\nu^2+(2hp)^2]^2} F_{-1}(\sqrt{\nu^2+(2hp)^2}) d\nu}{\dfrac{d}{d\sigma} \dfrac{\sigma}{\sigma^2+(2hp)^2}} =$$

$$= \lim_{\sigma\to 0} F_{-1}(\sqrt{\sigma^2+(2hp)^2}) = 1.$$

In this case the equation (XI.36) will be transformed into the equation that corresponds to the steady-state motion of the hydrofoil. The equation (XI.36) is not quite suitable for obtaining the solution, because functions $N_n(\sigma)$ are not expressed through the known special functions. Due to this fact, we can perform partial integration of only the first integral containing the divergent nucleus, while the second integral, which contains a regular nucleus, will be kept unchanged. Using this method the equation will acquire the following form:

$$\Gamma(y) = 2\pi v_0 a(y) f_1(k) \Bigg\{ B(y) - \frac{1}{4\pi v_0} \Bigg[\int_{-b}^{+b} \frac{d\Gamma}{d\eta} \frac{N_{-1}(p(y-\eta))}{y-\eta} d\eta -$$

$$- \int_{-b}^{+b} \Gamma(\eta) \Bigg[\frac{(y-\eta)^2-(2h)^2}{[(y-\eta)^2+(2h)^2]^2} F_{-1}(p\sqrt{(y-\eta)^2+(2h)^2}) -$$

$$- \frac{(2h)^2}{[(y-\eta)^2 + (2h)^2]^2} F_0 (p \sqrt{(y-\eta)^2 + (2h)^2}) \Bigg] d\eta \Bigg\}. \qquad (XI.40)$$ [446

However, we can completely avoid the derivation of the functions $N_n(\sigma)$ if we write the equation in the following form:

$$\Gamma(y) = 2v_0 a(y) \pi f_1(k) \Bigg\{ B(y) - \frac{1}{4\pi v_0} \Bigg[\int_{-b}^{+b} \frac{\Gamma'(\eta)}{y-\eta} d\eta -$$

$$- \int_{-b}^{+b} \Gamma(\eta) \Bigg(\frac{F_{-1}(p(y-\eta)) - 1}{(y-\eta)^2} + \frac{(y-\eta)^2 - (2h)^2}{[(y-\eta)^2 + (2h)^2]^2} \times$$

$$\times F_{-1}[p\sqrt{(y-\eta)^2 + (2h)^2}] - \frac{(2h)^2}{[(y-\eta)^2 + (2h)^2]^2} \times$$

$$F_0[p\sqrt{(y-\eta)^2 + (2h)^2}\Bigg) d\eta \Bigg] \Bigg\}. \qquad (XI.41)$$

Now the nucleus of the second integral is regular and the integral will exist.

It was already mentioned in Ch. VIII that the integro-differential equation for the airplane wing in an unsteady flow can be easily obtained from formulas (VIII.22)-(VIII. 24). The functions f(z, ζ) for this case will be as follows:

$$f(z, \eta) = \int_0^\infty e^{-ipu\sqrt{(y-\eta)^2 + z^2}} \frac{du}{\sqrt{u^2+1}}.$$

It is not difficult to demonstrate that, in this case, the function satisfies the equation

$$\frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial \eta^2} = p^2 f(z, \eta). \qquad (XI.42)$$

Computing, we obtain

$$f(z, \zeta) = \frac{\pi}{2} [H_0(ip(y-\eta)) - N_0(ip(y-\eta))],$$

$$\frac{df}{d\zeta} = \frac{i\pi p}{2} [H_{-1}(ip(y-\eta)) - N_{-1}(ip(y-\eta))]; \qquad (XI.43)$$

$$\mathrm{Re} \frac{df}{d\zeta} = -pK_1(p(y-\eta)); \quad \mathrm{Im} \frac{df}{d\zeta} = \frac{\pi}{2} p [I_1(p(y-\eta)) + L_1(p(y-\eta))]. \qquad (XI.44)$$

Then, by using the designations given in [92], we obtain: [447

$$B(y)=\frac{\Gamma(y)}{2\pi a(y) f_1(k) v_0}+\frac{1}{4\pi v_0}\left[\int_{-b}^{+b}\frac{d\Gamma}{d\eta}\frac{F_1(p(y-\eta))}{y-\eta}d\eta-\right.$$

$$\left.-\int_{-b}^{+b}\Gamma(\eta)p^2 f(0,\eta)\,d\eta\right],$$

$$F_1(p\sigma)=\sigma k_1(\sigma)-\frac{i\pi}{2}\sigma[I_1(\sigma)+L_{-1}(\sigma)], \qquad \text{(XI.45)}$$

$$\operatorname{Re} f(0,\eta)=k_0(p(y-\eta)); \quad \operatorname{Im} f(0,\eta)=-\frac{\pi}{2}[J_0\Big(p(y-\eta)\Big)+$$

$$+L_0(p(y-\eta))].$$

Using the formula (VIII.24) we can write the equation (XI.45) in the form of Küssner's equation and obtain a new expression for the Küssner function. However, there is no need for this, because the second integrating operator does not require regularization.

Let us present the approximate solution of equation (XI.41) for an elliptical load distribution in an infinite fluid considered earlier in Chapters VIII-X and let us try to find the solution in the following form:

$$\Gamma(y)=\Phi\sqrt{1-\bar{y}^2}.$$

Let us use the dimensionless form and assume that

$$B(\bar{y})=B;\ \bar{a}(y)=\frac{2}{\pi\lambda_0}\sqrt{1-\bar{y}^2}. \qquad \text{(XI.41) [sic]}$$

Then, from the equation (XI.41) we will obtain the following:

$$\Phi=\frac{2f_1(k)B}{\pi\lambda\left[1+\frac{8\bar{f}_1(k)}{\pi\lambda}\zeta(k_\lambda)\right]}, \qquad \text{(XI.46)}$$

where $\bar{f}_1(k)=2\pi f_1(k)$; $k_\lambda=pb$ is the Strouhal number; and function $\zeta(k_\lambda)$ is defined by the formula

$$\zeta(k_\lambda)=1-\frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\left[\frac{F_{-1}(k_\lambda(\bar{y}-\bar{\eta}))-1}{(\bar{y}-\bar{\eta})^2}+\right.$$

$$+\frac{(\bar{y}-\bar{\eta})^2-16\bar{h}^2}{[(\bar{y}-\bar{\eta})^2+16\bar{h}^2]^2}F_{-1}\left(k_\lambda\sqrt{(\bar{y}-\bar{\eta})^2+16\bar{h}^2}\right)-$$

$$\left.-\frac{16\bar{h}^2}{[(\bar{y}-\bar{\eta})^2+16\bar{h}^2]^2}F_0\left(k_\lambda\sqrt{(\bar{y}-\bar{\eta})^2+16\bar{h}^2}\right)\right]d\bar{\eta}\,d\bar{y}. \qquad \text{(XI.47)}$$

11.4. The Integro-Differential Equation for the Submerged Hydrofoil Moving with Arbitrary Velocities [448

In the assumptions of the Prandtl lifting line theory the corresponding values of φ and φ_z, obtained for $x = \xi$, will be found from (XI.2) as follows:

$$\varphi_\lambda = -\frac{e^{ipx}}{4\pi}\int_{-b}^{+b}\Gamma(\eta)\Bigg[-\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}\int_0^\infty \frac{e^{-ipu\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{3/2}}\,du +$$

$$+\frac{i}{\pi}\mathrm{Re}\int_0^\pi\int_0^\infty \frac{e^{\lambda[(z+\zeta)+i(y-\eta)\sin\theta]}\,\lambda(\lambda^2\cos^2\theta+\lambda\nu(1-2\tau_0\cos\theta)+\nu^2\tau_0^2)}{(\lambda\cos\theta-p)\,[\lambda^2\cos^2\theta-\lambda\nu(1+2\tau_0\cos\theta)+\nu^2\tau_0^2]}\,d\theta\, d\lambda +$$

$$+ 2\mathrm{Re}\int_0^{\frac{\pi}{2}} \frac{\lambda_1^2 e^{\lambda_1[z+\zeta+i(y-\eta)\sin\theta]}\,d\theta}{(\lambda_1\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}} +$$

$$+ 2\mathrm{Re}\int_0^{\frac{\pi}{2}} \frac{\bar\lambda_1^2 e^{\bar\lambda_1[z+\zeta+i(y-\eta)\sin\theta]}\,d\theta}{(\bar\lambda_1\cos\theta+p)\sqrt{1-4\tau_0\cos\theta}} -$$

$$- 2\mathrm{Re}\int_0^{\pi-\theta_{0}} \frac{\lambda_2^2 e^{\lambda_2[z+\zeta+i(y-\eta)\sin\theta]}\,d\theta}{(\lambda_2\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}} +$$

$$+ \mathrm{Re}\int_p^\infty \frac{e^{\lambda\left[z+\zeta+i(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}\right]}\left(p^2+\lambda\nu\left(1-\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right)}{\sqrt{1-\frac{p^2}{\lambda^2}}\left(p^2-\lambda\nu\left(1+\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right)}\,d\lambda\Bigg]d\eta; \qquad \text{(XI.48)}$$

$$\varphi_{z\lambda} = -\frac{e^{ipx}}{4\pi}\int_{-b}^{+b}\Gamma(\eta)\Bigg[\frac{\partial}{\partial z}\left(-\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\times\right.$$

$$\left.\times\int_0^\infty \frac{e^{-ipu\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{3/2}}\,du\right)+\frac{i}{\pi}\mathrm{Re}\int_0^\pi d\theta\,\times$$

$$\times\int_0^\infty \frac{e^{\lambda[z+\zeta+i(y-\eta)\sin\theta]}\lambda^2[\lambda^2\cos^2\theta+\lambda\nu(1-2\tau_0\cos\theta)+\nu^2\tau_0^2]}{(\lambda\cos\theta-p)[\lambda^2\cos^2\theta-\lambda\nu(1+2\tau_0\cos\theta)+\nu^2\tau_0^2]}\,d\lambda +$$

$$+ 2\mathrm{Re}\int_0^{\frac{\pi}{2}} \frac{\lambda_1^3 e^{\lambda_1[z+\zeta+i(y-\eta)\sin\theta]}}{(\lambda_1\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}}\,d\theta + \qquad [449$$

$$+ 2\mathrm{Re}\int_0^{\frac{\pi}{2}\prime} \frac{\bar{\lambda}_2^3 e^{\bar{\lambda}_1[z+\zeta+i(y-\eta)\sin\theta]}}{(\bar{\lambda}_1\cos\theta + p)\sqrt{1-4\tau_0\cos\theta}}\,d\theta -$$

$$- 2\mathrm{Re}\int_0^{\pi-\theta_0} \frac{\lambda_2^3 e^{\lambda_1[z+\zeta+i(y-\eta)\sin\theta]}}{(\lambda_2\cos\theta - p)\sqrt{1+4\tau_0\cos\theta}}\,d\theta +$$

$$+ \mathrm{Re}\int_p^{\infty} \frac{\lambda e^{\lambda[z+\zeta+i(y-\eta)\sin\theta]}\left[p^2+\lambda\nu\left(1-\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right]}{\sqrt{1-\frac{p^2}{\lambda^2}}\left[p^2-\lambda\nu\left(1+\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right]}\,d\lambda\Bigg]\,d\eta. \quad \text{(XI.49)}$$

Now it is easy to obtain the basic integro-differential equation:

$$\Gamma(y) = 2v_0 a(y)\bar{f}_1(k)\Bigg[B(y) - \frac{1}{4\pi v_0}\Bigg(\int_{-b}^{+b}\frac{\Gamma'(\eta)}{y-\eta}\,d\eta -$$

$$-\int_{-b}^{+b}\Gamma(\eta)\,\frac{F_{-1}(p|y-\eta|)-1}{(y-\eta)^2}\,d\eta - \frac{i}{\pi}\,\mathrm{Re}\int_0^{\pi}d\theta\int_0^{\infty}e^{\lambda[-2h+i(y-\eta)\sin\theta]}\times$$

$$\times\frac{\lambda^2[\lambda^2\cos^2\theta+\lambda\nu(1-2\tau_0\cos\theta)+\nu^2\tau_0^2]}{(\lambda\cos\theta-p)[\lambda^2\cos^2\theta-\lambda\nu(1+2\tau_0)+\nu^2\tau_0^2]}\,d\lambda - 2\mathrm{Re}\times$$

$$\times\int_0^{\frac{\pi}{2}}\frac{\lambda_2^3 e^{\lambda_1[-2h+i(y-\eta)\sin\theta]}}{(\lambda_1\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}}\,d\theta -$$

$$-2\mathrm{Re}\int_0^{\frac{\pi}{2}\prime}\frac{\bar{\lambda}_1^3 e^{\bar{\lambda}_1[-2h+i(y-\eta)\sin\theta]}}{(\bar{\lambda}_1\cos\theta+p)\sqrt{1-4\tau_0\cos\theta}}\,d\theta +$$

$$+2\mathrm{Re}\int_0^{\pi-\theta_1}\frac{\lambda_2^3 e^{\lambda_1[2h+i(y-\eta)\sin\theta]}}{(\lambda_2\cos\theta-p)\sqrt{1+4\tau_0\cos\theta}}\,d\theta -$$ [450

$$-\mathrm{Re}\int_p^{\infty}\lambda\,\frac{e^{\lambda\left[-2h+i(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}\right]}\left[p^2+\lambda\nu\left(1-\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right]}{\sqrt{1-\frac{p^2}{\lambda^2}}\left[p^2-\lambda\nu\left(1+\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right]}\,d\lambda\Bigg)\,d\eta\Bigg]. \quad \text{(XI.50)}$$

In the dimensionless form this equation can be written as:

$$\Gamma(y) = \frac{\bar{f}_1(k)}{2\lambda(y)}\left\{b(y) - \frac{1}{2\pi}\left[\int_{-1}^{+1}\frac{\bar{\Gamma}'(\bar{\eta})}{\bar{y}-\bar{\eta}}\,d\eta - \int_{-1}^{+1}\bar{\Gamma}(\bar{\eta})\,G_k(y-\eta)\,d\bar{\eta}\right]\right\}, \quad (XI.51)$$

where the nucleus $G_k(\bar{y} - \bar{\eta})$ is as follows:

$$G_k(y-\eta) = \frac{F_{-1}(k_\lambda(\bar{y}-\bar{\eta}))-1}{(\bar{y}-\bar{\eta})^2} - \frac{i}{\pi}\,\mathrm{Re}\int_0^{\pi} d\theta \int_0^{\infty} e^{\lambda[-4\bar{h}+i(y-\eta)\sin\theta]} \times$$

$$\times \frac{\lambda^2[\lambda^2\cos^2\theta + \lambda\omega(1 - 2\tau_0\cos\theta) + \omega^2\tau_0^2]}{(\lambda\cos\theta - k_\lambda)[\lambda^2\cos^2\theta - \lambda\omega(1 + 2\tau_0\cos\theta) + \omega^2\tau_0^2]}\,d\lambda -$$

$$-2\,\mathrm{Re}\int_0^{\frac{\pi}{2}} \frac{\lambda_1^3 e^{\lambda_1[-4\bar{h}+i(\bar{y}-\bar{\eta})\sin\theta]}}{(\lambda_1\cos\theta - k_\lambda)\sqrt{1 + 4\tau_0\cos\theta}}\,d\theta -$$

$$-2\,\mathrm{Re}\int_0^{\frac{\pi}{2}} \frac{\bar{\lambda}_1^3 e^{\bar{\lambda}_1[-4\bar{h}+i(\bar{y}-\bar{\eta})\sin\theta]}}{(\bar{\lambda}_1\cos\theta - k_\lambda)\sqrt{1 - 4\tau_0\cos\theta}}\,d\theta +$$

$$+2\,\mathrm{Re}\int_0^{\pi-\theta_1} \frac{\lambda_2^3 e^{\lambda_2[-4\bar{h}+i(\bar{y}-\bar{\eta})\sin\theta]}}{(\lambda^2\cos\theta - k_\lambda)\sqrt{1 + 4\tau_0\cos\theta}}\,d\theta -$$

$$-\mathrm{Re}\int_{k_\lambda}^{\infty} \frac{\lambda e^{\lambda\left[-4\bar{h}+i(\bar{y}-\bar{\eta})\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\right]}\left[k_\lambda^2 + \lambda\omega\left(1 - \frac{2\tau_0 k_\lambda}{\lambda}\right) + \omega^2\tau_0^2\right]}{\sqrt{1 - \frac{k_\lambda^2}{\lambda^2}}\left[k_\lambda^2 - \lambda\omega\left(1 + \frac{2\tau_0 k_\lambda}{\lambda}\right) + \omega^2\tau_0^2\right]}\,d\lambda. \quad (XI.52)$$

With $k_\lambda = 0$ and $\tau_0 = 0$, the nucleus is in the form of [451

$$G_0(y-\eta) = \mathrm{Re}\int_0^{\infty} \lambda e^{\lambda[-4\bar{h}+i(y-\eta)]}\,d\lambda -$$

$$-4\omega^2\,\mathrm{Re}\int_0^{\frac{\pi}{2}} e^{\omega\sec^2\theta[-4\bar{h}+i(y-\eta)\sin\theta]}\sec^5\theta\,d\theta. \quad (XI.53)$$

The equation with this nucleus corresponds to the equation (VIII.17).

Let us transform the nucleus $G_k(y - \eta)$ into the following form:

$$G_k(y-\eta) = \frac{F_{-1}[k_\lambda(\bar{y}-\bar{\eta})]-1}{\bar{y}-\bar{\eta}} - \frac{i}{\pi}\,\mathrm{Re}\int_0^{\pi} d\theta\int_0^{\infty} e^{\lambda[-4\bar{h}+i(\bar{y}-\bar{\eta})\sin\theta]} \times$$

$$\times \frac{\lambda^2 [\lambda^2 \cos^2\theta + \lambda\omega(1 - 2\tau_0 \cos\theta) + \omega^2\tau_0^2]}{(\lambda\cos\theta - k_\lambda)[\lambda^2\cos^2\theta - \lambda\omega(1 + 2\tau_0\cos\theta) + \omega^2\tau_0^2]} d\lambda -$$

$$-\mathrm{Re}\sqrt{\omega}\int_{\omega}^{\infty} \frac{e^{\lambda_1\left[-4\bar h + i(\bar y - \bar\eta)\sqrt{1-\frac{\omega}{u}}\right]}\lambda_1^3}{\left(\lambda_1\sqrt{\frac{\omega}{u}} - k_\lambda\right)\sqrt{1 + 4\tau_0\sqrt{\frac{\omega}{u}}}\sqrt{1-\frac{\omega}{u}}\,u^{3/2}} du -$$

$$-2\mathrm{Re}\sqrt{\omega}\int_{u_0}^{\infty} \frac{e^{\bar\lambda_1\left[-4\bar h + i(\bar y - \bar\eta)\sqrt{1-\frac{\omega}{u}}\right]}\bar\lambda_1^3}{\left(\bar\lambda_1\sqrt{\frac{\omega}{u}} + k_\lambda\right)\sqrt{1 - 4\tau_0\sqrt{\frac{\omega}{u}}}\sqrt{1-\frac{\omega}{u}}\,u^{3/2}} du +$$

$$+2\mathrm{Re}\sqrt{\omega}\int_{\omega}^{\infty} \frac{e^{\lambda_2\left[-4\bar h + i(\bar y - \bar\eta)\sqrt{1-\frac{\omega}{u}}\right]}\lambda_2^3}{\left(\lambda_2\sqrt{\frac{\omega}{u}} - k_\lambda\right)\sqrt{1 + 4\tau_0\sqrt{\frac{\omega}{u}}}\sqrt{1-\frac{\omega}{u}}\,u^{3/2}} du -$$

$$-2\mathrm{Re}\sqrt{\omega}\int_{u_0}^{\infty} \frac{e^{\bar\lambda_2\left[-4\bar h + i(\bar y - \bar\eta)\sqrt{1-\frac{\omega}{u}}\right]}\bar\lambda_2^3}{\left(\bar\lambda_2\sqrt{\frac{\omega}{u}} + k_\lambda\right)\sqrt{1 - 4\tau_0\sqrt{\frac{\omega}{u}}}\sqrt{1-\frac{\omega}{u}}\,u^{3/2}} du -$$

$$-\mathrm{Re}\int_{k_\lambda}^{\infty} \lambda \frac{c^{\lambda\left[-4\bar h + i(\bar y - \bar\eta)\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\right]}\left[k_\lambda^2 + \lambda\omega\left(1 - \frac{2\tau_0 k_\lambda}{\lambda}\right) + \omega^2\tau_0^2\right]}{\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\left[k_\lambda^2 - \lambda\omega\left(1 + 2\tau_0\frac{k_\lambda}{\lambda}\right) + \omega^2\tau_0^2\right]} d\lambda. \qquad \text{(XI.54)}$$

$$u_0 = \begin{cases} \omega \text{ at } \tau_0 < \frac{1}{4}, \\ 16\varepsilon^2\omega \text{ at } \tau_0 > \frac{1}{4}, \end{cases}$$ [452

$$\lambda_1 = \frac{u}{2}\left(1 + 2\tau_0\sqrt{\frac{\omega}{u}} + \sqrt{1 + 4\tau_0\sqrt{\frac{\omega}{u}}}\right),$$

$$\lambda_2 = \frac{u}{2}\left(1 + 2\tau_0\sqrt{\frac{\omega}{u}} - \sqrt{1 + 4\tau_0\sqrt{\frac{\omega}{u}}}\right). \qquad \text{(XI.55)}$$

The parameter τ_0 is related to k_λ and ω by the expression $\,_0 = \frac{k_\lambda}{\omega}$.

Let us examine now the limiting values of the nucleus

$$G_h'(\bar y - \bar\eta) = G_k(y - \eta) - \frac{F_{-1}[k_\lambda(\bar y - \bar\eta)] - 1}{(y-\eta)^2}.$$

For $\omega \to 0$ and a finite k_λ

$$G_{h_0}(y-\eta) = -\frac{i}{\pi}\,\mathrm{Re}\int_0^{\pi} d\theta \int_0^{\infty} \frac{e^{\lambda[-4\bar{h}+i(\bar{y}-\bar{\eta})\sin\theta]}}{\lambda\cos\theta - k_\lambda}\,d\lambda -$$

$$-\,\mathrm{Re}\int_{k_\lambda}^{\infty} \frac{\lambda e^{\lambda\left[-4h+i(y-\eta)\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\right]}}{\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}}\,d\lambda. \qquad \text{(XI.56)}$$

This nucleus corresponds to the boundary condition $\theta_x = 0$ on the free surface.

For $\omega \to \infty$ and a finite k_λ

$$G_{h\infty}(y-\eta) = \frac{i}{\pi}\,\mathrm{Re}\int_0^{\pi} d\theta \int_0^{\infty} e^{\lambda[-4\bar{h}+i(\bar{y}-\bar{\eta})\sin\theta]} +$$

$$+\,\mathrm{Re}\int_{k_\lambda}^{\infty} \frac{\lambda e^{\lambda\left[-4\bar{h}+i(y-\eta)\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\right]}}{\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}}\,d\lambda. \qquad \text{(XI.57)}$$

This nucleus corresponds to the boundary condition $\theta_y = 0$ on the free surface. Formulas (XI.56) and (XI.57) yield the following result known from Chapters VIII-X: [453

$$G_{h_0}(y-\eta) = -G_{h\infty}(y-\eta).$$

The following interesting result is obtained when the Strouhal number approaches infinity:

$$G_h(y-\eta) = 0,$$

from where an important conclusion follows, namely, that for large values of Strouhal's numbers ($k_\lambda \to \infty$) the free surface does not affect the induced inclination angles of the flow due to the submerged hydrofoil.

It is interesting to note that in the steady-state case the single integrals with respect to u (XI.54) determine the effect of the Fr number, while the last integral determines the effect of the solid wall. During the process of examination of the limiting values with $\omega \to \infty$, the integrals which take into account the number k_λ approach zero, while with $\omega \to \infty$ they approach the double value of

the last integral with an opposite sign. However, for the limiting value of k_λ the first four single integrals approach zero in both limiting instances, while the limiting values are produced only by the last integral. However, if k_λ approaches zero in the limit then the steady-state case is re-established.

Let us examine the question of identity of the approximations considered above. Let us examine the integral

$$J = \int_p^\infty \frac{e^{\lambda\left[z+iy\sqrt{1-\frac{p^2}{\lambda^2}}\right]}}{\sqrt{1-\frac{p^2}{\lambda^2}}}\,d\lambda = p\int_1^\infty \frac{se^{p(sz+iy\sqrt{s^2-1})}}{\sqrt{s^2-1}}\,ds, \qquad \text{(XI.58)}$$

where $z<0$.

Substituting the variables s = ch t, one may write the integral as follows:

$$J = \int_p^\infty e^{ip\sqrt{z^2+y^2}\operatorname{sh}(t-i\varphi)}\operatorname{ch} + dt,$$

where $\sin\varphi = \frac{z}{\sqrt{z^2+y^2}};\quad \cos\varphi = \frac{y}{\sqrt{z^2+y^2}}.$

Let us introduce a new variable:

$$\operatorname{sh}(t-i\varphi) = \mu,\ \mu = \frac{y}{\sqrt{y^2+z^2}}\operatorname{sh}t - \frac{iz}{\sqrt{y^2+z^2}}\operatorname{ch}t,$$

$$\operatorname{ch}t = \frac{iz\mu \pm |y|\sqrt{1+\mu^2}}{\sqrt{y^2+z^2}}.$$

$$J = \frac{p}{\sqrt{y^2+z^2}}\int_{\frac{-iz}{\sqrt{y^2+z^2}}}^{\infty\left(\frac{y}{\sqrt{y^2+z^2}}-\frac{iz}{\sqrt{y^2+z^2}}\right)} \frac{(iz\mu \pm |y|\sqrt{1+\mu^2})}{\sqrt{1+\mu^2}}e^{ip\sqrt{z^2+y^2}\mu}\,d\mu,$$

Since $d\mu = \operatorname{ch}(t-i\varphi)\,dt,\ \operatorname{ch}(t-i\varphi) = \pm\sqrt{1+\mu^2},$ [454

$$\operatorname{ch}(t-i\varphi) = \frac{y\operatorname{ch}t}{\sqrt{z^2+y^2}} + \frac{iz\operatorname{sh}t}{\sqrt{y^2+z^2}}.$$

For $y>0$ $\operatorname{Re}\operatorname{ch}(t-i\varphi)>0$ and $\operatorname{ch}(t-i\varphi) = +\sqrt{1+\mu^2}$, for $y<0$ $\operatorname{Re}\operatorname{ch}(t-i\varphi)<0$ and $\operatorname{ch}(t-i\varphi) = -\sqrt{1+\mu^2}$.

Let us examine the positive values of y:

$$J = p \int_{\frac{-iz}{\sqrt{y^2+z^2}}}^{\infty(y-iz)} e^{ip\sqrt{z^2+y^2}\mu} \frac{(iz\mu \pm y\sqrt{1+\mu^2}}{\sqrt{y^2+z^2}} \frac{d\mu}{\sqrt{1+\mu^2}} =$$

$$= \frac{p}{\sqrt{y^2+z^2}} \left\{ \pm \frac{y}{ip\sqrt{z^2+y^2}} e^{ip\sqrt{z^2+\mu^2}\mu} \Big|_{-\frac{iz}{\sqrt{y^2+z^2}}}^{\infty(y-iz)} + \right.$$

$$\left. + iz \int_{\frac{-iz}{\sqrt{y^2+z^2}}}^{\infty\left(\frac{y}{\sqrt{y^2+z^2}} - \frac{iz}{\sqrt{y^2+z^2}}\right)} e^{ip\sqrt{y^2+z^2}\mu} \frac{\mu d\mu}{\sqrt{1+\mu^2}} \right\}.$$

Performing integration by parts we obtain

$$J = \pm \frac{iye^{pz}}{y^2+z^2} + \frac{z}{y^2+z^2} \frac{\dfrac{e^{pz}\, iz}{\sqrt{y^2+z^2}}}{\sqrt{1 - \dfrac{z^2}{y^2+z^2}}} -$$

$$- \frac{z}{y^2+z^2} \int_{\frac{-iz}{\sqrt{y^2+z^2}}}^{\infty(y-iz)} \frac{e^{ip\mu\sqrt{y^2+z^2}}}{(1+\mu^2)^{3/2}} d\mu.$$

Let us write the real part of the integral

$$\operatorname{Re} J = -\frac{z}{y^2+z^2} \operatorname{Re} \int_{\frac{-iz}{\sqrt{y^2+z^2}}}^{\infty(y-iz)} \frac{e^{ip\mu\sqrt{y^2+z^2}}}{(1+\mu^2)^{3/2}} d\mu.$$

Let us change the integration path. We have the following: [455

$$\operatorname{Re} J = -\frac{z}{y^2+z^2} \operatorname{Re} \int_{\frac{-iz}{\sqrt{y^2+z^2}}}^{\infty(y-iz)} \frac{e^{ip\mu\sqrt{y^2+z^2}}}{(1+\mu^2)^{3/2}} d\mu -$$

$$- \frac{z}{y^2+z^2} \operatorname{Re} \int_0^{\infty} \frac{e^{ip\mu\sqrt{y^2+z^2}}}{(1+\mu^2)^{3/2}} d\mu.$$

Since the first integral will be purely imaginary, then it follows

$$\mathrm{Re}\int_{p}^{\infty}\frac{e^{\lambda\left[z+iy\sqrt{1-\frac{p^2}{\lambda^2}}\right]}}{\sqrt{1-\frac{p^2}{\lambda^2}}}\,d\lambda=-\frac{pz}{y^2+z^2}\,\mathrm{Re}\int_{0}^{\infty}\frac{e^{-ipu\sqrt{y^2+z^2}}}{(u^2+1)^{3/2}}\,du. \quad \text{(XI.59)}$$

These integrals determine the real parts of the potential φ_λ given in formula (XI.48) for Fr $\to\infty$, i.e., parts obtained through the use of the approximations considered. Thus, the identity of the approximations has been proven.

The formula (XI.59) makes it possible to draw a general conclusion about the behavior of the nucleus of the equation. If we define the function $F_{-1}[k_\lambda(\bar{y}-\bar{\eta})]$ by means of the formula (XI.59), then it will follow that with $k_\lambda\to\infty$, $F_{-y}[k_\lambda\sqrt{(\bar{y}-\bar{\eta})+\xi}]_{\xi\to 0}\to 0$, and hence

$$G_{\substack{h\\ k_\lambda\to\infty}}(y-\eta)\to-\frac{1}{(y-\eta)^2},$$

therefore, we can draw the conclusion that for large values of k_λ, the submerged hydrofoil in an unsteady flow will behave as a foil with a large relative span and with $k_\lambda\to\infty$, the hydromechanical properties of a vibrating submerged foil of a finite span will be determined by the properties of the foil in a plane flow.

The solution of equation (XI.5) for the foil with the elliptical distribution of circulation will also be given by formulas (XI.47), in which function $\zeta(k_\lambda,\ \mathrm{Fr})$ is determined by the formula

$$\zeta(k_\lambda,\ \mathrm{Fr})=1-\frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\int_{-1}^{+1}\sqrt{1-\bar{\eta}^2}\,G_k(\bar{y}-\bar{\eta})\,d\eta\,dy;$$

$G_k(\bar{y}-\bar{\eta})$ is defined by formula (XI.53).

The integral equation for the lifting line of the type (VII.25) is derived from the general expression of (XI.5). [456

Let us consider that $v_z=-B_0(y)e^{ipx}$ and $(x-\xi)=-a(y)$; then from the expressions (XI.5) and (XI.27) we obtain

$$
\frac{1}{4\pi}\int_{-b}^{+b}\Gamma(\eta)\left[\frac{\partial}{\partial z}\left(-\frac{(z-\zeta)}{(y-\eta)+(z-\zeta)}\int_{\frac{a(y)}{\sqrt{(y-\eta)^2+(z-\zeta)^2}}}^{\infty} e^{-ip\lambda\sqrt{(y-\eta)^2+(z-\zeta)^2}}\times\right.\right.
$$

$$
\times\frac{d\lambda}{(\lambda^2+1)^{3/2}}+\frac{i}{2\pi}\int_{-\pi}^{\pi}d\theta\int_0^{\infty}\frac{\begin{array}{c}e^{\lambda[z+\zeta+i(y-\eta)\sin\theta]}\lambda^2[\lambda^2\cos^2\Theta+\\+\lambda\nu(1-2\tau_0\cos\Theta)+\nu^2\tau_0^2]\times\\\times e^{-ia(y)\sqrt{(y-\eta)^2+(z-\zeta)^2}}\end{array}}{\begin{array}{c}(\lambda\cos\Theta-p)[\lambda^2\cos^2\Theta-\lambda\nu\times\\\times(1-2\tau_0\cos\Theta)+\nu^2\tau_0^2]\end{array}}d\lambda+
$$

$$
+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\lambda_1^3 e^{\lambda_1[z+\zeta+i(y-\eta)\sin\theta]}\,e^{-ia(y)(\lambda_1\cos\theta-p)}}{(\lambda_1\cos\Theta-p)\sqrt{1+4\tau_0\cos\Theta}}d\Theta+
$$

$$
+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\bar{\lambda}_1^3 e^{\bar{\lambda}_1[z+\zeta+i(y-\eta)\sin\theta]}\,e^{+ia(y)(\bar{\lambda}_1\cos\theta+p)}}{(\bar{\lambda}_1\cos\Theta+p)\sqrt{1-4\tau_0\cos\Theta}}d\Theta-
$$

$$
-\int_{-\pi+\theta_0}^{+\pi-\theta_0}\frac{\lambda_2^3 e^{\lambda_2[z+\zeta+i(y-\eta)\sin\theta]}e^{-ia(y)(\lambda_2\cos\theta-p)}}{(\lambda\cos\Theta-p)\sqrt{1+4\tau_0\cos\Theta}}d\Theta+
$$

$$
\left.\left.+\operatorname{Re}\int^{\infty}\frac{\lambda e^{\lambda[z+\zeta+i(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}}\left(p^2+\lambda\nu\left(1-\frac{2\tau_0 p}{\lambda}\right)+\nu^2\tau_0^2\right)}{\sqrt{1-\frac{p^2}{\lambda^2}}\left[p^2-\lambda\nu\left(1+\frac{2\tau_0 p}{\lambda}\right)+\nu\tau_0^2\right]}d\lambda\right)\right]=
$$

$$
=B(y). \qquad z+\zeta=-2\bar{h} \tag{XI.60}
$$

11.5. The Integral Equation for the Submerged Hydrofoil in a Plane-Parallel Flow [457

The theory of the submerged hydrofoil in a plane-parallel unsteady flow was given in Ch. V. The potential and the integral equation for the plane problem are derived below by applying limits to the general expressions.

Let us assume that the span of a hydrofoil approaches infinity and the circulation is constant along the span.

It is then clear that the potentials θ and φ and the properties at each cross section, perpendicular to the longitudinal axis of the hydrofoil, will be identical. They will be defined by the following formulas:

$$\Theta(x, z) = \frac{v_0}{4\pi}\int_{-a}^{+a}\gamma(\xi)\,d\xi\int_{-\infty}^{+\infty}\frac{\partial}{\partial\zeta}G(x, y, z, \xi, \eta, \zeta)\,d\eta, \qquad \text{(XI.61)}$$

$$\varphi = -\frac{e^{ipx}}{4\pi}\int_{-a}^{+a}\gamma(\xi)\,d\xi\int_{\infty}^{x}\int_{-\infty}^{+\infty}\frac{\partial}{\partial\zeta}G(\tau, y, z, \xi, \eta, \zeta)\,e^{-ip\tau}d\tau d\eta. \qquad \text{(XI.62)}$$

The integral equation for determining density $\gamma(\xi)$ is also found from the expression (XI.5). If $v_z = -\bar{v}_z(x)$, then the equation will be as follows:

$$e^{ipx}\int_{-a}^{+a}\gamma(\xi)\,d\xi\int_{-\infty}^{+\infty}d\eta\,\frac{\partial}{\partial z}\int_{+\infty}^{x}\frac{\partial}{\partial\zeta}G(\tau, y, z, \xi, \eta, \zeta)\,e^{-ip\tau}d\tau = 4\pi\bar{v}_z(x). \qquad \text{(XI.63)}$$

For a foil in an infinite fluid

$$\frac{\partial}{\partial z}\int_{-\infty}^{+\infty}\frac{\partial}{\partial\zeta}\frac{1}{r}\,d\eta = \frac{2[(x-\xi)^2-(z-\zeta)^2]}{[(x-\xi)^2+(z-\zeta)^2]^2}.$$

Then, for a foil in an infinite fluid the equation (XI.63) will be in the following form:

$$e^{ipx}\int_{-a}^{+a}\gamma(\xi)\,d\xi\int_{\infty}^{x}\frac{e^{-ip\tau}}{(\tau-\xi)^2}\,d\tau = 2\pi\bar{v}_z(x). \qquad \text{(XI.64)}$$

For the steady-state motion $\int_{\infty}^{x}\frac{dx}{(\tau-\xi)^2} = -\frac{1}{x-\xi}$ and then we obtain the singular equation derived earlier

$$\int_{-a}^{+a}\frac{\gamma(\xi)}{x-\xi}\,d\xi = -2\pi f'(x).$$

Performing the integration by parts in the expression (XI.64), we obtain a different expression for the equation: [458

$$\int_{-a}^{+a}\frac{\gamma(\xi)}{x-\xi}\,d\xi + ipe^{ipx}\int_{-a}^{+a}\gamma(\xi)\int_{\infty}^{x}\frac{e^{-ip\tau}}{(\tau-a)}\,d\xi d\tau = -2\pi\bar{v}_z(x). \qquad \text{(XI.65)}$$

This equation reduces to the Birnbaum's equation [92].

By substituting u = τ = ζ, the equation (XI.65) can be written as follows:

$$\int_{-a}^{+a} \frac{\gamma(\xi)}{x-\xi} d\xi + ip \int_{-a}^{+a} \gamma(\xi) e^{-ip(\xi - x)} \int_{\infty}^{x-\xi} \frac{e^{-ipu} du}{u} = -2\pi \bar{v}_z(x).$$

Substituting the variable u by the variable x', (u = = x - x'), we obtain the equation

$$\int_{-a}^{+a} \frac{\gamma(\xi)}{(x-\xi)} d\xi - ip \int_{-a}^{+a} \gamma(\xi) e^{-ip\xi} \int_{-\infty}^{\xi} \frac{e^{ipx'}}{x-x'} dx d\xi = -2\pi \bar{v}_z(x). \quad \text{(XI.66)}$$

This equation can be written in the form

$$\int_{-a}^{+a} \frac{\gamma(\xi)}{(x-\xi)} d\xi - ip \left[\int_{-a}^{+a} \gamma(\xi) e^{-ip\xi} \int_{-\infty}^{+a} \frac{e^{ipx'}}{x-x'} dx d\xi + \right.$$

$$\left. + \int_{-a}^{+a} \gamma(\xi) e^{-ip\xi} \int_{a}^{\xi} \frac{e^{ipx'}}{x-x'} dx d\xi \right] = -2\pi \bar{v}_z(x).$$

If we apply Dirichlet's formula to the last double integral, we will obtain Birnbaum's equation

$$\int_{-a}^{+a} \frac{\gamma(\xi)}{(x-\xi)} d\xi - ip \left[\int_{-a}^{+a} \gamma(\xi) e^{-ip\xi} \int_{-\infty}^{+a} \frac{e^{ipx'}}{x-x'} dx d\xi + \right.$$

$$\left. + \int_{-a}^{+a} \frac{e^{-ip\xi}}{x-\xi} d\xi \int_{\xi}^{a} \gamma(x') e^{ipx'} dx' \right] = -2\pi \bar{v}_z(x). \quad \text{(XI.67)}$$

Thus, the acceleration potential methods produce the same results as the theory which is based on the physical picture of the Küssner-Birnbaum vortex formation.

Let us show how to solve the integral $\int_{-\infty}^{+\infty} \frac{\partial}{\partial \xi} \frac{1}{r'} d\eta$ [459

when $\frac{1}{r'}$ is determined with the aid of the integral expression (VII.35):

$$J = \int_{-\infty}^{+\infty} \frac{\partial}{\partial \xi} \frac{1}{r'} d\eta =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\Theta \int_{0}^{\infty}\int_{-\infty}^{\infty} e^{\lambda(z+\zeta+i\omega)}\, d\lambda d\eta =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\Theta \int_{0}^{\infty} \frac{e^{\lambda(z+\zeta+i\omega)}}{i\sin\theta}\, d\lambda \Bigg|_{-\infty}^{+\infty}.$$

The integrand expression contains specific points. Computing the remainders at these points, we obtain

$$J = +\, 2\mathrm{Re}\int_{0}^{\infty} e^{\lambda[z+\zeta+i(x-\xi)]} d\lambda.$$

Thus, for the potential φ and the induced velocity along the Oz axis we have

$$\varphi = -\frac{1}{2\pi}\int_{-a}^{+a}\gamma(\xi)\, e^{ipx}\int_{\infty}^{x} e^{ip\tau}\Bigg[\frac{(z-\zeta)}{[(\tau-\xi)^2+(z-\zeta)^2]^2} -$$

$$-\mathrm{Re}\int_{0}^{\infty} e^{\lambda[z+\zeta+i(\tau-\xi)]} d\lambda - \nu_1\int_{0}^{\infty}\frac{\lambda e^{\lambda[z-\zeta+i(\tau-\xi)]}}{\tau_0^2\lambda_0^2-2\tau_0\nu_1(1-i\beta)\lambda-\nu_1\lambda+\nu_1^2(1-2i\beta)} d\lambda -$$

$$-\nu_1\int_{0}^{\infty}\frac{\lambda e^{\lambda[z+\zeta-i(\tau-\xi)]}}{\tau_0^2\lambda^2+2\tau_0\nu_1(1-i\beta)\lambda-\lambda\nu_1+\nu_1^2(1-2i\beta)} d\lambda\Bigg] d\tau d\xi, \qquad \text{(XI.68)}$$

$$\varphi_z = -\frac{1}{2\pi}\int_{-a}^{+a}\gamma(\xi)\, e^{ipx}\int_{\infty}^{x} e^{ip\tau}\Bigg[\frac{(\tau-\xi)^2-(z-\zeta)^2}{[(\tau-\xi)^2+(z-\zeta)^2]^2} -$$

$$-\mathrm{Re}\int_{0}^{\infty} e^{\lambda[z+\zeta-i(\tau-\xi)]}\lambda d\lambda -$$

$$-\nu_1\int_{0}^{\infty}\frac{e^{\lambda[z+\zeta+i(x-\xi)]}\lambda^2 d\lambda}{\tau_0^2\lambda^2-2\tau_0\nu_1(1-i\beta)\lambda-\nu_1\lambda+\nu_1^2(1-2i\beta)} - \qquad [460$$

$$-\nu_1\int_{0}^{\infty}\frac{e^{\lambda[z+\zeta-i(\tau-\xi)]}\lambda^2 d\lambda}{\tau_0^2\lambda^2-2\tau_0\nu_1(1-i\beta)\lambda-\nu_1\lambda+\nu_1^2(1-2i\beta)}\Bigg] d\tau d\xi. \qquad \text{(XI.69)}$$

For the steady-state motion

$$\varphi_z = -\frac{1}{2\pi}\int_{-a}^{+a}\gamma(\xi)\int_{\infty}^{x}\Bigg(\frac{(\tau-\zeta)^2-(z-h)^2}{[(\tau-\zeta)^2+(z-h)^2]^2} +$$

$$+ \operatorname{Re}\int_0^\infty e^{\lambda[z+\zeta-i(\tau-\xi)]}\lambda d\lambda - \nu\int_{0(L_1)}^\infty \frac{e^{[z+\zeta+i(\tau-\xi)]\lambda}}{\lambda+i\mu-\nu}d\lambda +$$

$$+ \nu\int_{0(L_2)}^\infty \frac{e^{\lambda[z+\zeta-i(x-\xi)]}}{\lambda-i\mu-\nu}d\lambda\Bigg)d\tau d\xi. \qquad \text{(XI.70)}$$

For $z = \zeta = -h$ we obtain the following equation:

$$\int_{-a}^{+a}\gamma(\xi)\Bigg[\frac{1}{x-\xi} + \operatorname{Re} i\int_0^\infty \frac{(\lambda+\nu)e^{\lambda[-2h-i(x-\xi)]}}{\lambda-\nu} -$$

$$- 2\pi\nu\cos\nu(x-\xi)e^{-2\nu h}\Bigg]d\xi = -2\pi\bar{f}'(x). \qquad \text{(XI.71)}$$

The equation (XI.71) is easily transformed into the equation given in Ch. I. The specific points of the integrand expressions in formulas (XI.68) and (XI.69) will be derived as the roots of the equations

$$\tau_0^2\lambda^2 - 2\tau_0\nu_1(1-i\beta)\lambda - \nu_1\lambda + \nu_1^2(1-2i\beta) = 0,$$

$$\tau_0^2\lambda^2 + 2\tau_0\nu_1(1-i\beta)\lambda - \nu_1\lambda + \nu_1^2(1-2i\beta) = 0.$$

The roots of these equations have already been studied in Ch. V and they are as follows:

$$\nu'_{12} = \nu_1\frac{1-2\tau_0(1-i\beta) \pm \sqrt{1-4\tau_0-4\tau_0 i\beta-4\tau_0^2\beta^2}}{2\tau_0^2},$$

$$\nu'_{34} = \nu_1\frac{1+2\tau_0(1-i\beta) \pm \sqrt{1+4\tau_0+4i\beta\tau_0-4\tau_0^2\beta^2}}{2\tau_0^2}.$$

Selecting the remainders, the expressions (XI.68) and (XI.69) can be transformed to the following: [461

$$\varphi = -\frac{1}{2\pi}\int_{-a}^{+a}\gamma(\xi)d\xi e^{ipx}\int_\infty^x e^{-ip\tau}\Bigg(\frac{z-\zeta}{(\tau-\xi)^2+(z-\zeta)^2} -$$

$$- \operatorname{Re}\int_0^\infty e^{\lambda[z+\zeta-i(\tau-\xi)]}d\lambda - \int_0^\infty \frac{\lambda e^{\lambda[z+\zeta+i(\tau-\xi)]}}{(\lambda-\nu_3)\sqrt{1+4\tau_0}}d\lambda +$$

$$+ \int_0^\infty \frac{\lambda e^{\lambda[z+\zeta+i(\tau-\xi)]}}{(\lambda-\nu_4)\sqrt{1+4\tau_0}}d\lambda - \int_0^\infty \frac{\lambda e^{\lambda[z+\zeta-i(\tau-\xi)]}}{(\lambda-\nu_1)\sqrt{1-4\tau_0}}d\lambda +$$

$$
+\int_0^\infty \frac{\lambda e^{\lambda[z+\zeta-i(\tau-\xi)]}}{(\lambda-\nu_2)\sqrt{1-4\tau_0}}\,d\lambda+
$$

$$
+\frac{i\pi\left(\nu_3 e^{\nu_3[z+\zeta+i(\tau-\xi)]}-\nu_4 e^{\nu_4[z+\zeta+i(\tau-\xi)]}\right)}{\sqrt{1+4\tau_0}}-
$$

$$
-\frac{i\pi\left(\nu_1 e^{\nu_1[z+\zeta-i(\tau-\xi)]}+\nu_2 e^{\nu_2[z+\zeta-i(\tau-\xi)]}\right)}{\sqrt{1-4\tau_0}}\Bigg)d\tau, \qquad \text{(XI.72)}
$$

$$
\varphi_z=-\frac{1}{2\pi}\int_{-a}^{+a}\gamma(\xi)\,d\xi e^{ipx}\int_\infty^x e^{-ip\tau}\left[\frac{(\tau-\xi)^2-(z-\zeta)^2}{[(\tau-\xi)^2-(z-\zeta)^2]^2}-\right.
$$

$$
-\operatorname{Re}\int_0^\infty \lambda e^{\lambda[z+\zeta-i(\tau-\xi)]}\,d\lambda-\frac{1}{\sqrt{1+4\tau_0}}\left(\int_0^\infty \frac{e^{\lambda(z+\zeta+i(\tau-\xi)]}\lambda}{(\lambda-\nu_3)}\,d\lambda-\right.
$$

$$
\left.-\int_0^\infty \frac{\lambda e^{\lambda[z+\zeta+i(\tau-\xi)]}}{\lambda-\nu_4}\,d\lambda\right)+\frac{1}{\sqrt{1-4\tau_0}}\left(\int_0^\infty \frac{\lambda e^{\lambda[z+\zeta-i(\tau-\xi)]}}{\lambda-\nu_1}\,d\lambda-\right.
$$

$$
\left.-\int_0^\infty \frac{\lambda e^{\lambda[z+\zeta-i(\tau-\xi)]}}{\lambda-\nu_2}\,d\lambda\right)+\frac{i\pi}{\sqrt{1+4\tau_0}}\left(\nu_3^2 e^{\nu_3[z+\zeta+i(\tau-\xi)]}-\right.
$$

$$
\left.\left.-\nu_4^2 e^{\nu_4[z+\zeta+i(\tau-\xi)]}\right)-\frac{i\pi}{\sqrt{1-4\tau_0}}\left(\nu_1^2 e^{\nu_1[z+\zeta-i(\tau-\xi)]}+\nu_2^2 e^{\nu_2[z+\zeta-i(\tau-\xi)]}\right)\right]d\tau, \qquad \text{(XI.73)}
$$

where [462

$$
\nu_{12}=\nu_1\frac{1-2\tau_0\pm\sqrt{1-4\tau_0}}{2\tau_0^2},
$$

$$
\nu_{34}=\nu_1\frac{1+2\tau_0\pm\sqrt{1+4\tau_0}}{2\tau_0^2}. \qquad \text{(XI.74)}
$$

From the expression (XI.73) we will obtain the desired integral equation

$$
\int_{-a}^{+a}\gamma(\xi)\left[\frac{1}{(x-\xi)}+ipe^{ipx}\int_\infty^x \frac{e^{-ip\tau}}{(\tau-\xi)}\,d\tau+\right.
$$

$$
+e^{ipx}\int_\infty^x e^{ip\tau}\left\{\operatorname{Re}\int_0^\infty \lambda e^{\lambda[-2h-i(\tau-\xi)]}+\frac{1}{\sqrt{1+4\tau_0}}\left(\int_0^\infty \frac{\lambda e^{\lambda[-2h+i(\tau-\xi)]}}{(\lambda-\nu_3)}\,d\lambda-\right.\right.
$$

$$
\left.-\int_0^\infty \frac{\lambda e^{\lambda[-2h+i(\tau-\xi)]}}{\lambda-\nu_4}\,d\lambda\right)-\frac{1}{\sqrt{1-4\tau_0}}\left(\int_0^\infty \frac{\lambda e^{\lambda[-2h-i(\tau-\xi)]}}{\lambda-\nu_1}\,d\lambda-\right.
$$

$$-\int_0^\infty \frac{\lambda e^{\lambda[-2h-i(\tau-\xi)]}}{\lambda-\nu_2}\,d\lambda\Bigg)-\frac{i\pi}{\sqrt{1+4\tau_0}}\left(\nu_3^2 e^{\nu_3[-2h+i(\tau-\xi)]}-\right.$$

$$\left.-\nu_4^2 e^{\nu_4[-2h+i(\tau-\xi)]}\right)+\frac{i\pi}{\sqrt{1-4\tau_0}}\left(\nu_1^2 e^{\nu_1[-2h-i(\tau-\xi)]}+\right.$$

$$\left.\left.+\nu_2 e^{\nu_2[-2h-i(\tau-\xi)]}\right)\right]d\tau=-2\pi\bar{\nu}_z(x). \qquad \text{(XI.75)}$$

This equation can also be transformed to the form of (V.29).

With $\tau_0 > \frac{1}{4}$ the roots $\nu_{1,2}$ will be complex and the corresponding remainders will be equal to zero. Then, for $\tau_0 > \frac{1}{4}$, one should, in the expression (XI.75), make $\nu_{1,2}$ equal to zero in those terms which were obtained in the evaluation of the remainders.

The equation (XI.75) can be given a new form which will be of interest to us in future studies and in solving this equation:

$$\int_{-a}^{+a}\gamma(\xi)\left[\frac{1}{x-\xi}+ipe^{ipx}\int_0^x\frac{e^{-ip\tau}}{\tau-\xi}\,d\tau-\frac{1}{2}i\int_0^\infty\frac{e^{-2\lambda h}\lambda e^{-i\lambda(x-\xi)}}{p+\lambda}\,d\lambda+\right.$$

$$+\frac{1}{2}i\int_0^\infty\frac{e^{-\lambda 2h}\lambda e^{i\lambda(x-\xi)}}{p-\lambda}\,d\lambda+\frac{i}{\sqrt{1+4\tau_0}}\left(\int_0^\infty\frac{e^{-\lambda 2h}\lambda e^{i\lambda(x-\xi)}}{(p-\lambda)(\lambda-\nu_3)}\,d\lambda-\right.$$ [463

$$\left.-\int_0^\infty\lambda\frac{e^{-\lambda 2h}e^{i\lambda(x-\xi)}}{(p-\lambda)(\lambda-\nu_4)}\,d\lambda\right)-\frac{i}{\sqrt{1-4\tau_0}}\left(\int_0^\infty\frac{e^{-\lambda 2h}\lambda e^{-i\lambda(x-\xi)}}{(p+\lambda)(\lambda-\nu_1)}\,d\lambda-\right.$$

$$\left.-\int_0^\infty\frac{e^{-2h\lambda}e^{i\lambda(x-\xi)}}{(p+\lambda)(\lambda-\nu_2)}\,d\lambda\right)-\frac{\pi}{\sqrt{1+4\tau_0}}\left(\frac{\nu_3^2}{p-\nu_3}e^{\nu_3(-2h+i(x-\xi))}-\right.$$

$$\left.-\frac{\nu_4^2}{p-\nu_4}e^{\nu_4(-2h+i(x-\xi))}\right)-\frac{\pi}{\sqrt{1-4\tau_0}}\left(\frac{\nu_1^2}{p+\nu_1}e^{\nu_1[-2h-i(x-\xi)]}+\right.$$

$$\left.\left.+\frac{\nu_2^2}{p+\nu_2}e^{\nu_2[-2h-i(x-\xi)]}\right)\right]d\xi=-2\pi\bar{\nu}_z(x). \qquad \text{(XI.76)}$$

11.6. Determination of the Velocity Potential by Using the General Integral Formula

The function $G(x, y, z, \xi, \eta, \zeta)$ can be found with the aid of the integral expression (VII.34). Let us look for G_1 in the following form

$$G_1 = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}A(k, \lambda)\, e^{\sqrt{k^2+\lambda^2}(z+\zeta)} e^{i\lambda(x-\xi)+i(y-\eta)k}\, d\lambda.$$

From the differential equation (XI.8) we find

$$A(k, \lambda) = -\frac{\nu}{\left[\tau_0^2\frac{\lambda^2}{\sqrt{k^2+\lambda^2}} - \frac{2\tau_0\nu}{\sqrt{k^2+\lambda^2}}(1-i\beta) - \nu + \frac{\nu^2(1-2i\beta)}{\sqrt{k^2+\lambda^2}}\right]}.$$

Then

$$G(x, y, z) = \frac{1}{r} - \frac{1}{r'} - \frac{\nu}{\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}e^{\sqrt{k^2+\lambda^2}(z+\zeta)}\times$$

$$\times\frac{e^{i(x-\xi)\lambda}e^{i(y-\eta)k}}{\left[\tau_0^2\frac{k^2}{\sqrt{k^2+\lambda^2}} - \frac{2\tau_0\nu k(1-i\beta)}{\sqrt{k^2+\lambda^2}} - \nu + \frac{\nu^2(1-2i\beta)}{\sqrt{k^2+\lambda^2}}\right]}d\lambda \qquad \text{(XI.77)}$$

or [464

$$G(x, y, z) = \frac{1}{r} - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{\sqrt{k^2+\lambda^2}}\int_{-\infty}^{+\infty}e^{\sqrt{k^2+\lambda^2}(z+\zeta)}e^{i(x-\xi)\lambda}e^{i(y-\eta)k}\times$$

$$\times\frac{\frac{\tau_0^2\lambda^2}{\sqrt{k^2+\lambda^2}} - \frac{2\tau_0\nu\lambda}{\sqrt{k^2+\lambda^2}} + \nu + \frac{\nu^2}{\sqrt{k^2+\lambda^2}}}{\frac{\tau_0^2k^2}{\sqrt{k^2+\lambda^2}} - \frac{2\tau_0\nu k(1-i\beta)}{\sqrt{k^2+\lambda^2}} - \nu + \frac{\nu^2(1-2i\beta)}{\sqrt{k^2+\lambda^2}}}. \qquad \text{(XI.78)}$$

In accordance with formula (VII.45)

$$N(k, \lambda) = -\frac{e^{(z+\zeta)\sqrt{k^2+\lambda^2}}e^{i(y-\eta)k}\left(-\frac{\tau_0^2k^2}{\sqrt{k^2+\lambda^2}} - \frac{2\tau_0\nu\lambda}{\sqrt{k^2+\lambda^2}} + \nu + \frac{\nu^2}{\sqrt{k^2+\lambda^2}}\right)}{2\sqrt{k^2+\lambda^2}},$$

$$Q(k, \lambda) = \frac{\tau_0^2\lambda^2}{\sqrt{k^2+\lambda^2}} - \frac{2\tau_0\nu\lambda(1-i\beta)}{\sqrt{k^2+\lambda^2}} - \nu + \frac{\nu(1-2i\beta)}{\sqrt{k^2+\lambda^2}},$$

$$Q'(k, \lambda) = \frac{\tau_0^2\lambda^2(2k^2+\lambda^2) - 2\tau\nu k^2 - \nu^2\lambda^2}{(\lambda^2-k^2)^{3/2}}.$$

Let us examine the equation $Q(k, \lambda) = 0$:

$$\tau_0^2\lambda^2 - 2\tau_0\nu\lambda(1-i\beta) - \nu(\sqrt{k^2+y^2} - \nu(1-2i\beta) = 0,$$

$$\lambda = \frac{2\tau_0\nu(1-i\beta) \pm \sqrt{4\tau_0^2\nu^2(1-i\beta)^2 - 8\tau_0\nu^2\left[\sqrt{\lambda^2+k^2}-\nu(1-2i\beta)\right]}}{2\tau_0^2}, \quad \text{(XI.79)}$$

from which we obtain two equations for determining λ_1 and λ_2:

$$\lambda_1' = \frac{2\tau_0\nu(1-i\beta) + \sqrt{4\tau_0^2\nu^2(1-i\beta)^2 + 4\tau_0^2\nu^2\left[\sqrt{\lambda_1^2+k^2}-\nu(1-2i\beta)\right]}}{2\tau_0^2},$$

$$\text{(XI.80)}$$

$$\lambda_2' = \frac{2\tau_0(1-i\beta) - \sqrt{4\tau_0^2\nu^2(1-i\beta)^2 + 4\tau_0^2\nu^2\left[\sqrt{\lambda_1^2+k^2}-\nu(1-2i\beta)\right]}}{2\tau_0^2}.$$

With an accuracy involving terms of the order of β^2

$$\lambda_1' = \frac{2\tau_0\nu + \sqrt{4\tau^2\nu^2\sqrt{k^2+\lambda_1^2}} - 2\tau_0\nu i\beta}{2\tau_0^2},$$

$$\lambda_2' = \frac{2\tau_0\nu - \sqrt{4\tau^2\nu^2\sqrt{k^2+\lambda_2^2}} - 2\tau_0\nu i\beta}{2\tau^2}.$$

It follows that $\operatorname{Sign}\operatorname{Im}\lambda_1' = \operatorname{Sign}\operatorname{Im}\lambda_2' = -1$ and we can use the formula (VII.45): [465

$$\varphi = -\frac{e^{ipx}}{4\pi}\iint_s \gamma(Q)\Bigg[\int_\infty^x e^{-ip\tau}\frac{\partial}{\partial\zeta}\frac{1}{r}\,d\tau +$$

$$+\frac{\partial}{\partial\zeta}\left(\operatorname{Sign}p\int_{-\infty}^{+\infty}\frac{e^{\sqrt{k^2+p^2}(z-\zeta)}e^{i(y-\eta)k}\left(\frac{\tau_0 p^2}{\sqrt{k^2+p^2}} - \frac{2\tau_0\nu p}{\sqrt{k^2+p^2}} + \nu + \frac{\nu^2}{\sqrt{k^2+p^2}}\right)}{2\sqrt{k^2+p^2}\left(\frac{\tau_0^2p^2}{\sqrt{k^2+p^2}} - \frac{2\tau_0\nu p}{\sqrt{k^2+p^2}} - \nu + \frac{\nu^2}{\sqrt{k^2+p^2}}\right)}dk + \right.$$

$$+\sum_{l=1}^{2}\int_{-\infty}^{+\infty}\frac{e^{\sqrt{k^2+\lambda_l^2}(z-\zeta)}e^{i(y-\eta)k}e^{ix}(\lambda_l-p)(k^2+\lambda_l^2)\left(\frac{\tau_0\lambda_l^2}{\sqrt{k^2+\lambda_l^2}} - \frac{2\tau_0\nu\lambda_l}{\sqrt{k^2+\lambda_l^2}} + \nu + \frac{\nu^2}{\sqrt{k^2+\lambda_l^2}}\right)}{2(\lambda_l-p)\left[\tau_0^2\lambda_l^2(2k^2+\lambda_l^2) - 2\tau_0\nu k^2 - \nu^2\lambda_l^2\right]}dk +$$

$$e^{\sqrt{k^2+\lambda_l^2}(z+\zeta)}e^{i(y-\eta)}e^{-ix(\bar\lambda_l+p)}(k^2+\bar\lambda_l^2)\left(\frac{\tau_0\lambda_l^2}{\sqrt{k^2+\lambda_l^2}} + \right.$$

$$+\sum_{i=1}^{2}\int_{-\infty}^{\infty}\frac{+\frac{2\tau_0\nu\bar{\lambda}_j}{\sqrt{k^2+\lambda_j^2}}+\nu+\frac{\nu^2}{\sqrt{k^2+\lambda_j^2}}\Big)}{2(\bar{\lambda}_1+p)[\tau_0^2\bar{\lambda}_i^2(2k^2+\lambda_i^2)-2\tau_0\nu k^2-\nu\bar{\lambda}_i^2]}\,dk+$$

$$+\frac{i}{\pi}\int_{-\infty}^{+\infty}d\lambda\int_{-\infty}^{+\infty}\frac{e^{\sqrt{k^2+\lambda^2}(z+\zeta)}e^{i(y-\eta)k}e^{ix(\lambda-p)}}{(\lambda-p)\left[+\frac{\tau_0^2\lambda^2}{\sqrt{k^2+\lambda^2}}-\frac{2\tau_0\nu\lambda}{\sqrt{k^2+\lambda^2}}-\nu+\frac{\nu^2}{\sqrt{k^2+\lambda^2}}\right]}\times$$

$$\times\frac{\frac{\tau_0^2\lambda^2}{\sqrt{k^2+\lambda^2}}-\frac{2\tau_0\nu\lambda}{\sqrt{k^2+\lambda^2}}+\nu+\frac{\nu^2}{\sqrt{k^2+\lambda^2}}}{2\sqrt{k^2+\lambda^2}}\,dk\Bigg)\Bigg]ds. \qquad \text{(XI.81)}$$

The roots $\bar{\lambda}_j$ are the roots of the equation

$$\tau_0^2\lambda^2+2\tau_0\nu\lambda-\nu(\sqrt{k^2+\lambda^2}-\nu)=0; \qquad \text{(XI.82)}$$

$$\lambda_{1,2}=\frac{-2\nu\pm\sqrt{\lambda_{1,2}^2+k^2}\sqrt{\nu}}{\tau_0}.$$

11.7. The Submerged Hydrofoil in an Unsteady Flow of Finite Depth

For a fluid of finite depth h_0, we will look for the function $G(x, y, z)$ in the form of

$$G=\frac{1}{r}+\frac{1}{r'}+G_1, \qquad \text{(XI.83)}$$

where

$$r'=\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta+2h_0)^2};$$

G_1 is the harmonic function in the range of $0 \geqslant z \geqslant -h_0$.

We have two conditions for evaluating the function G

$$\frac{\tau_0^2}{\nu_1}G_{1xx}-2i\tau_0(1-i\beta)G_{1x}+G_{1z}-\nu(1-2i\beta)G_1=F,$$

$$F=-\left[\frac{\tau_0^2}{\nu_1}\left(\frac{1}{r}+\frac{1}{r_1'}\right)_{xx}-2i\tau_0\left(\frac{1}{r}+\frac{1}{r_1'}\right)_{xx}-\nu\left(\frac{1}{r}+\frac{1}{r_1'}\right)_x+\right.$$

$$\left.+\left(\frac{1}{r}+\frac{1}{r_1'}\right)_z\right]; \quad (z=0) \qquad \text{(XI.84)}$$

$$G_{1z}=0. \quad (z=-h_0). \tag{XI.85}$$

With the aid of the integral expression (VII.35), function F can be represented as follows:

$$F=\frac{1}{\pi}\int_{-\pi}^{+\pi}d\theta\int_0^{\infty}e^{-\lambda h_0}e^{i\lambda\omega}\operatorname{ch}\lambda(\zeta+h_0)\times$$
$$\times\left[\frac{\tau^2}{\nu_1}\lambda^2\cos^2\theta-2\tau\cos\theta\lambda+\lambda+\nu\right]d\lambda. \tag{XI.86}$$

Satisfying the condition in the relation (XI.85), we can represent the function G_1 as follows:

$$G_1=\frac{1}{\pi}\int_{-\pi}^{+\pi}d\theta\int_0^{\infty}A(\lambda,\theta)e^{-\lambda h_0}e^{i\lambda\omega}\operatorname{ch}\lambda(z+h_0)\times$$
$$\times\frac{\operatorname{ch}\lambda(\zeta+h_0)}{\operatorname{ch}\lambda h_0}\left[\frac{\tau^2}{\nu_1}\lambda^2\cos^2\theta-2\tau\cos\theta\lambda+\nu+\lambda\right]d\lambda.$$

From equation (XI.84) [467

$$A(\lambda,\theta)=\frac{1}{-\frac{\tau^2}{\nu_1}\cos^2\Theta\lambda+2\tau(1-i\beta)\lambda+\lambda\operatorname{th}\lambda h_0-\nu(1-2i\beta)}.$$

Then

$$G_1=\frac{1}{\pi}\int_{-\pi}^{+\pi}d\Theta\int_0^{\infty}\frac{e^{-\lambda h_0}e^{i\lambda\omega}\operatorname{ch}\lambda(z+h_0)\operatorname{ch}\lambda(\zeta+h_0)\times\left[\frac{\tau^2}{\nu_1}\lambda^2\cos^2\Theta-2\tau\cos\lambda+\lambda+\nu\right]}{\operatorname{ch}\lambda h_0\left[-\frac{\tau^2}{\nu}\cos^2\Theta\lambda+2\tau(1-i\beta)\cos\Theta\lambda+\lambda\operatorname{th}\lambda h_0-\nu(1-2i\beta)\right]}d\lambda. \tag{XI.87}$$

For $\tau=0$ and $\frac{\tau^2}{\nu_1}=\frac{1}{\nu}$, from formula (XI.87) it is easy to obtain the formula (IX.4).

With $\tau=0$ and $\frac{\tau^2}{\nu_1}=0$, from formula (XI.87) we obtain the M. D. Khaskind formula for the pulsating source

$$G_1=\frac{1}{\pi}\int_{-\pi}^{+\pi}d\Theta\int_0^{\infty}\frac{e^{-\lambda h_0}e^{i\lambda\omega}\operatorname{ch}\lambda(z+h_0)\operatorname{ch}\lambda(\xi+h_0)(\lambda+\nu)\,d\lambda}{\operatorname{ch}\lambda h[\lambda\operatorname{th}\lambda h_0-\nu(1-2i\beta)]}. \tag{XI.88}$$

Let us analyze the roots of the equation

$$\tau_0^2\cos^2\Theta\lambda^2 - \lambda\nu_1[2\tau_0(1-i\beta)\cos\Theta + \operatorname{th}\lambda h_0] + \nu^2(1-2i\beta) = 0, \quad (XI.89)$$

$$\lambda'_{1,2} = \nu_1 \frac{[\operatorname{th}\lambda h_0 + 2\tau_0(1-i\beta)\cos\Theta] \pm \pm\sqrt{\operatorname{th}^2\lambda h_0 + 4\tau_0\cos\Theta\operatorname{th}\lambda h_0 - 4\tau_0 i\beta\cos\Theta\operatorname{th}\lambda h_0}}{2\tau^2\cos^2\Theta} \quad (XI.90)$$

and let us obtain two transcendental equations for determining the roots λ_1 and λ_2

$$\lambda_1 = \frac{\nu_1}{2\tau_0\cos^2\Theta}[\operatorname{th}\lambda_1 h_0 + 2\tau_0\cos\Theta + \sqrt{\operatorname{th}^2\lambda_1 h_0 + 4\tau_0\cos\Theta\lambda_1 h_0}], \quad (XI.91)$$

$$\lambda_2 = \frac{\nu_1}{2\tau_0\cos^2\Theta}[\operatorname{th}\lambda_2 h_0 + 2\tau_0\cos\Theta - \sqrt{\operatorname{th}^2\lambda_2 h_0 + 4\tau_0\cos\Theta\operatorname{th}\lambda_2 h_0}]. \quad (XI.92)$$

With $\cos\theta > 0$ the roots λ_1 and λ_2 are always real and [468
with $\cos\theta < 0$, λ_1 and λ_2 will be real only when

$$\operatorname{th}\bar{\lambda}_j h_0 > 4\tau_0|\cos\Theta|. \quad (XI.93)$$

The condition $\operatorname{th}\bar{\lambda}_j h_0 = 4\tau_0|\cos\Theta|$ leads us to the equation

$$\bar{\lambda}_j = \frac{\nu_1}{4\tau_0^2\cos^2\Theta}\operatorname{th}\bar{\lambda}_j h, \quad (XI.94)$$

from which we obtain another equation for determining the real roots in the equations (XI.91) and (XI.92) as follows:

$$\frac{\nu h_0}{4\cos^2\Theta} > 1. \quad (XI.95)$$

It follows, then, that the roots $\bar{\lambda}_1$ and $\bar{\lambda}_2$ will be real only for those values of θ which satisfy the inequalities (XI.93) and (XI.95). The signs of the imaginary terms λ_j are determined in the same manner as in (XI.80). Now the function $G(x, y, z)$ can be written as follows:

$$G_1 = -\frac{\nu_1}{\pi\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Theta \int_{\substack{0\\(L_1)}}^{\infty} \frac{e^{-\lambda h_0}e^{i\lambda\omega}\operatorname{ch}\lambda(z+h_0)\operatorname{ch}\lambda(\zeta+h_0)}{\operatorname{ch}\lambda h_0\cos^2\Theta(\lambda_1-\lambda_2)} \times$$

$$\times\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta - 2\tau_0\cos\Theta\lambda + \lambda + \nu\right)\left(\frac{1}{\lambda-\lambda_1} - \frac{1}{\lambda-\lambda_2}\right)d\lambda -$$

$$-\frac{\nu_1}{\pi\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Theta \int\limits_{\substack{0\\(L_2)}}^{\infty} \frac{e^{-\lambda h_0} e^{i\lambda\omega}\,\mathrm{ch}\,\lambda(z+h_0)\,\mathrm{ch}\,\lambda(\zeta+h_0)}{\mathrm{ch}\,\lambda h_0 \cos^2\Theta\,(\lambda_1-\lambda_2)(\lambda-\bar{\lambda}_1)} \times$$

$$\times\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta + 2\tau_0\cos\Theta\lambda+\lambda+\nu\right)d\lambda -$$

$$-\frac{\nu_1}{\pi\tau^2}\int\limits_{-\Theta_0}^{+\Theta_0} d\Theta \int\limits_{0}^{\infty} \frac{e^{-\lambda h_0} e^{i\lambda\omega}\,\mathrm{ch}\,\lambda(z+h_0)\,\mathrm{ch}\,\lambda(\zeta+h_0) \times \left(\frac{\tau_0^2}{\nu}\lambda^2\cos^2\Theta - 2\tau\cos\Theta\lambda+\lambda+\nu\right)}{\mathrm{ch}\,\lambda h_0\,(\lambda-\lambda_1)(\lambda-\lambda_2)\cos^2\Theta}\,d\lambda +$$

$$+\frac{\nu_1}{\pi\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Theta \int\limits_{0}^{\infty} \frac{e^{-\lambda h_0} e^{i\lambda\omega}\,\mathrm{ch}\,\lambda(z+h_0)\,\mathrm{ch}\,\lambda(\zeta+h_0)}{\mathrm{ch}\,\lambda h_0 \cos^2\Theta\,(\lambda-\bar{\lambda}_2)(\lambda_1-\bar{\lambda}_2)} \times$$ [469

$$\times\left(\frac{\tau_0^2}{\nu}\lambda^2\cos^2\Theta + 2\tau_0\cos\Theta\lambda+\lambda+\nu\right)d\lambda, \qquad \text{(XI.96)}$$

where θ_0 is the larger angle, which is determined by the equalities

$$\Theta_0 = \arccos\sqrt{\frac{\nu h_0}{4}},$$

$$\mathrm{th}\,\frac{\nu_1 h_0}{\tau_0\cos\Theta_0} = 4\tau_0\cos\Theta_0. \qquad \text{(XI.97)}$$

Computing the remainders at the points λ_i, we may write the formula for G as follows:

$$G_1 = -\frac{1}{\pi}\int\limits_{-\pi}^{+\pi} d\Theta \int\limits_{0}^{\infty} \frac{e^{-\lambda h_0} e^{i\lambda\omega}\,\mathrm{ch}\,\lambda(z+h_0)\,\mathrm{ch}\,\lambda(\zeta+h_0)}{\mathrm{ch}\,\lambda h_0\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta - 2\tau_0\cos\Theta\lambda - \lambda\,\mathrm{th}\,\lambda h_0 + \nu_1\right]} \times$$

$$\times\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta - 2\tau_0\cos\Theta_0\lambda+\lambda+\nu\right)d\lambda +$$

$$+\frac{\nu_1}{\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{-\lambda_1 h_0}e^{i\lambda_1\omega}\operatorname{ch}\lambda_1(z+h_0)\operatorname{ch}\lambda_1(\zeta+h_0)\times\left(\frac{\tau_0^2}{\nu_1}\lambda_1^2\cos^2\Theta-2\tau_0\cos\Theta\lambda_1+\lambda_1+\nu_2\right)}{\operatorname{ch}\lambda_1 h_0\cos^2\Theta(\lambda_1-\lambda_2)}d\Theta-$$

$$-\frac{\nu_1}{\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{-\lambda_2 h_0}e^{i\lambda_2\omega}\operatorname{ch}\lambda_2(z+h_0)\operatorname{ch}\lambda_2(\zeta+h_0)\times\left(\frac{\tau_0^2}{\nu_1}\lambda_2^2\cos^2\Theta-2\tau_0\cos\Theta\lambda_2+\lambda_2+\nu_1\right)}{\operatorname{ch}\lambda_2 h_0\cos^2\Theta(\lambda_1-\lambda_2)}d\Theta-$$

$$-\frac{\nu_1}{\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{-\bar\lambda_1 h_0}e^{-i\bar\lambda_1\omega}\operatorname{ch}\bar\lambda_1(z+h_0)\operatorname{ch}\bar\lambda_1(\zeta+h_0)\times\left(\frac{\tau_0^2}{\nu_1}\bar\lambda_1^2\cos^2\Theta+2\tau_0\cos\Theta\bar\lambda_1+\bar\lambda_1+\nu_2\right)d\Theta}{\operatorname{ch}\bar\lambda_1 h_0\cos^2\Theta(\lambda_1-\lambda_2)}- \quad [470$$

$$-\frac{\nu_1}{\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{-\bar\lambda_2 h_0}e^{-i\bar\lambda_2\omega}\operatorname{ch}\bar\lambda_2(z+h_0)\operatorname{ch}\bar\lambda_2(\zeta+h_0)}{\operatorname{ch}\bar\lambda_2 h_0\cos^2\Theta(\lambda_1-\lambda_2)}\times$$

$$\times\left(\frac{\tau_0^2}{\nu_1}\bar\lambda_2^2\cos^2\Theta+2\tau_0\cos\Theta\bar\lambda_2+\bar\lambda_2+\nu_1\right)d\Theta. \qquad \text{(XI.98)}$$

Now it is not difficult to obtain the expression for the velocity potential φ:

$$\varphi=-\frac{1}{4\pi}\iint\limits_{s}\gamma(Q)\,e^{ip(\xi-x)}\left[\int\limits_{\infty}^{x}\frac{\partial}{\partial\zeta}\frac{1}{r}e^{-ip(\tau-\xi)}d\tau+\right.$$

$$+\frac{\partial}{\partial\zeta}\left[-\frac{i}{2\pi}\int\limits_{-\pi}^{+\pi}d\Theta\int\limits_{0}^{\infty}\frac{e^{\lambda[(z+\zeta+2h_0)-i(y-\eta)\sin\Theta]}e^{i(x-\xi)(\lambda\cos\Theta-p)}}{(\lambda\cos\Theta-p)}d\lambda+\right.$$

$$+\frac{i}{\pi}\int\limits_{-\pi}^{+\pi}d\Theta\int\limits_{0}^{\infty}\frac{e^{-\lambda h_0}e^{i\lambda(y-\eta)\sin\Theta}e^{i(x-\xi)(\lambda\cos\Theta-p)}\operatorname{ch}\lambda(z+h_0)\operatorname{ch}\lambda(\zeta+h_0)}{(\lambda\cos\Theta-p)\operatorname{ch}\lambda h_0}\times$$

$$\times\frac{\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta-2\tau_0\cos\Theta\lambda+\lambda+\nu_1\right)}{\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta-2\tau_0\cos\Theta\lambda-\lambda\operatorname{th}\lambda h_0+\nu_1\right)}d\lambda+$$

$$+\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_1 h_0} e^{i\lambda_1(y-\eta)\sin\Theta} e^{i(x-\xi)(\lambda_1\cos\Theta-p)} \operatorname{ch}\lambda_1(z+h_0)\operatorname{ch}\lambda_1(\zeta+h_0)}{(1-\lambda_1')\operatorname{ch}\lambda_1 h_0(\lambda_1\cos\Theta-p)\cos^2\Theta(\lambda_1-\lambda_2)}\times$$

$$\times\left(\frac{\tau_0^2}{\nu_1}\lambda_1^2\cos^2\Theta-2\tau_0\cos\Theta\lambda_1+\lambda_1+\nu_1\right)d\Theta-$$

$$-\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{\lambda_2 h_0} e^{i\lambda_2(y-\eta)\sin\Theta} e^{i(x-\xi)(\lambda_2\cos\Theta-p)} \operatorname{ch}\lambda_2(z+h_0)\operatorname{ch}\lambda_2(\zeta+h_0)}{(1+\lambda_2')\operatorname{ch}\lambda_2 h_0\cos^2\Theta(\lambda_1-\lambda_2)(\lambda_2\cos\Theta-p)}\times$$ [471

$$\times\left(\frac{\tau_0^2}{\nu_1}\lambda_2^2\cos^2\Theta-2\tau_0\cos\Theta\lambda_2+\lambda_2+\nu_1\right)d\Theta+$$

$$+\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\bar\lambda_1 h_0} e^{i\bar\lambda_1(y-\eta)\sin\Theta} e^{-i(x-\xi)(\bar\lambda_1\cos\Theta+p)} \operatorname{ch}\bar\lambda_1(z+h_0)\operatorname{ch}\lambda_1(\zeta+h_0)}{\operatorname{ch}\bar\lambda_1 h_0\cos^2\Theta(\bar\lambda_1-\bar\lambda_2)(\bar\lambda_1\cos\Theta+p)(1-\bar\lambda_1')}\times$$

$$\times\left(\frac{\tau_0^2}{\nu_1}\bar\lambda_1^2\cos^2\Theta+2\tau_0\cos\Theta\bar\lambda_1+\bar\lambda_1+\nu_1\right)d\Theta+$$

$$+\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\bar\lambda_2 h_0} e^{i\bar\lambda_2(y-\eta)\sin\Theta} e^{-i(x-\xi)(\bar\lambda_2\cos\Theta+p)} \operatorname{ch}\bar\lambda_2(z+h_0)\operatorname{ch}\lambda_2(\zeta+h_0)}{\operatorname{ch}\bar\lambda_2 h_0\cos^2\Theta(\bar\lambda_1-\bar\lambda_2)(\bar\lambda_2\cos\Theta+p)(1-\bar\lambda_2')}\times$$

$$\left.\left.\times\left(\frac{\tau_0^2}{\nu_1}\bar\lambda_2^2\cos^2\Theta+2\tau_0\cos\Theta\bar\lambda_2+\bar\lambda_2+\nu_1\right)d\Theta\right]\right\}ds. \qquad \text{(XI.99)}$$

Let us write this expression in a different form by separating the remainders at the point $\cos\theta_0 = \frac{p}{\lambda}$ and writing them as double integrals:

$$\varphi=-\frac{1}{4\pi}\iint_s \gamma(Q)e^{-ip(\xi-x)}\left\{-\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}\times\right.$$

$$\times\int_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z-\zeta)^2}}}^{\infty}\frac{e^{ipu\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{3/2}}du+\frac{\partial}{\partial\zeta}\left[-\frac{i}{2\pi}\times\right.$$

$$\times\int_{-\pi}^{+\pi}d\Theta\int_0^{\infty}\frac{e^{\lambda[(z+\zeta+2h_0)-i(y-\eta)\sin\Theta]}e^{i(x-\xi)(\lambda\cos\Theta-p)}}{(\lambda\cos\Theta-p)}d\lambda-$$

$$
-\operatorname{Re}\int_{p}^{\infty}\frac{e^{\lambda\left[-(z+\zeta+2h_0)-i(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}\right]}d\lambda}{\lambda\sqrt{1-\frac{p^2}{\lambda^2}}}+
$$ [472

$$
+\frac{i}{2\pi}\int_{-\pi}^{+\pi}d\Theta\int_{0}^{\infty}\frac{\begin{array}{c}e^{-\lambda h_0}e^{i\lambda(y-\eta)\sin\Theta}\operatorname{ch}\lambda(z+h_0)\operatorname{ch}\lambda(\zeta+h_0)\times\\ \times\left[\frac{\tau_0^2}{\nu_1}\lambda^2-2\tau_0\cos\Theta\lambda+\lambda+\nu\right]\end{array}}{\begin{array}{c}\operatorname{ch}\lambda h_0\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta+2\tau_0\cos\Theta\lambda-\right.\\ \left.-\lambda\operatorname{th}\lambda h_0+\nu_1\right](\lambda\cos\Theta-p)\end{array}}\times
$$

$$
\times e^{(x-\xi)(\lambda\cos\Theta-p)}d\lambda+
$$

$$
+2\operatorname{Re}\int_{p}^{\infty}\frac{e^{-\lambda h_0}e^{i(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}}\operatorname{ch}\lambda(z+h_0)\operatorname{ch}\lambda(\zeta+h_0)\left[\frac{\tau_0^2}{\nu_1}p^2-2\tau_0p+\lambda+\nu_1\right]}{\operatorname{ch}\lambda h_0\lambda\left[\frac{\tau_0^2}{\nu_1}p^2-2\tau_0p-\lambda\operatorname{th}\lambda h_0+\nu_1\right]\sqrt{1-\frac{p^2}{\lambda^2}}}d\lambda+
$$

$$
+\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\begin{array}{c}e^{-\lambda_1h_0}e^{i\lambda_1(y-\eta)\sin\Theta}e^{i(x-\xi)(\lambda_1\cos\Theta-p)}\operatorname{ch}\lambda_1(z+h_0)\times\\ \times\operatorname{ch}\lambda_1(\zeta+h_0)\left[\frac{\tau_0^2}{\nu_1}\lambda_1^2\cos^2\Theta-2\tau_0\cos\Theta\lambda_1+\lambda_1+\nu_1\right]\end{array}}{\operatorname{ch}\lambda_1h_0\cos^2\Theta(\lambda_1-\lambda_2)(\lambda_1\cos\Theta-p)(1-\lambda_1')}d\Theta-
$$

$$
-\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\begin{array}{c}e^{-\lambda_2h_0}e^{i\lambda_2(y-\eta)\sin\Theta}e^{i(x-\xi)(\lambda_2\cos\Theta-p)}\operatorname{ch}\lambda_2(z+h_0)\times\\ \times\operatorname{ch}\lambda_2(\zeta+h_0)\left[\frac{\tau_0^2}{\nu_1}\lambda_2^2\cos^2\Theta-2\tau_0\cos\Theta\lambda_2+\lambda_2+\nu_1\right]\end{array}}{\operatorname{ch}\lambda_2h_0\cos^2\Theta(\lambda_1-\lambda_2)(\lambda_2\cos\Theta-p)(1-\lambda_2')}d\Theta+
$$

$$
+\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\begin{array}{c}e^{-\bar\lambda_1h_0}e^{i\bar\lambda_1(y-\eta)\sin\Theta}e^{-i(x-\xi)(\bar\lambda_1\cos\Theta+p)}\operatorname{ch}\bar\lambda_1(z+h_0)\times\\ \times\operatorname{ch}\bar\lambda_1(\zeta+h_0)\left[\frac{\tau_0^2}{\nu_1}\bar\lambda_1^2\cos^2\Theta-2\tau_0\cos\Theta\bar\lambda_1+\bar\lambda_1+\nu_1\right]\end{array}}{\operatorname{ch}\bar\lambda_1h_0\cos^2\Theta(\bar\lambda_1-\bar\lambda_2)(\bar\lambda_1\cos\Theta+p)(1-\bar\lambda_1')}d\Theta+
$$

$$
+\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{-\bar\lambda_2h_0}e^{i\bar\lambda_2(y-\eta)\sin\Theta}e^{-i(x-\xi)(\bar\lambda_2\cos\Theta+p)}\operatorname{ch}\bar\lambda_2(z+h_0)\operatorname{ch}\bar\lambda_2(\zeta+h_0)}{\operatorname{ch}\bar\lambda_2h_0\cos^2\Theta(\bar\lambda_1-\bar\lambda_2)(\bar\lambda_2\cos\Theta+p)(1-\bar\lambda_2')}\times
$$ [473

$$
\times\left(\frac{\tau_0^2}{\nu_1}\bar\lambda_2^2\cos^2\Theta+2\tau_0\cos\Theta\bar\lambda_2+\bar\lambda_2+\nu_1\right)d\Theta\Bigg]\Bigg\}ds. \qquad \text{(XI.100)}
$$

11.8. The Integral Equations for the Submerged Hydrofoil in a Fluid of Finite Depth

The general integral equation for the problem is obtained in the ordinary way by using the condition (31) as follows:

$$\frac{1}{4\pi}\iint_{s}\gamma(Q)\,e^{-ip(\xi-x)}\times$$

$$\times\left\{\frac{\partial}{\partial z}\left[-\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}\int\limits_{\frac{x-\xi}{\sqrt{(y-\eta)^2+(z-\zeta)^2}}}\frac{e^{-ip\lambda\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{\frac{3}{2}}}\,du\right]-\right.$$

$$-\frac{i}{\pi}\int_{-\pi}^{+\pi}d\Theta\int_{0}^{\infty}\frac{e^{\lambda[-(z+\zeta+2h_0)+i(y-\eta)\sin\Theta]}e^{i(x-\xi)(\lambda\cos\Theta-p)}\lambda^2}{(\lambda\cos\Theta-p)}\,d\lambda-$$

$$-\operatorname{Re}\int_{p}^{\infty}\frac{e^{\lambda[-(z+\zeta+2h_0)-i(y-\eta)\sin\Theta)}\lambda}{\sqrt{1-\frac{p^2}{\lambda^2}}}\,d\lambda+$$

$$+\frac{i}{2\pi}\int_{-\pi}^{+\pi}d\Theta\int_{0}^{\infty}\frac{\begin{gathered}e^{-\lambda h_0}e^{i\lambda(y-\eta)\sin\Theta}e^{i(x-\xi)(\lambda\cos\Theta-p)}\lambda^2\operatorname{sh}\lambda(z+h_0)\times\\ \times\operatorname{sh}\lambda(\zeta+h_0)\left[\frac{\tau_0^2}{\nu_1}\tau_0^2\cos^2\Theta-2\tau_0\cos\Theta\lambda+\lambda+\nu_1\right]\end{gathered}}{\begin{gathered}\operatorname{ch}\lambda h_0\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta-2\tau_0\cos\Theta\lambda-\lambda\operatorname{th}\lambda h_0+\right.\\ \left.+\nu_1\right](\lambda\cos\Theta-p)\end{gathered}}\,d\lambda+$$

$$+2\operatorname{Re}\int_{p}^{\infty}\frac{\begin{gathered}e^{-\lambda h_0}e^{i\lambda(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}}\lambda\operatorname{sh}\lambda(z+h_0)\operatorname{sh}\lambda(\zeta+h_0)\times\\ \times\left[\frac{\tau_0^2}{\nu_1}p^2-2\tau_0p+\lambda+\nu_1\right]\end{gathered}}{\operatorname{ch}\lambda h_0\left[\frac{\tau_0^2}{\nu_1}p^2-2\tau_0p-\lambda\operatorname{th}\lambda h_0+\nu_1\right]\sqrt{1-\frac{p^2}{\lambda^2}}}\,d\lambda+$$

$$+\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\begin{gathered}e^{-\lambda_1h_0}e^{i\lambda_1(y-\eta)\sin\Theta}e^{i(x-\xi)(\lambda_1\cos\Theta-p)}\lambda_1^2\operatorname{sh}\lambda_1(z+h_0)\operatorname{sh}\lambda_1(\zeta+h_0)\times\\ \times\left[\frac{\tau_0^2}{\nu_1}\lambda_1^2\cos^2\Theta-2\tau_0\cos\Theta\lambda_1+\lambda_1+\nu_1\right]\end{gathered}}{\operatorname{ch}\lambda_1h_0\cos^2\Theta\,(\lambda_1-\lambda_2)\,(\lambda_1\cos\Theta-p)(1-\lambda_1')}\,d\Theta-$$ [474

$$-\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\begin{gathered}e^{-\lambda_2h_0}e^{i\lambda_2(y-\eta)\sin\Theta}e^{i(x-\xi)(\lambda\cos\Theta-p)}\lambda_2^2\operatorname{sh}\lambda_2(z+h_0)\operatorname{sh}\lambda_2(\zeta+h_0)\times\\ \times\left[\frac{\tau_0^2}{\nu_1}\lambda_2^2\cos^2\Theta-2\tau_0\cos\Theta\lambda_2+\lambda_2+\nu_2\right]\end{gathered}}{\operatorname{ch}\lambda_2h_0\cos^2\Theta\,(\lambda_1-\lambda_2)\,(\lambda_2\cos\Theta-p)\,(1-\lambda_2')}\,d\Theta+$$

$$
+\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\bar{\lambda}_1 h_0} e^{i\bar{\lambda}_1 (y-\eta)\sin\Theta} e^{-i(x-\xi)(\bar{\lambda}_1\cos\Theta+p)}\bar{\lambda}_1^2 \operatorname{sh}\bar{\lambda}_1(z+h_0)\operatorname{sh}\bar{\lambda}_1(\zeta+h_0)\times \left[\frac{\tau_0^2}{\nu_1}\bar{\lambda}_1^2\cos^2\Theta - 2\tau_0\cos\Theta\bar{\lambda}_1+\bar{\lambda}_1+\nu_1\right]}{\operatorname{ch}\bar{\lambda}_1 h_0\cos^2\Theta(\bar{\lambda}_1-\bar{\lambda}_2)(\bar{\lambda}_1\cos\Theta+p)(1-\bar{\lambda}_1')}\, d\Theta -
$$

$$
-\frac{\nu_1}{\tau_0^2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\bar{\lambda}_2 h_0} e^{i\bar{\lambda}_2 (y-\eta)\sin\Theta} e^{-i(x-\xi)(\bar{\lambda}_1\cos\Theta+p)}\bar{\lambda}_2^2 \operatorname{sh}\bar{\lambda}_2(z+h_0)\operatorname{sh}\bar{\lambda}_2(\zeta+h_0)}{\operatorname{ch}\bar{\lambda}_2 h_0\cos^2\Theta(\bar{\lambda}_1-\bar{\lambda}_2)(\bar{\lambda}_2\cos\Theta+p)(1-\bar{\lambda}_2')}\times
$$

$$
\left.\times\left(\frac{\tau_0^2}{\nu_1}\bar{\lambda}_2^2\cos^2\Theta - 2\tau_0\cos\Theta\bar{\lambda}_2+\bar{\lambda}_2+\nu_1\right)d\Theta\Bigg]\Bigg]\right|_{z=\zeta=-h} ds=-v_z \quad . \qquad \text{(XI.101)}
$$

Using the theory of the lifting line the vertical velocity φ_{z_λ} will be determined by the formula

$$
\varphi_{z_\lambda} = -\frac{e^{ipx}}{4\pi}\int_{-b}^{+b}\Gamma(\eta)\left[\frac{\partial}{\partial z}\left(-\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}\int_0^\infty \frac{e^{-ip\lambda\sqrt{(y-\eta)^2+(z-\zeta)^2}}}{(u^2+1)^{\frac{3}{2}}}du\right)-\right.
$$

$$
-\frac{i}{\pi}\int_{-\pi}^{+\pi}d\Theta\int_0^\infty \frac{e^{\lambda[-(z+\zeta+2h_0)+i(y-\eta)\sin\Theta]}\lambda^2 d\lambda}{(\lambda\cos\Theta-p)}-
$$

$$
-\operatorname{Re}\int_p^\infty \frac{e^{\lambda\left[-(z+\zeta+2h_0)+i(y-\eta)\sqrt{1-\frac{p^2}{\lambda^2}}\right]}\lambda d\lambda}{\sqrt{1-\frac{p^2}{\lambda^2}}}+ \quad [475
$$

$$
+\frac{i}{2\pi}\int_{-\pi}^{+\pi}d\Theta\int_0^\infty \frac{e^{-\lambda h_0}e^{i\lambda(y-\eta)\sin\Theta}\lambda^2\operatorname{sh}\lambda(z+h_0)\operatorname{sh}\lambda(\zeta+h_0)\times\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta-2\tau_0\cos\Theta\lambda+\lambda+\nu_1\right)d\lambda}{\operatorname{ch}\lambda h_0\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\Theta-2\tau_0\cos\Theta\lambda-\lambda\operatorname{th}\lambda h-\nu_1\right](\lambda\cos\Theta-p)}+
$$

$$
+2\operatorname{Re}\int_p^\infty \frac{e^{-\lambda h_0}e^{i\lambda\sqrt{1-\frac{p^2}{\lambda^2}}(y-\eta)}\lambda\operatorname{sh}\lambda(z+h_0)\operatorname{sh}\lambda(\zeta+h_0)\times\left(\frac{\tau_0^2}{\nu_1}p^2-2\tau_0 p+\lambda+\nu_2\right)d\lambda}{\operatorname{ch}\lambda h_0\left[\frac{\tau_0^2}{\nu_1}p^2-2\tau_0 p-\lambda\operatorname{th}\lambda h_0+\nu_1\right]\sqrt{1+\frac{p^2}{\lambda^2}}}+
$$

$$+\frac{\nu_1}{\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_1 h_0} e^{i\lambda_1 (y-\eta)\sin\Theta} \lambda_1^2 \operatorname{sh}\lambda_1 (z+h_0) \operatorname{sh}\lambda_1(\zeta+h_0) \times \left(\frac{\tau_0^2}{\nu_1}\lambda_1^2\cos^2\Theta - 2\tau_0\cos\Theta\,\lambda_1 + \lambda_1 + \nu_1\right)}{\operatorname{ch}\lambda_1 h_0 \cos^2\Theta(\lambda_1-\lambda_2)(\lambda_1\cos\Theta - p)(1-\lambda_1')} d\Theta -$$

$$-\frac{\nu_1}{\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_2 h_0} e^{i\lambda_2 (y-\eta)\sin\Theta} \lambda_2^2 \operatorname{sh}\lambda_2 (z+h_0) \operatorname{sh}\lambda_2(\zeta+h_0) \times \left(\frac{\tau_0^2}{\nu_1}\lambda_2^2\cos^2\Theta - 2\tau_0\cos\Theta\lambda_2 + \lambda_2 + \nu_1\right)}{\operatorname{ch}\lambda_2 h_0 \cos^2\Theta(\lambda_1-\lambda_2)(\lambda_2\cos\Theta - p)(1-\lambda_2')} d\Theta +$$

$$+\frac{\nu_1}{\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\bar\lambda_1 h_0} e^{i\bar\lambda_1 (y-\eta)\sin\Theta} \bar\lambda_1^2 \operatorname{sh}\bar\lambda_1 (z+h_0) \operatorname{sh}\bar\lambda_1(\zeta+h_0) \times \left[\frac{\tau_0^2}{\nu_1}\bar\lambda_1^2\cos^2\Theta - 2\tau_0\cos\Theta\bar\lambda_1 + \bar\lambda_1 + \nu_1\right]}{\operatorname{ch}\bar\lambda_1 h_0 \cos^2\Theta(\bar\lambda_1-\bar\lambda_2)(\bar\lambda_1\cos\Theta + p)(1-\bar\lambda_1')} d\Theta -$$

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$$-\frac{\nu_1}{\tau_0^2}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\bar\lambda_2 h_0} e^{i\bar\lambda_2 (y-\eta)\sin\Theta} \bar\lambda_2^2 \operatorname{sh}\bar\lambda_2 (z+h_0) \operatorname{sh}\lambda(\zeta+h_0) \times \left[\frac{\tau_0^2}{\nu_1}\bar\lambda_2^2\cos^2\Theta - 2\tau_0\cos\Theta\bar\lambda_2 + \bar\lambda_2 + \nu_1\right]}{\operatorname{ch}\bar\lambda_2 h_0 \cos^2\Theta(\bar\lambda_1-\bar\lambda_2)(\bar\lambda_2\cos\Theta + p)(1-\bar\lambda_2')} d\Theta \Bigg] d\eta. \quad \text{(XI.102)}$$

Through the usual approach we arrive at the equation (XI.50) with the nucleus

$$G_k(y-\eta) = \frac{F_{-1}[k_\lambda(\bar y - \bar\eta)] - 1}{(y-\eta)^2} +$$

$$+\frac{i}{\pi}\operatorname{Re}\int\limits_0^{\pi} d\Theta \int\limits_0^{\infty} e^{\lambda[-4(\bar h_0 - \bar h)]}\frac{\lambda^2 e^{i(y-\eta)\sin\Theta}}{(\lambda\cos\Theta - k_\lambda)} d\lambda +$$

$$+\operatorname{Re}\int\limits_{k_\lambda}^{\infty} \frac{e^{\lambda\left[-4(\bar h_0 - \bar h) + i(y-\eta)\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\right]}}{\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}} d\lambda -$$

$$
-\frac{i}{\pi}\operatorname{Re}\int_0^{\pi} d\Theta \int_0^{\infty} e^{-2\lambda\bar{h}_0} e^{i\lambda(\bar{y}-\bar{\eta})\sin\Theta} \frac{\lambda^2 \operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h})[\lambda^2\cos^2\Theta + \lambda\omega(1-2\tau_0\cos\Theta)+\omega^2\tau_0^2]\,d\lambda}{\operatorname{ch} 2\lambda h_0(\lambda\cos\Theta - k_\lambda)[\lambda^2\cos^2\Theta - \lambda\omega(\operatorname{th} 2\lambda h_0 + 2\tau_0\cos\Theta)+\omega^2\tau_0^2]} -
$$

$$
-2\operatorname{Re}\int_{k_\lambda}^{\infty} \frac{e^{-2\lambda\bar{h}_0} e^{i\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}}\lambda \operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h}) \times \left[k_\lambda^2 + \lambda\omega\left(1-\frac{2\tau_0 k_\lambda}{\lambda}\right)+\omega^2\tau_0^2\right] d\lambda}{\operatorname{ch} 2\lambda h_0 \sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\left[k_\lambda^2 - \lambda\omega\left(\operatorname{th} 2\lambda h_0 + 2\tau_0\frac{k_\lambda}{\lambda}\right)+\omega^2\tau_0^2\right]} -
$$

$$
-\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-2\lambda_1 h_0} e^{i\lambda_1(\bar{y}-\bar{\eta})\sin\Theta}\lambda_2^2 \operatorname{sh}^2 2\lambda_1(\bar{h}_0-\bar{h})[\lambda_1^2\cos^2\Theta + \lambda_1\omega(1-2\tau_0\cos\Theta)+\omega^2\tau_0^2]}{\operatorname{ch} 2\lambda_1\bar{h}_0\cos^2\Theta(\lambda_1-\lambda_2)(\lambda_1\cos\Theta - k_\lambda)(1-\lambda_1)} d\Theta +
$$

$$
+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-2\lambda_2 h_0} e^{i\lambda_2(y-\eta)\sin\Theta}\lambda_2^2 \operatorname{sh}^2 2\lambda_2(\bar{h}_0-\bar{h}_1)[\lambda_2^2\cos^2\Theta + \lambda_2\omega(1-2\tau_0\cos\Theta)+\omega^2\tau_0^2]}{\operatorname{ch} 2\lambda_2 h_0\cos^2\Theta(\lambda_1-\lambda_2)(\lambda_2\cos\Theta - k_\lambda)(1-\lambda_2')} d\Theta -
$$

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$$
-\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-2\bar{\lambda}_1 h_0} e^{i\bar{\lambda}_1(y-\eta)\sin\Theta}\bar{\lambda}_1^2 \operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h})[\bar{\lambda}_1^2\cos^2\Theta + \lambda\omega(1+2\tau_0\cos\Theta)+\omega^2\tau_0^2]}{\operatorname{ch} 2\bar{\lambda}_1 h_0\cos^2\Theta(\bar{\lambda}_1-\bar{\lambda}_2)(\bar{\lambda}_1\cos\Theta + k_\lambda)[1-\bar{\lambda}_1')} d\Theta +
$$

$$
+\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-2\bar{\lambda}_2 h_0} e^{i\lambda_1(\bar{y}-\bar{\eta})\sin\Theta}\bar{\lambda}_2^2 \operatorname{sh}^2 2\bar{\lambda}_2(\bar{h}_0-\bar{h})[\bar{\lambda}_2^2\cos^2\Theta + \lambda\omega(1+2\tau_0\cos\Theta)+\omega^2\tau_0^2]}{\operatorname{ch} 2\bar{\lambda}_2 h_0\cos^2\Theta(\bar{\lambda}_1-\bar{\lambda}_2)(\bar{\lambda}_2\cos\Theta + k_\lambda)(1-\lambda_2)} d\Theta. \qquad \text{(XI.103)}
$$

For the steady-state motion

$$
G_0(\bar{y}-\bar{\eta}) = \operatorname{Re}\int_0^{\infty} \lambda e^{[-\lambda(\bar{h}_0-\bar{h})+i(y-\eta)]} + 2\operatorname{Re}\int_0^{\infty} \frac{e^{-2\lambda\bar{h}_0} e^{i\lambda(\bar{y}-\bar{\eta})}\lambda \operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h})}{\operatorname{sh} 2\lambda h_0} d\lambda -
$$

$$-2\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{2\lambda_1 h_0}e^{i\lambda(y-\eta)\sin\Theta}\lambda_1(\lambda_1\cos^2\Theta+\omega)\,\mathrm{sh}^2\,2\lambda_1(\bar{h}_0-\bar{h}_1)}{\mathrm{ch}\,2\lambda h_0\cos^3\Theta(1-\lambda_1')}d\Theta. \quad \text{(XI.104)}$$

Here $\lambda_1=\frac{\omega}{\cos^2\Theta}\,\mathrm{th}\,\lambda_1 h_0$ and the integration extends to θ, which is determined from the inequality $\frac{\omega h_0}{\cos^2\Theta}>1$. We can show that with $\omega \begin{smallmatrix}\to 0\\ \to\infty\end{smallmatrix}$ the single integrals with respect to θ approach zero and the limiting values will be determined by double integrals only.

For $\omega\to 0$

$$G_k(\bar{y}-\bar{\eta})=\frac{F_{-1}[k_\lambda(\bar{y}-\bar{\eta})]-1}{(\bar{y}-\bar{\eta})^2}+\frac{i}{\pi}\,\mathrm{Re}\int_0^\pi d\Theta\int_0^\infty\frac{e^{-\lambda(\bar{h}_0-\bar{h})}e^{i(\bar{y}-\bar{\eta})\sin\Theta}}{(\lambda\cos\Theta-k_\lambda)}d\lambda+$$

$$+\,\mathrm{Re}\int_{k_\lambda}^\infty\frac{e^{\lambda\left[-i(\bar{h}_0-\bar{h})+i(\bar{y}-\bar{\eta})\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\right]}}{\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}}d\lambda- \quad [478$$

$$-\frac{i}{\pi}\,\mathrm{Re}\int_0^\pi d\Theta\int_0^\infty\frac{e^{-2\lambda h_0}e^{i\lambda(\bar{y}-\bar{\eta})\sin\Theta}\lambda^2\,\mathrm{sh}^2\,2\lambda(\bar{h}_0-\bar{h}_1)}{\mathrm{ch}\,2\lambda h_0(\lambda\cos\Theta-k_\lambda)}d\lambda-$$

$$-2\,\mathrm{Re}\int_{k_\lambda}^\infty\frac{e^{-2\lambda h_0}e^{i\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}}\lambda\,\mathrm{sh}^2 2\lambda(\bar{h}_0-\bar{h})}{\mathrm{ch}\,2\lambda\bar{h}_0\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}}d\lambda. \quad \text{(XI.105}$$

For $\omega\to\infty$

$$G_k(\bar{y}-\bar{\eta})=\frac{F_{-1}[k_\lambda(\bar{y}-\bar{\eta})]-1}{(\bar{y}-\bar{\eta})^2}+$$

$$+\frac{i}{\pi}\,\mathrm{Re}\int_0^\pi d\Theta\int_0^\infty\frac{e^{-4\lambda(\bar{h}_0-\bar{h})}e^{i(\bar{y}-\bar{\eta})\sin\Theta}\lambda^2}{(\lambda\cos\Theta-k_\lambda)}d\lambda+$$

$$+\,\mathrm{Re}\int_{k_\lambda}^\infty\frac{e^{\lambda\left[-4(\bar{h}_0-\bar{\eta})+i(\bar{y}-\bar{\eta})\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}\right]}\lambda}{\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}}d\lambda+$$

$$+\frac{i}{\pi}\operatorname{Re}\int_0^{\pi} d\Theta \int_0^{\infty} \frac{e^{-2\lambda h_0} e^{i\lambda(\bar{y}-\bar{\eta})\sin\Theta} \lambda^2 \operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h}_1)}{\operatorname{sh} 2\lambda\bar{h}_0 (\lambda\cos\Theta - k_\lambda)} d\lambda +$$

$$+2\operatorname{Re}\int_{k_\lambda}^{\infty} \frac{e^{-2\lambda_0} e^{i\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{k_\lambda^2}{\lambda^2}}} \lambda \operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h})}{\operatorname{sh} 2\lambda\bar{h}_0 \sqrt{1-\frac{k_\lambda^2}{\lambda^2}}} d\lambda. \qquad \text{(XI.106)}$$

From formulas (XI.105) and (XI.106) with $k_\lambda \to 0$ we obtain the limiting results given in Ch. VIII, which may also be obtained from the expression (XI.104).

In the same way as in a fluid with finite depth, the nucleus $G_k(y - \eta)$ with $k_\lambda \to 0$ approaches $-\frac{1}{(\bar{y}-\bar{\eta})^2}$. The solution of the problem for the foil with the elliptical distribution of circulation in an infinite fluid will also be given by formula (XI.59), in which the nucleus should be determined by formula (XI.103). [479

11.9. <u>The Submerged Hydrofoil in a Two-Dimensional Fluid Flow of Finite Depth</u>

The velocity potential φ of the foil in a plane flow will be determined by formula (XI.62). Let us utilize the formula (XI.87) as follows:

$$\varphi = -\frac{1}{2\pi}\int_{-a}^{+a} \gamma(\xi) e^{ipx} \int_{\infty}^{x} e^{-ip\tau} \left(\frac{(z-\zeta)}{(\tau-\zeta)^2 + (z-\zeta)^2} - \operatorname{Re}\int_0^{\infty} e^{\lambda[-(z+\zeta+2h_0)]} \times \right.$$

$$\times e^{\lambda i(\tau-\xi)} d\lambda - \int_0^{\infty} \frac{e^{-\lambda h_0} e^{i\lambda(\tau-\xi)} \operatorname{ch}\lambda(z+h_0)\operatorname{sh}\lambda(\zeta+h_0) \times [\tau_0^2\lambda^2 + \lambda\nu_1(1-2\tau_0)+\nu_1^2] d\lambda}{\operatorname{ch}\lambda h_0 \{\tau_0^2\lambda^2 - \lambda\nu_1[\operatorname{th}\lambda h_0 + 2\tau_0(1-i\beta)] + \nu_1^2(1-2i\beta)\}} -$$

$$\left. - \int_0^{\infty} \frac{e^{-\lambda h_0} e^{-i\lambda(\tau-\xi)} \operatorname{ch}\lambda[z+h_0]\operatorname{sh}\lambda[\zeta+h_0][\tau_0^2\lambda^2 + \lambda\nu_1(1+2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda h_0 \{\tau_0^2\lambda^2 - \lambda\nu[\operatorname{th}\lambda h_0 - 2\tau_0(1-i\beta)] + \nu_1^2(1-2i\beta)\}} d\lambda \right) d\tau d\xi. \qquad \text{(XI.107)}$$

The formula for the induced vertical velocity has the form

$$
\begin{aligned}
\varphi_z = -\frac{1}{2\pi}\int_{-a}^{+a}\gamma(\xi)e^{ipx}\int_{\infty}^{x}e^{-ip\tau}\Bigg(&\frac{(\tau-\xi)^2-(z-\zeta)^2}{[(\tau-\xi)^2+(z-\zeta)^2]^2}+ \\
&+\operatorname{Re}\int_0^{\infty}e^{\lambda[-(z+\zeta+2h_0)+i(\tau-\xi)]}\lambda\,d\lambda+ \\
+\int_0^{\infty}\frac{e^{-\lambda h_0}e^{i\lambda(\tau-\xi)}\,\mathrm{sh}\,\lambda[z+h_0]\,\mathrm{sh}\,\lambda[\zeta+h_0][\tau_0^2\lambda^2+\lambda\nu_1(1-2\tau_0)+\nu_1^2]}{\mathrm{ch}\,\lambda h_0\{\tau_0^2\lambda^2-\lambda\nu_1[\mathrm{th}\,\lambda h_0+2\tau_0(1-i\beta)]+\nu_1^2(1-2i\beta)\}}\,d\lambda&- \\
-\int_0^{\infty}\frac{e^{-\lambda h_0}e^{-i\lambda(\tau-\xi)}\,\mathrm{sh}\,\lambda(z+h_0)\,\mathrm{sh}\,\lambda[\zeta+h_0][\tau_0^2\lambda^2+\lambda\nu_1(1+2\tau_0)+\nu_1^2]}{\mathrm{ch}\,\lambda h_0\{\tau_0^2\lambda^2-\lambda\nu_1[\mathrm{th}\,\lambda h_0-2\tau_0(1-i\beta)]+\nu_1^2(1-2i\beta)\}}\,d\lambda&\Bigg)d\tau\,d\xi. \qquad \text{(XI.108)}
\end{aligned}
$$

For determining the roots of the equations [480

$$
\begin{aligned}
&\tau_0^2\lambda^2-\lambda\nu_1[\mathrm{th}\,\lambda h_0+2\tau_0(1-i\beta)]+\nu_1^2(1-2i\beta)=0, \\
&\tau_0^2\lambda^2-\lambda\nu_1[\mathrm{th}\,\lambda h_0-2\tau_0(1-i\beta)]+\nu_1^2(1-2i\beta)=0
\end{aligned} \qquad \text{(XI.109)}
$$

we have the equations

$$
\left.
\begin{aligned}
&\lambda'_{1,4}=\frac{\nu_1}{2\tau_0^2}\Big[\mathrm{th}\,\lambda h_0+2\tau_0\pm\sqrt{\mathrm{th}^2\lambda h_0+4\tau_0\,\mathrm{th}\,\lambda h_0-} \\
&\qquad -2i\beta\tau\Big(1\pm\frac{\mathrm{th}\,\lambda h_0}{\sqrt{\mathrm{th}^2\lambda h_0}}\Big], \\
&\lambda'_{2,3}=\frac{\nu_1}{2\tau_0^2}\Big[\mathrm{th}\,\lambda h_0-2\tau_0\pm\sqrt{\mathrm{th}^2\lambda h_0-4\tau_0\,\mathrm{th}\,\lambda h_0+} \\
&\qquad +2i\beta\tau\Big(1\pm\frac{\mathrm{th}\,\lambda h_0}{\sqrt{\mathrm{th}^2\lambda h_0-4\tau_0\mathrm{th}_0\,\tau}}\Big), \\
&\operatorname{Sign}\operatorname{Im}\lambda'_2=\operatorname{Sign}\operatorname{Im}\lambda'_3=\operatorname{Sign}\operatorname{Im}\lambda'_4=-1, \\
&\operatorname{Sign}\operatorname{Im}\lambda'_1=+1
\end{aligned}
\right\} \qquad \text{(XI.110)}
$$

and therefore, during the integration, the specific points λ_2, λ_3, λ_4 should be bypassed from above along the path L_1, while the point λ_1 should be bypassed from below along the path L_2. The roots $\lambda_{3,4}$ are real for all values of τ_0, while the roots $\lambda_{1,2}$ will be real only for the τ_0 values which are determined from the relationships $\mathrm{th}\,\lambda_i h_0 > 4\tau_0$ and $\frac{\nu_{1h}}{4\tau_0^2} > 1$. It follows from these relationships that $\lambda_{1,2}$ will

be real only for the values of τ_0 smaller than those determined by the equation $\operatorname{th} \nu h \tau_0 = 4\tau_0$ under the condition that $\frac{\nu h}{4} > 1.$ (XI.111)

By determining the remainders at the λ_i points, the formulas can be transformed into the following form:

$$\varphi = -\frac{1}{2\pi}\int_{-a}^{+a} \gamma(\xi)\, d\xi e^{ipx} \int_{\infty}^{x} e^{-ip\tau} \left\{ \frac{(z-\zeta)}{(\tau-\xi)^2+(z-\zeta)^2} - \right.$$

$$- \operatorname{Re} \int_0^{\infty} e^{\lambda[-(z+\zeta+2h_0)+i(\tau-\xi)]} d\lambda -$$

$$- \int_0^{\infty} \frac{e^{-\lambda h_0} e^{i\lambda(\tau-\xi)} \operatorname{ch} \lambda[z+h_0] \operatorname{sh} \lambda(\zeta+h_0)[\tau_0^2\lambda^2+\lambda_1\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch} \lambda h_0 [\tau_0^2\lambda^2 - \lambda\nu_1 [\operatorname{th} \lambda h_0 + 2\tau_0] + \nu_1^2]} d\lambda -$$

$$- \int_0^{\infty} \frac{e^{-\lambda h_0} e^{-i\lambda(\tau-\xi)} \operatorname{ch} \lambda(z+h_0) \operatorname{sh} \lambda(\zeta+h_0)[\tau_0^2\lambda^2+\lambda\nu_1(1+2\tau_0)+\nu_1^2] d\lambda}{\operatorname{ch} \lambda h_0 [\tau_0^2\lambda^2 - \lambda\nu_1 (\operatorname{th} \lambda h_0 - 2\tau_0] + \nu_1^2]} +$$ [481

$$+ \frac{i\pi}{\tau_0^2(\lambda_3-\lambda_4)} \left(\frac{e^{-\lambda_3 h_0} e^{i\lambda_3(\tau-\xi)} \operatorname{ch} \lambda_3(z+h_0) \operatorname{sh} \lambda_3(\zeta+h_0) \times \times [\tau_0^2\lambda_3^2+\lambda_3\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch} \lambda_3 h_0 (1-\lambda_{3\lambda})} - \right.$$

$$\left. - \frac{e^{-\lambda_4 h_0} e^{i\lambda_4(\tau-\xi)} \operatorname{ch} \lambda_4(z+h_0) \operatorname{sh} \lambda_4(\zeta+h_0)[\tau_0^2\lambda_4^2+\lambda_4\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch} \lambda_4 h_0 (1-\lambda_{4\lambda})} \right) -$$

$$- \frac{i\pi}{\lambda_0^2(\lambda_1-\lambda_2)} \left[\frac{e^{-\lambda_1 h_0} e^{-i\lambda_1(\tau-\xi)} \operatorname{ch} \lambda_1(z+h_0) \operatorname{sh} \lambda_1(\zeta+h_0) \times \times [\tau_0^2\lambda_1^2+\lambda_1\nu_1(1+2\tau_0)+\nu_1^2]}{\operatorname{ch} \lambda_1 h_0 (1-\lambda_{1\lambda})} + \right.$$

$$\left. \left. + \frac{e^{-\lambda_2 h_0} e^{-i\lambda_2(\tau-\xi)} \operatorname{ch} \lambda_2(z+h_0) \operatorname{sh} \lambda(\zeta+h_0) \times \times [\tau_0^2\lambda_2^2+\lambda_2\nu_1(1+2\tau_0)+\nu_1^2]}{\operatorname{ch} \lambda_2 h_0 (1-\lambda_{2\lambda})} \right] \right\} d\tau,$$

$$\varphi_z = -\frac{1}{2\pi}\int_{-a}^{+a} \gamma(\xi)\, e^{ipx} d\xi \int_{\infty}^{x} e^{-ip\tau} \left\{ \frac{(\tau-\xi)^2-(z-\zeta)^2}{[(x-\xi)^2+(z-\zeta)^2]^2} + \right.$$

$$+ \operatorname{Re} \int_0^{\infty} e^{\lambda[-(z+\zeta+2h_0)+i(\tau-\xi)]} \lambda\, d\lambda -$$

$$- \int_0^{\infty} \frac{\lambda e^{-\lambda h_0} e^{i\lambda(\tau-\xi)} \operatorname{sh} \lambda(z+h_0) \operatorname{sh} \lambda(\zeta+h_0)[\tau_0^2\lambda^2+\lambda\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch} \lambda h_0 [\tau_0^2\lambda^2 - \lambda\nu_1 (\operatorname{th} \lambda h_0 + 2\tau_0) + \nu_1^2]} d\lambda -$$

$$-\int_0^\infty \frac{\lambda e^{-\lambda h_0} e^{-i\lambda(\tau-\xi)} \operatorname{sh}\lambda(z+h_0)\operatorname{sh}\lambda(\zeta+h_0)[\tau_0^2\lambda^2+\lambda\nu_1(1+2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda h_0[\tau_0^2\lambda^2-\lambda\nu_1(\operatorname{th}\lambda h_0-2\tau_0)+\nu_1^2]}d\lambda+$$

$$+\frac{i\pi}{\tau_0^2(\lambda_3-\lambda_4)}\left[\frac{e^{-\lambda_3 h_0}e^{i\lambda_3(\tau-\xi)}\operatorname{sh}\lambda_3(z+h_0)\operatorname{sh}\lambda_3(\zeta+h_0)[\tau_0^2\lambda_3^2+\lambda_3\nu_1(1-2\tau_0)+\nu_1^2]\lambda_3}{\operatorname{ch}\lambda_3 h_0(1-\lambda_{3\lambda})}-\right.$$

$$\left.-\frac{e^{-\lambda_4 h_0}e^{i\lambda_4(\tau-\xi)}\operatorname{sh}\lambda_4(z+h_0)\operatorname{sh}\lambda_4(\zeta+h_0)[\tau_0^2\lambda_4^2+\lambda_4\nu_1(1-2\tau_0)+\nu_1^2]\lambda_4}{\operatorname{ch}\lambda_4 h_0(1-\lambda_{4\lambda})}\right]-$$

$$-\frac{i\pi}{\tau_0^2(\lambda_1-\lambda_2)}\left[\frac{e^{-\lambda_1 h_0}e^{-i\lambda_1(\tau-\xi)}\operatorname{sh}\lambda_1(z+h_0)\operatorname{sh}\lambda_1(\zeta+h_0)[\tau_0^2\lambda_1^2+\lambda_1\nu_1(1+2\tau_0)+\nu_1^2]\lambda_1}{\operatorname{ch}\lambda_1 h_0(1-\lambda_{1\lambda})}+\right.$$ [482

$$\left.\left.+\frac{\lambda_2 e^{-\lambda_2 h_0}e^{-i\lambda_2(\tau-\xi)}\operatorname{sh}\lambda_2(z+h_0)\operatorname{sh}\lambda_2(\zeta+h_0)[\tau_0^2\lambda_2^2+\lambda_2\nu_1(1+2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda_2 h_0(1-\lambda_{2\lambda})}\right]\right\}d\tau. \qquad \text{(XI.112)}$$

The integral equation for the two-dimensional problem will be

$$\int_{-a}^{+a}\gamma(\xi)\left\{\frac{1}{(x-\xi)}+ipe^{ipx}\int_\infty^x\frac{e^{-ip\tau}}{(\tau-\xi)}d\tau+\right.$$

$$+e^{ipx}\int_\infty^x e^{-ip\tau}\left[-\operatorname{Re}\int_0^\infty e^{\lambda[-2(h_0-h)-i(\tau-\xi)]}\lambda d\lambda+\right.$$

$$+\int_0^\infty\frac{e^{-\lambda h_0}e^{i\lambda(\tau-\xi)}\operatorname{sh}^2\lambda(h_0-h)[\tau_0^2\lambda^2+\lambda\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda h_0[\tau_0^2\lambda^2-\lambda\nu_1(\operatorname{th}\lambda h_0+2\tau_0)+\nu_1^2]}d\lambda+$$

$$+\int_0^\infty\frac{e^{-\lambda h_0}e^{-i\lambda(\tau-\xi)}\operatorname{sh}^2\lambda(h_0-h)[\tau_0^2\lambda^2+\lambda\nu_1(1+2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda h_0[\tau_0^2\lambda^2-\lambda\nu_1(\operatorname{th}\lambda h_0-2\tau_0)+\nu_1^2]}d\lambda-$$

$$-\frac{i\pi}{\tau_0^2(\lambda_3-\lambda_4)}\left(\frac{e^{-\lambda_3 h_0}e^{i\lambda_3(\tau-\xi)}\operatorname{sh}^2\lambda_3(h_0-h)[\tau_0^2\lambda_3^2+\lambda_3\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda_3 h_0(1-\lambda_{3\lambda})}-\right.$$

$$\left.-\frac{e^{-\lambda_4 h_0}e^{i\lambda_4(\tau-\xi)}\operatorname{sh}^2\lambda_4(h_0-h)[\tau_0^2\lambda_4^2+\lambda_4\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda_4 h_0(1-\lambda_{4\lambda})}\right)+$$

$$+\frac{i\pi}{\tau_0^2(\lambda_1-\lambda_2)}\left(\frac{e^{-\lambda_1 h_0}e^{-i\lambda_1(\tau-\xi)}\operatorname{sh}^2\lambda_1(h_0-h)[\tau_0^2\lambda_1^2+\lambda_1\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda_1 h_0(1-\lambda_{1\lambda})}+\right.$$

$$+\frac{e^{-\lambda_2 h_0} e^{-i\lambda_2(\tau-\xi)} \operatorname{sh}^2 \lambda_2(h_0-h)[\tau_0^2\lambda_2^2+\lambda_2\nu_1(1-2\tau_0)+\nu_1^2]}{\operatorname{ch}\lambda_2 h_0(1-\lambda_{2\lambda})}\Bigg)\Bigg]d\tau\Bigg\}d\xi =$$

$$= -2\pi\bar{v}_z(x). \qquad \text{(XI.114)}$$

Here $\lambda_{i\lambda} = \frac{d\lambda_i}{d\lambda}$.

As in the case of equation (XI.75), we can write the equation (XI.114) in a different form. The derivative given above is also designated as λ_i'.

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$$\int_{-a}^{+a}\gamma(\xi)\Bigg\{\frac{1}{x-\xi}+ipe^{ipx}\int_{\infty}^{x}\frac{e^{-ip\tau}}{\tau-\xi}d\tau-\frac{i}{2}\int_{0}^{\infty}\frac{e^{-2(h_0-h)}e^{-i\lambda(x-\xi)}\lambda}{p+\lambda}d\lambda-$$

$$-\frac{i}{2}\int_{0}^{\infty}e^{\frac{-2(h_0-h)}{}}\frac{e^{i\lambda(x-\xi)}}{p-\lambda}d\lambda+$$

$$+i\int_{0}^{\infty}\frac{e^{-\lambda h_0}e^{i\lambda(x-\xi)}\operatorname{sh}^2\lambda(h_0-h)[\tau_0^2\lambda^2+\lambda\nu_1(1-2\tau_0)+\nu_1^2]}{(p-\lambda)\operatorname{ch}\lambda h_0[\tau_0^2\lambda^2-\lambda\nu_1(\operatorname{th}\lambda h_0+2\tau_0)+\nu_1^2]}d\lambda+$$

$$+i\int_{0}^{\infty}\frac{e^{-\lambda h_0}e^{-i\lambda(x-\xi)}\operatorname{sh}^2\lambda(h_0-h)[\tau_0^2\lambda^2+\lambda\nu_1(1+2\tau_0)+\nu_1^2]}{(p+\lambda)\operatorname{ch}\lambda h_0[\tau_0^2\lambda^2-\lambda\nu_1(\operatorname{th}\lambda h_0-2\tau_0)+\nu_1^2]}d\lambda+$$

$$+\frac{\pi}{\tau_0^2(\lambda_3-\lambda_4)}\Bigg[\frac{e^{-\lambda_3 h_0}e^{i\lambda_3(x-\xi)}\operatorname{sh}^2\lambda_3(h_0-h)[\tau_0^2\lambda_3^2+\lambda_3\nu_1(1-2\tau_0)+\nu_1^2]}{(p-\lambda_3)(1-\lambda_{3\lambda})\operatorname{ch}\lambda_3 h_0}-$$

$$-\frac{e^{-\lambda_4 h_0}e^{i\lambda_4(x-\xi)}\operatorname{sh}^2\lambda_4(h_0-h)[\tau_0^2\lambda_4^2+\lambda_4\nu_1(1-2\tau_0)+\nu_1^2]}{(p-\lambda_4)(1-\lambda_{4\lambda})\operatorname{ch}\lambda_4 h_0}\Bigg]-$$

$$-\frac{\pi}{\tau_0^2(\lambda_1-\lambda_2)}\Bigg[\frac{e^{-\lambda_1 h_0}e^{-i\lambda_1(x-\xi)}\operatorname{sh}^2\lambda_1(h_0-h)[\tau_0^2\lambda_1^2+\lambda_1\nu_1(1+2\tau_0)+\nu_1^2]}{(p+\lambda_1)(1-\lambda_{1\lambda})\operatorname{ch}\lambda_1 h_0}+$$

$$+\frac{e^{-\lambda_2 h_0}e^{-i\lambda_2(x-\xi)}\operatorname{sh}^2\lambda_2(h_0-h)[\tau_0^2\lambda_2^2+\lambda_2\nu_1(1+2\tau_0)+\nu_1^2]}{(p+\lambda_2)(1-\lambda_{2\lambda})\operatorname{ch}\lambda_2 h_0}\Bigg]\Bigg\}d\xi=$$

$$=-2\pi\bar{v}_z(x). \qquad \text{(XI.115)}$$

For the steady-state motion

$$p=0,\ \tau=0,\ \lambda_3=\lambda_1=\lambda_0=\nu\operatorname{th}\lambda_0 h_0,\ \lambda_4=\lambda_2=0,\ \lambda_{0\lambda}=\frac{\nu h_0}{\operatorname{ch}^2\lambda_0 h_0}$$

and the equation (XI.115) will acquire the form

$$\int_{-a}^{+a} \gamma(\xi) \left[\frac{1}{x-\xi} - \operatorname{Re} i \int_0^\infty e^{-2\lambda(h_0-h)} e^{-i\lambda(x-\xi)} d\lambda - \right.$$

$$- \operatorname{Re} i \int_0^\infty \frac{e^{-\lambda h_0} e^{i\lambda(x-\xi)} \operatorname{sh}^2 \lambda (h_0 - h)(\lambda + \nu)}{\operatorname{ch} \lambda h_0 (\lambda - \nu \operatorname{th} \lambda h_0)} d\lambda -$$

$$\left. - 2\pi \frac{e^{-\lambda_0 h_0} \operatorname{sh}^2 \lambda_0 (h_0 - h)(\lambda_0 + \nu) \cos \lambda_0 (x - \xi)}{\operatorname{ch} \lambda_0 h_0 \left(1 - \frac{\nu h_0}{\operatorname{ch}^2 \lambda_0 h_0}\right)} \right] d\xi = -2\pi f'(x) \qquad \text{(XI.116)}.$$

CHAPTER XII. MOTION OF LIFTING SYSTEMS NEAR THE INTERFACE OF FLUIDS WITH DIFFERENT DENSITIES [484

In Ch. VI two-dimensional problems for the hydrofoil motion near the interface of fluids with different densities were examined. This chapter is devoted to the study of the three-dimensional motion of lifting systems near the interface of fluids with different densities. The solutions are also formed by employing the method of the velocity potential.

For simplicity, let us examine only those fluid flows which have equal velocities at infinity. For the general case of flows with different velocities at infinity, the solutions differ only slightly from those given below; they will be only somewhat more complex due to the more general boundary conditions at the interface.

For the problems dealing with the steady-state motion, the boundary conditions for the acceleration potentials at the interface are obtained from the conditions (VII.8).

$$\bar{\varrho}(\theta_{1xx} - \mu Q_{1x} + \nu Q_z) - (\theta_{2xx} - \mu\theta_{2x} + \nu\theta_{2z}) = 0, \qquad \text{(XII.1)}$$

$$\theta_{1z} = \theta_{2z} \qquad z = 0 \qquad \text{(XII.2)}$$

The other boundary conditions for the problem are those given in (VII.9)-(VII.11).

Here, as in Ch. VI, the index "1" will designate the upper and the index "2" the lower half-space (half-plane). If one of the fluids has a finite width (depth), then the acceleration potential will satisfy the additional condition on the limiting surface

$$\theta_{jz} = 0, \qquad z = h_j. \qquad \text{(XII.3)}$$

In studying the unsteady-state flows we will, as in Ch. XI, analyze such unsteady motions which are characterized by a finite steady-state forward velocity and infinitely small unsteady-state velocity increments, periodically varying with time.

If $\theta_j(x, y, z, t) = \theta_{j0}(x, y, z) + \bar{\theta}_{j1}(x, y, z)e^{i\omega t}$ is the total acceleration potential, then the potentials $\theta_{j0}(x, y, z)$ will be determined by the boundary conditions (XII.1), (XII.2). For dedetermining the potentials $\bar{\theta}_j(x, y, z)$, the conditions at the interface will be in the following form: [485

$$\bar{\varrho}\left[\bar{\theta}_{1z} - 2i\tau_0(1 - i\beta)\bar{\theta}_{1x} - \nu_1(1 - 2i\beta)\bar{\theta}_1 + \frac{\tau_0^2}{\nu_1}\bar{\theta}_{1xx}\right] -$$

$$-\left[\bar{\theta}_{2z} - 2i\tau_0(1 - i\beta)\bar{\theta}_{2x} - \nu_1(1 - 2i\beta)\bar{\theta}_2 + \frac{\nu_0^2}{\nu_1}\theta_{2xx}\right] = 0,$$

$$\bar{\theta}_{1z} = \bar{\theta}_{2z} \qquad \text{(XII.4)}$$

The methods developed in Ch. VII-XI produce effective solutions for a wide variety of problems dealing with the motion of lifting surfaces near the interface. In this chapter, the prime attention will be directed at the derivation of the general expressions for the Green function, the velocity potentials, and general integral equations.

For establishing the general patterns and obtaining simple formulas, an approximate solution of the lifting line equation for the hydrofoil with the elliptical distribution of circulation in an infinite flow is derived. This solution is used in all chapters discussing the hydrodynamics of the finite-span submerged hydrofoil. This solution is, to some extent, a universal solution making it possible, also in this chapter, to use the expansion in parameter τ, formulas and tables for the $G_{sp}(\lambda)$ functions, and to avoid time-consuming calculations.

The integral equations corresponding to Weissinger theory approximations can easily be obtained from the general integral equations for the arbitrary lifting surface, if we draw together the chord to a point and assume that $x - \xi = -a(y)$ (a is the half-chord of the foil). Therefore, we will not consider them separately; we will analyze only the integro-differential equations for Prandtl's lifting line.

12.1. Motion of the Lifting Surface Above the Interface of Fluids with Different Densities

Let us examine the problem of steady-state motion of a lifting surface above the interface of fluids with different densities.

Using the Fourier method the following equation was obtained for the potentials of a three-dimensional source located at the point ζ in the upper half-space: /486

$$G_1 = \frac{1}{r} + \operatorname{Re} i\bar{\nu}(1-a)\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\bar{\nu}}{\cos^2\theta}[-(z+\zeta)+i\omega]}\sec^2\theta d\theta +$$

$$+\frac{1}{2\pi}\operatorname{Re} a\int\limits_{-\pi}^{-\pi} d\theta \int\limits_{0}^{\infty} e^{\lambda[-(z+\zeta)+i\omega]}\frac{\cos^2\theta\lambda-\nu}{\cos^2\theta\lambda-\bar{\nu}}d\lambda, \qquad \text{(XII.5)}$$

$$G_2 = -\frac{2\operatorname{Re} i\bar{\nu}\bar{\varrho}}{1+\bar{\varrho}}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\bar{\nu}}{\cos^2\theta}[(z-\zeta)+i\omega]}\sec^2\theta d\theta +$$

$$+\frac{\operatorname{Re}}{\pi}\frac{\bar{\varrho}}{1+\bar{\varrho}}\int\limits_{-\pi}^{+\pi} d\theta\int\limits_{0}^{\infty} e^{\lambda[(z-\zeta)+i\omega]}\frac{\lambda\cos^2\theta}{\cos^2\theta\lambda-\bar{\nu}}d\lambda,$$

$$r = \sqrt{(x-\zeta)^2+(y-)\eta^2+(z-\zeta)^2} \quad \omega = i(y-\eta)\sin\theta + i(x-\zeta)\cos\theta. \qquad \text{(XII.6)}$$

From the general formulas (VIII.3)-(VIII.33) we obtain the expressions for the potentials θ_i and φ_i:

$$\theta_1 = \frac{v_0}{4\pi}\iint\limits_s \gamma(\theta)\left(\frac{(z-\zeta)}{r^{\frac{3}{2}}} - \operatorname{Re} i\bar{\nu}^2(1-a)\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\bar{\nu}}{\cos^2\theta}[-(z+\zeta)+i\omega]}\sec^4\theta d\theta -\right.$$

$$\left.-\frac{1}{2\pi}\operatorname{Re} a\int\limits_{-\pi}^{+\pi} d\theta\int\limits_{0}^{\infty}\frac{e^{\lambda[-(z+\zeta)+i\omega]}\lambda(\cos^2\theta\lambda-\nu)}{\cos^2\theta\lambda-\bar{\nu}}d\lambda\right)ds, \qquad \text{(XII.7)}$$

$$\theta_2 = \frac{v_0}{4\pi}\iint\limits_s \gamma(\theta)\left(\frac{2\operatorname{Re} i\bar{\nu}^2\bar{\varrho}_1}{1+\bar{\varrho}}\int\limits_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{\nu}}{\cos^2\theta}[(z-\zeta)+i\omega]}\sec^4\theta d\theta -\right.$$

$$\left.-\frac{\operatorname{Re}}{\pi}\frac{\bar{\varrho}}{1+\bar{\varrho}}\int\limits_{-\pi}^{+\pi} d\theta\int\limits_{0}^{\infty} e^{\lambda[(z-\zeta)+i\omega]}\frac{\lambda^2\cos^2\theta}{\cos^2\theta\lambda-\bar{\nu}}d\lambda\right)ds. \qquad \text{(XII.8)}$$

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$$\varphi_1 = -\frac{1}{4\pi}\iint\limits_s \gamma(\Theta)\left[\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{(x-\xi)}{r}-1\right)-\right.$$

$$- \operatorname{Re} \bar{v} (1-a) \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{v}}{\cos^2\theta}[-(z+\zeta)+i\omega]} \sec^3\theta d\theta +$$

$$+ \frac{a}{2\pi} \operatorname{Re} i \int_{-\pi}^{+\pi} \frac{d\theta}{\cos\theta} \int_0^\infty \frac{e^{\lambda[-(z+\zeta)+i\omega]}(\cos^2\theta\lambda - v)}{\cos^2\theta\lambda - \bar{v}} d\lambda +$$

$$\left. + \operatorname{Re} \int_0^\infty e^{\lambda[-(z+\zeta)+i(y-\eta)]} d\lambda \right] ds, \qquad \text{(XII.9)}$$

$$\varphi_2 = -\frac{1}{4\pi} \iint_s \gamma(\theta) \left(\frac{2 \operatorname{Re} \bar{v} \bar{\varrho}}{1+\bar{\varrho}} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{v}}{\cos^2\theta}[(z-\zeta)+i\omega]} \sec^3\theta d\theta + \right.$$

$$\left. + \frac{\operatorname{Re}}{\pi} \frac{i\bar{\varrho}}{1+\varrho} \int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{[(z-\zeta)+i\omega]} \frac{\lambda^2 \cos\theta}{\cos^2\theta\lambda - v} d\lambda \right) ds. \qquad \text{(XII.10)}$$

From formula (XII.9) it is easy to obtain the following:

$$\varphi_{1z} = -\frac{1}{4\pi} \iint_s \gamma(\theta) \left[\frac{\partial}{\partial z} \frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2} \left(\frac{(x-\xi)}{r} - 1 \right) + \right.$$

$$+ \operatorname{Re} \bar{v}^2 (1-a) \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\bar{v}}{\cos^2\theta}[z+\zeta+i\omega]} \sec^5\theta d\theta +$$

$$+ \frac{a}{2\pi} \operatorname{Re} i \int_{-\pi}^{+\pi} \frac{d\theta}{\cos\theta} \int_0^\infty \frac{e^{\lambda[-(z+\zeta)+i\omega]}\lambda(\cos^2\theta\lambda - v)}{\cos^2\theta\lambda - \bar{v}} d\lambda -$$

$$\left. - \operatorname{Re} \int_0^\infty e^{\lambda[-(z+\zeta)+i(y-\eta)]} d\lambda \right] ds, \qquad \text{(XII.11)}$$ [488

the general integral equation will then be

$$\frac{1}{4\pi} \iint \gamma(\theta) \left[\frac{1}{(y-\eta)^2} \left(\frac{(x-\xi)}{\sqrt{(x-\xi)^2+(y-\eta)^2}} - 1 \right) + \right.$$

$$+ Re\,\bar{v}^2(1-a)\int\limits_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{v}}{\cos^2\theta}(-2h+i\omega)} \sec^5\theta d\theta +$$

$$+\frac{a}{2\pi}\operatorname{Re} i\int\limits_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\int\limits_{0}^{\infty} e^{\lambda[-2h+i\omega]}\frac{\lambda(\cos^2\theta\lambda-\nu)}{\cos^2\theta\lambda-\bar{\nu}}d\lambda -$$

$$-\operatorname{Re}\int\limits_{0}^{\infty} e^{\lambda[-2h+i(y-\eta)]}d\lambda\Bigg]ds = v_0\alpha. \qquad \text{(XII.12)}$$

Using the lifting line theory we can obtain the integro-differential equation in the form

$$\Gamma(\bar{y}) = \frac{a_h}{2\lambda(y)}\left\{\alpha(y) - \frac{\operatorname{Re}}{2\pi}\int\limits_{-1}^{+1}\Gamma'(\eta)\left[\frac{1}{\bar{y}-\bar{\eta}} + G(\bar{y}-\bar{\eta})\right]\right\},$$

where function $G(\bar{y} - \bar{\eta})$ is defined by the formula

$$G(\bar{y}-\bar{\eta}) = \operatorname{Re}\left[\bar{\omega}(1-a)\int\limits_{0}^{\infty}\frac{\lambda+1}{\lambda}e^{-\bar{\omega}i(\lambda+1)\left[(\bar{y}-\bar{\eta})\sqrt{\frac{\lambda}{\lambda+1}}-4i\bar{h}\right]}d\lambda -\right.$$

$$\left.-\frac{1}{\bar{y}-\bar{\eta}-4i\bar{h}}\right], \qquad \text{(XII.13)}$$

where $\bar{\omega} = \bar{\nu}l$.

With $a = \frac{1-\bar{\varrho}}{1+\varrho} = 0$, the integro-differential equation (XII.13) transforms into the equation for an airplane wing. Let us write the function $G(\bar{y} - \bar{\eta})$ in such a form from which the following result follows easily:

$$G(\bar{y}-\eta) = (1-a)\int\limits_{\bar{\omega}}^{\infty} e^{-4\lambda\bar{h}}\frac{\lambda}{\lambda-\omega}\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\bar{\omega}}{\lambda}}\,d\lambda -$$ [489

$$-\int\limits_{0}^{\infty} e^{-4\lambda\bar{h}}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda. \qquad \text{(XII.14)}$$

With $\bar{\omega}\to 0$

$$G_0(\bar{y}-\bar{\eta}) = -a\int\limits_{0}^{\infty} e^{-4\lambda\bar{h}}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda, \qquad \text{(XII.15)}$$

with $\omega\to\infty$

$$G_{\infty}(\bar{y}-\bar{\eta}) = -\int_0^{\infty} e^{-4\lambda \bar{H}} \sin\lambda(\bar{y}-\bar{\eta})\, d\lambda.$$

It is clear that functions $G_0(\bar{y} - \bar{\eta})$ and $G_{\bar{\omega}}(\bar{y} - \bar{\eta})$ have the same sign and differ only by a factor "a", which for air and water is close to unity. It follows then that the characteristics of the hydrofoil of finite span will be close to those of the hydrofoil near a solid wall.

The equation (XII.13) can be solved by the methods used in Ch. VIII by developing the expansion of the function $G(\bar{y} - \bar{\eta})$ in powers of parameter τ. This expansion can also be written in the form given in (VIII.33), by using the function $G'_{s,p}(\frac{\bar{\omega}}{\tau})$ instead of $G_{s,p}(\frac{\omega}{\tau})$. The functions $G'_{s,p}(\frac{\bar{\omega}}{\tau})$ are expressed by the formula

$$G'_{s,p}\left(\frac{\bar{\omega}}{\tau}\right) = \frac{(1-a)\, e^{-\frac{\bar{\omega}}{\tau}} \left(\frac{\omega}{\tau}\right)^{s-p}}{p!\,(s-p-1)!\,(p+1)!} \int_0^{\infty} e^{-\frac{\bar{\omega}}{\tau}\lambda} (\lambda+1)^{\frac{s}{2}+\frac{1}{2}} \lambda^{\frac{s}{2}-p-\frac{3}{2}}\, d\lambda \quad \text{(XII.16)}$$

and are related to the function $G_{s,p}(\frac{\bar{\omega}}{\tau})$ by the following formula:

$$G'_{s,p}\left(\frac{\bar{\omega}}{\tau}\right) = (1-a)\left[G_{s,p}\left(\frac{\bar{\omega}}{\tau}\right) + 1\right] - 1. \quad \text{(XIII.17)}$$

For a hydrofoil with the elliptical distribution of circulation in an infinite fluid, the solution of equation (XII.13) can be obtained by the methods which were discussed in the preceding chapters. If the coefficients of the lifting force and drag in an ideal liquid are determined by formulas (VIII.39), then the function ζ can be determined from the formula similar to that given in (VIII.40)

With $\bar{\omega} \to \infty$, the formula (XII.18), which corresponds to the case of a hydrofoil moving near a solid wall, is in the form [490

$$\zeta_{\infty} = 1 - (0{,}5\tau^2 + 0{,}25\tau^4 + 0{,}0625\tau^6 + 0{,}0469\tau^8 + \\ + 0{,}0237\tau^{10} + 0{,}0188\tau^{12} + \ldots). \quad \text{(XIII.18)}$$

With $\omega \to 0$, ζ_0 is determined by the formula

$$\zeta_0 = 1 - a - a\zeta_{\infty}. \quad \text{(XII.19)}$$

12.2. Motion of the Lifting Surface Below the Interface Between Fluids with Different Densities

The solution of this problem is developed in the same manner as was done for the problem in Section 1.

Again, the expression for the potential of the three-dimensional source, located at point ζ in the lower half-space, is

$$G_1(x, y, z, \xi_2, \eta_2, \zeta_2) = -\frac{\mathrm{Re}\,2i\bar{\nu}}{\pi(1+\bar{\varrho})}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{\nu}}{\cos^2\theta}[-(z-\zeta_2)+i\omega_2]}\sec^2\theta\,d\theta + \frac{\mathrm{Re}}{\pi}\frac{1}{1+\bar{\varrho}}\int_{-\pi}^{+\pi} d\Theta\int_0^{\infty} e^{\lambda[-(z-\zeta_2)+i\omega_2]}\frac{\lambda\cos^2\theta}{\lambda\cos^2\theta-\bar{\nu}}\,d\lambda, \qquad \text{(XII.20)}$$

$$G_2(x, y, z, \xi_2, \eta_2, \zeta_2) = \frac{1}{r} + \mathrm{Re}\,i\bar{\nu}(1+a)\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{\nu}}{\cos^2\theta}[(z+\zeta_2)+i\omega_2]}\sec^2\theta\,d\theta - \frac{\mathrm{Re}}{2\pi}a\int_{-\pi}^{+\pi}\int_0^{\infty} e^{\lambda[(z+\zeta_2)+i\omega_2]}\frac{\lambda\cos^2\theta+\nu}{\lambda\cos^2\theta-\nu}\,d\lambda\,d\theta. \qquad \text{(XII.21)}$$

The expressions for the potentials have the following form:

$$Q_1 = \frac{v_0}{4\pi}\iint_s \gamma(\theta)\left(-\frac{\mathrm{Re}\,2i\bar{\nu}^2}{\pi(1+\bar{\varrho})}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{\nu}}{\cos^2\theta}[-(z-\zeta)+i\omega]}\sec^4\theta\,d\theta + \frac{\mathrm{Re}}{\pi(1+\bar{\varrho})}\int_{-\pi}^{+\pi} d\Theta\int_0^{\infty} e^{\lambda[-(z-\zeta)+i\omega]}\frac{\lambda^2\cos^2\theta}{\lambda\cos^2\theta-\bar{\nu}}\,d\lambda\right)ds, \qquad \text{(XII.22)}$$

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$$Q_2 = \frac{v_0}{4\pi}\iint_s \gamma(\theta)\left(\frac{(z-\zeta)}{r^{3/2}} + \mathrm{Re}\,i\bar{\nu}^2(1+a)\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\bar{\nu}}{\cos^2\theta}[(z+\zeta)+i\omega]}\sec^4\theta\,d\theta - \frac{\mathrm{Re}}{2\pi}a\int_{-\pi}^{+\pi} d\Theta\int_0^{\infty} e^{\lambda[(z+\zeta)+i\omega]}\frac{\lambda(\lambda\cos^2\theta+\nu)}{\lambda\cos^2\theta-\bar{\nu}}\,d\lambda\right)ds, \qquad \text{(XII.23)}$$

$$\varphi_1 = -\frac{1}{4\pi}\iint_s \gamma(\theta)\Bigg(-\frac{\mathrm{Re}\,2\bar{\nu}}{\pi(1+\bar{\varrho})}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{\frac{\bar{\nu}}{\cos^2\theta}[-(z-\zeta)+i\omega]}\sec^3\theta\, d\theta -$$

$$-\frac{\mathrm{Re}}{\pi(1+\bar{\varrho})}\, i\int_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\int_0^{\infty} e^{\lambda[-(z-\zeta)+i\omega]}\frac{\lambda\cos\theta}{\lambda\cos^2\theta-\bar{\nu}}\,d\lambda\Bigg)\,ds. \qquad \text{(XII.24)}$$

$$\varphi_2 = -\frac{1}{4\pi}\iint_s \gamma(\theta)\Bigg[\frac{(z-\zeta)}{(y-\eta)^2-(z+\zeta)^2}\left(\frac{(x-\xi)}{r}-1\right)+$$

$$+\mathrm{Re}\,\bar{\nu}(1+a)\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{\nu}}{\cos^2\theta}[z+\zeta+i\omega]}\sec^3\theta\, d\theta +$$

$$+\frac{\mathrm{Re}\,ai}{2\pi}\int_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\int_0^{\infty} e^{\lambda[(z+\zeta)+i\omega]}\frac{\lambda\cos^2\theta+\nu}{\lambda\cos^2\theta-\bar{\nu}}-$$

$$-\mathrm{Re}\int_0^{\infty} e^{\lambda[z+\zeta+i(y-\eta)]}\,d\lambda\Bigg]\,ds. \qquad \text{(XII.25)}$$

With a = 1, φ_2 is transformed into the expression (VIII.8) for the potential of the hydrofoil submerged under the free surface.

The induced vertical velocity in the lower half-space is given by the formula

$$\varphi_{2z} = -\frac{1}{4\pi}\iint_s \gamma(\theta)\Bigg[\frac{\partial}{\partial z}\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{(x-\xi)}{r}-1\right)+$$ [492

$$+\mathrm{Re}\,\bar{\nu}^2(1+a)\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{\nu}}{\cos^2\theta}[z+\zeta+i\omega]}\sec^5\theta\, d\theta +$$

$$+\frac{\mathrm{Re}\,ai}{2\pi}\int_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\int_0^{\infty} e^{\lambda[(z+\zeta)+i\omega]}\frac{\lambda(\lambda\cos^2\theta+\nu)}{\lambda\cos^2\theta-\bar{\nu}}\,d\lambda-$$

$$-\mathrm{Re}\int_0^{\infty} e^{\lambda[z+\zeta+i(y-\eta)}\lambda\,d\lambda\Bigg]\,ds. \qquad \text{(XII.26)}$$

The integral equation of the lifting surface arbitrary in shape will be found from the expression (XII.16) as follows:

$$\frac{1}{4\pi}\iint_{s}\gamma(\theta)\left[\frac{1}{(y-\eta)^2}\left(\frac{(x-\varepsilon)}{\sqrt{(x-\varepsilon)^2+(y-\eta)^2}}-1\right)+\right.$$

$$+\operatorname{Re}\nu^2(1+a)\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{\frac{\bar{\nu}}{\cos^2\theta}[-2h+i\omega]}\sec^5\theta d\theta+$$

$$+\operatorname{Re}\frac{ai}{2\pi}\int_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\int_0^\infty e^{\lambda(-2h+i\omega)}\frac{\lambda(\lambda\cos^2\theta+\nu)}{\lambda\cos^2\theta-\nu}d\lambda-$$

$$\left.-\operatorname{Re}\int_0^\infty e^{\lambda[-2h+i(y-\eta)]}\lambda d\lambda\right]ds=v_0\alpha. \qquad \text{(XII.27)}$$

From the lifting line theory, the regular part of the nucleus of the integro-differential equation of the form given in (VIII.15) will in this case have the following form:

$$G(y-\eta)=\operatorname{Re}\left[\bar{\omega}(1+a)\,i\int_0^\infty\frac{\lambda+1}{\lambda}e^{-\bar{\omega}/(\lambda+1)[(\bar{y}-\bar{\eta})\sqrt{\frac{\lambda}{\lambda+1}}-4i\bar{h}]}d\lambda-\right.$$

$$\left.-\frac{1}{\bar{y}-\bar{\eta}-4i\bar{h}}\right], \qquad \text{(XIII.28)}$$

or [493

$$G(y-\eta)=(1+a)\int_{\bar{\omega}}^\infty e^{-4\lambda\bar{h}}\frac{\lambda}{\lambda-\omega}\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\bar{\omega}}{\lambda}}\,d\lambda-$$

$$-\int_0^\infty e^{-4\lambda\bar{h}}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda. \qquad \text{(XII.29)}$$

The expansion of the nuclei (XII.28) and (XII.29) in powers of the parameter can also be written in the form of (VIII.33) by using the functions $G''_{s,p}\left(\frac{\bar{\omega}}{\tau}\right)$ instead of $G_{s,p}\left(\frac{\omega}{\tau}\right)$:

$$G''_{s,p}\left(\frac{\bar{\omega}}{\tau}\right)=(1+a)\frac{e^{-\frac{\bar{\omega}}{\tau}}\left(\frac{\bar{\omega}}{\tau}\right)^{s-p}}{p!\,(s-p-1)\ldots(p+1)}\times$$

$$\times \int_0^\infty e^{-\frac{\bar{\omega}}{\tau}\lambda} (\lambda + 1)^{\frac{s}{2} + \frac{1}{2}} \lambda^{\frac{s}{2} - p - 3/2} d\lambda - 1, \qquad \text{(XII.30)}$$

$$G''_{s,p}\left(\frac{\bar{\omega}}{\tau}\right) = (1 + a)\left[G_{s,p}\left(\frac{\bar{\omega}}{\tau}\right) - 1\right] + 1.$$

Be determining the characteristics of the hydrofoil with the elliptical distribution of circulation in an infinite fluid we will arrive at the formula (VIII.40) for ζ_{Fr}, in which we should use the functions $G''_{s,p}\left(\frac{\bar{\omega}}{\tau}\right)$ instead of $G'_{s,p}\left(\frac{\bar{\omega}}{\tau}\right)$. With $\bar{\omega} \to \infty$, the function ζ_∞ will be determined from the formula (VIII.41), while with $\bar{\omega} \to 0$

$$\zeta_0 = 1 + a\,(0{,}5\tau^2 + 0{,}25\tau^4 + 0{,}0625\tau^6 + 0{,}0469\tau^8 + \\ + 0{,}0237\tau^{10} + 0{,}0188\tau^{12}). \qquad \text{(XII.31)}$$

For a = 1, all the results of this problem will correspond to those of the problem of a hydrofoil submerged under the surface of a heavy fluid.

12.3. Motion of a Hydrofoil System Near the Interface Between Fluids of Different Densities [494

The problem of interaction of hydrofoils moving through different fluids is of great theoretical and practical interest. The solution of this problem for a two-dimensional parallel flow is given in Ch. VII.

Now we will examine a three-dimensional problem of motion of two foils in fluids of different densities.

We will look for the acceleration potentials in the following form:

$$\theta_j = \theta_{1j} + \theta_{2j}. \qquad \text{(XII.32)}$$

where θ_{1j} - the harmonic function everywhere in space, with the exception of the surface s_j;

θ_{2j} - the harmonic function in the j-th half-space.

When crossing the s_j surface, the potentials θ_{1j} experience a jump:

$$\Theta_{1j-}-\Theta_{1j+}=-v_0\gamma_i(\theta).$$

Using the known jump $\gamma_i(\theta)$, one may determine the function θ_{1j} from the following expression:

$$\Theta_j=\frac{v_0}{4\pi}\iint_{s_j}\gamma_j(\theta)G_{1j}(x, y, z, \xi, \eta, \zeta)\,ds, \qquad \text{(XII.33)}$$

where $G_{1j}(x, y, z, \xi, \eta, \zeta)$ is the harmonic function in the i-th half-space, with the exception of points $\theta_j(\xi, \eta, \zeta)$ on the s_j surface.

Let us find functions θ_{2j} in the following form:

$$\Theta_{2j}=\frac{v_0}{4\pi}\iint_{s_{\bar j}}\gamma_{\bar j}(\theta)G_{2j}(x, y, z, \xi_{\bar j}, \eta_{\bar j}, \zeta_j)\,ds. \qquad \text{(XII.34)}$$

For the steady-state motion θ_j are related to the value $\frac{\partial\varphi_i}{\partial x}$ by an expression in the form

$$\Theta_i=-v_0\frac{\partial\varphi_i}{\partial x}.$$

We obtain the expressions for the velocity potentials by introducing the integrals given in the expressions (XII.33) and (XII.34), which satisfy the condition ahead, at infinity:

$$\varphi_j=\frac{1}{4\pi}\iint_{s}\gamma_j(\theta)\int_{\infty}^{x}G_{1j}(t, y, z, \xi_j, \eta_j, \zeta_j)\,dt\,ds+$$

$$+\frac{1}{4\pi}\iint_{s_{\bar j}}\gamma_{\bar j}(\theta)\int_{\infty}^{x}G_{2j}(t, y, z, \xi_{\bar j}, \eta_{\bar j}, \zeta_{\bar j})\,dt\,ds. \qquad \text{(XII.35)}$$ [495

$$j=1,2$$

The functions $G_{ij}(x, y, z, \xi, \eta, \zeta)$ are related to $G_j(x, y, z, \xi_j, \eta_j, \zeta_j)$ by the expressions

$$G_{1j}(x, y, z, \xi_j, \eta_j, \zeta_j)=-\frac{\partial}{\partial\zeta_j}G_j(x, y, z, \xi_j, \eta_j, \zeta_j),$$

$$G_{2j}(x, y, z, \xi_j, \eta_j, \zeta_j)=-\frac{\partial}{\partial\zeta_{\bar j}}G_j(x, y, z, \xi_{\bar j}, \eta_{\bar j}, \zeta_{\bar j}). \qquad \text{(XII.36)}$$

We obtain the system of integral equations for this problem from the condition (31):

$$\frac{1}{4\pi}\frac{\partial}{\partial z}\iint_{s_j}\gamma_j(\theta)'\int_{\infty}^{x}G_{1j}(t,y,z,\xi_j,\eta_j,\zeta_j)\,dtds+$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial z}\iint_{s_{\bar j}}\gamma_{\bar j}(\theta)\int_{\infty}^{x}G_{2j}(t,y,z,\xi_{\bar j},\eta_{\bar j},\zeta_j)\,dtds\Big|_{z=\zeta_j}=-v_0\alpha_j.$$

$$j=1,2 \qquad \text{(XII.37)}$$

The system of integral equations (XII.37) together with the relations (XII.5), (XII.6), (XII.20), (XII.26) and (XII.36) solves the problem of motion of a system of arbitrarily-shaped foils near the interface.

As was already mentioned, it is very unlikely that one can obtain the analytical solution of the system (XII.37) for an arbitrarily shaped hydrofoil; however, with the aid of high-speed computers the solution can be obtained using the numerical methods. For the special shapes of hydrofoils the system can be simplified by making additional assumptions which would make it possible to achieve the solution to the problem in a simpler way. For foils with larger spans we can introduce the assumptions which are valid in the lifting line theory.

Let us write the system (XII.37) as follows:

$$\Bigg|\frac{1}{4\pi}\frac{\partial}{\partial z}\iint_{s_j}\gamma_j(\theta)\int_{\xi_j}^{x}G_{1j}(t,y,z,\xi_j,\eta_j,\zeta_j)\,dtds+$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial z}\iint_{s_j}\gamma_{\bar j}(\theta)\int_{\xi_{\bar j}}^{x}G_{2j}(t,y,z,\xi_{\bar j},\eta_{\bar j},\zeta_{\bar j})\,dtds\Big|_{z=\zeta_j}+$$

$$+\Bigg|+\frac{1}{4\pi}\frac{\partial}{\partial z}\iint_{s_j}\gamma_j(\theta)\int_{\infty}^{\xi_j}G_{1j}(t,y,z,\xi_j,\eta_j,\zeta_j)\,dtds+$$ [496

$$+\frac{1}{4\pi}\frac{\partial}{\partial z}\iint_{s_j}\gamma_{\bar j}(\theta)\int_{\infty}^{\xi_{\bar j}}G_{2\bar j}(t,y,z,\xi_{\bar j},\eta_{\bar j},\zeta_j(dtds\Big|_{z=\zeta_j}=-v_0\alpha_j. \quad \text{(XII.38)}$$

$$(j=1,2)$$

It follows from Ch. VII that when $\gamma(\theta)=\gamma(\xi)$ and the span tends to infinity the first two terms in the equation (XII.38) describe the two-dimensional parallel flow, while the remaining terms will, according to the lifting line theory, define the flow downwash at the point $x=\xi$.

Determining the first terms by using the hypothesis of plane sections, we obtain

$$\Gamma_j(y) = v_0 a_j(y) a_{h_j} \Big[\alpha_j(y) +$$

$$+ \frac{1}{4\pi v_0} \int_{-b_j}^{+b_j} \Gamma_j(\eta) \frac{\partial}{\partial z} \int_{\infty}^{\xi_j} G_{1j}(t, y, z, \xi_j, \eta_j, \zeta_j)\, dt\, d\eta +$$

$$+ \frac{1}{4\pi v_0} \int_{-b_{\bar{j}}}^{+b_{\bar{j}}} \Gamma_{\bar{j}}(\eta) \frac{\partial}{\partial z} \int_{\infty}^{\xi_{\bar{j}}} G_{2\bar{j}}(t, y, z, \xi_{\bar{j}}, \eta_{\bar{j}}, \zeta_{\bar{j}})\, dt\, d\eta \Big]\Big|_{z=\zeta_j}, \quad (j = 1,2) \quad \text{(XII.39)}$$

a_{hj} is determined from the formulas in Ch. VI.

Performing computations on the system (XII.39), we obtain the following system of integro-differential equations:

$$\Gamma_1(y) = v_0 a_1(y) a_{h_1} \Big\{ \alpha_1(y) - \frac{\mathrm{Re}}{4\pi v_0} \int_{-b_1}^{+b_1} \frac{d\Gamma_1}{d\eta} \Big[\frac{1}{y-\eta} - \frac{1}{y-\eta-2ih_1} +$$

$$+ \bar{\nu}(1-a) i \int_0^{\infty} \frac{\lambda+1}{\lambda} e^{-\bar{\nu} i(\lambda+1)\left[(y-\eta)\sqrt{\frac{\lambda}{\lambda+1}} - 2ih_1\right]} d\lambda \Big] d\eta -$$

$$- \frac{\mathrm{Re}}{4\pi v_0} \int_{-b_2}^{+b_2} \frac{d\Gamma_2}{d\eta} \frac{2\bar{\nu} i}{1+\bar{\varrho}} \int_0^{\infty} \frac{\lambda+1}{\lambda} e^{-\bar{\nu} i(\lambda+1)\left[(y-\eta)\sqrt{\frac{\lambda}{\lambda+1}} - 2i(h_1+h_2)\right]} d\lambda\, d\eta \Big\}. \quad \text{(XII.40)}$$

$$\Gamma_2(y) = v_0 a_2(y) a_{h_2} \Big\{ \alpha_2(y) - \frac{\mathrm{Re}}{4\pi v_0} \int_{-b_2}^{+b_2} \frac{d\Gamma_2}{d\eta} \Big[\frac{1}{y-\eta} -$$ [497

$$- \frac{1}{y-\eta-2ih_2} + \bar{\nu}(1+a) i \int_0^{\infty} e^{-\bar{\nu} i(\lambda+1)\left[(y-\eta)\sqrt{\frac{\lambda}{\lambda+1}} - 2ih_2\right]} d\lambda \Big] d\eta -$$

$$- \frac{\mathrm{Re}}{4\pi v_0} \int_{-b_1}^{+b_1} \frac{d\Gamma_1}{d\eta} 2\bar{\nu} \frac{\bar{\varrho}}{1+\bar{\varrho}} i \int_0^{\infty} e^{-\bar{\nu} i(\lambda+1)\left[(y-\eta)\sqrt{\frac{\lambda}{\lambda+1}} - 2i(h_1+h_2)\right]} d\lambda\, d\eta \Big\}$$

or in the dimensionless form

$$\bar{\Gamma}_1(\bar{y}) = \frac{a_{h_1}}{2\lambda(\bar{y})} \Big\{ \alpha_1(\bar{y}) - \frac{\mathrm{Re}}{2\pi} \int_{-1}^{+1} \bar{\Gamma}_1'(\eta) \Big[\frac{1}{\bar{y}-\bar{\eta}} - \frac{1}{\bar{y}-\bar{\eta}-4i\bar{H}_1} +$$

$$+ \bar{\omega}_1(1+a) i \int_0^{\infty} \frac{\lambda+1}{\lambda} e^{\bar{\omega}_1(\lambda+1)\left[(\bar{y}-\bar{\eta})\sqrt{\frac{\lambda}{\lambda+1}} - 4i\bar{H}_1\right]} d\bar{\eta} -$$

$$-\frac{\mathrm{Re}}{2\pi}\int_{-1}^{+1}\bar{\Gamma}_2'(\eta)\frac{2\bar{\omega}_2}{1+\bar{\varrho}}i\int_0^{\infty}\frac{\lambda+1}{\lambda}e^{-\bar{\omega}_2 i(\lambda+1)\left[(\bar{y}-\bar{\eta})\sqrt{\frac{\lambda}{\lambda+1}}-2i(\bar{H}_1+\bar{H}_2)k\right]}d\lambda\, d\bar{\eta}\Bigg\}, \quad \text{(XII.41)}$$

$$\bar{\Gamma}_2(\bar{y})=\frac{a_{h_2}}{2\lambda(\bar{y})_2}\Bigg\{\alpha_2(y)-\frac{\mathrm{Re}}{2\pi}\int_{-1}^{+1}\bar{\Gamma}_2'(\eta)\Bigg[\frac{1}{\bar{y}-\eta}-\frac{1}{\bar{y}-\bar{\eta}-4i\bar{H}_2 k}+$$

$$+\bar{\omega}_2(1+a)i\int_0^{\infty}\frac{\lambda+1}{\lambda}e^{-\bar{\omega}_2 i(\lambda+1)\left[(\bar{y}-\bar{\eta})\sqrt{\frac{\lambda}{\lambda+1}}-4i\bar{H}_2 k\right]}d\lambda\Bigg]d\bar{\eta}-$$

$$-\frac{\mathrm{Re}}{2\pi}\int_{-1}^{+1}\bar{\Gamma}_1'(\eta)\frac{2\bar{\omega}_1 i\bar{\varrho}}{1+\bar{\varrho}}\int_0^{\infty}\frac{\lambda+1}{\lambda}e^{\bar{\omega}_1 i(\lambda+1)\left[(\bar{y}-\bar{\eta})\sqrt{\frac{\lambda}{\lambda+1}}-2i(\bar{H}_1+\bar{H}_2)\right]}d\lambda\, d\eta\Bigg\},$$

$$\bar{\Gamma}_1(y)=\frac{\Gamma(y)}{b_1 v_0},\quad \bar{\Gamma}_2(y)=\frac{\Gamma(y)}{b_2 v_0},\quad \bar{H}_1=\frac{h}{b_1},\quad \bar{H}_2=\frac{h}{b_2},$$

$$k_1=\frac{b_1}{b_2}\quad \bar{\omega}_i=\nu b_i\,.$$

From the system (XII.41) the integro-differential equations for foils above and below the interface are easily obtained: [498

$$\Gamma(y)=\frac{a_h}{2\lambda(y)}\Bigg\{\alpha(\bar{y})-\frac{\mathrm{Re}}{2\pi}\int_{-1}^{+1}\Gamma'(\bar{\eta})\Bigg[\frac{1}{\bar{y}-\eta}+\bar{\omega}(1\pm a)\int_0^{\infty}\frac{\lambda+1}{\lambda}\times$$

$$\times e^{\bar{\omega}i(\lambda+1)\left[(\bar{y}-\bar{\eta})\sqrt{\frac{\lambda}{\lambda+1}}-4i\bar{H}\right]}d\lambda-\frac{1}{\bar{y}-\bar{\eta}-4i\bar{H}}\Bigg]d\bar{\eta}\Bigg\}. \quad \text{(XII.42)}$$

For the hydrofoil above the interface we have to use the minus sign in the equation (XII.42), while for that below the interface the plus sign.

For the upper foil when a = 1 we obtain, from equation (XII.42), the equation for a hydrofoil near a solid wall, while for the lower hydrofoil, we obtain the basic integro-differential equation for a hydrofoil submerged under the free surface.

We can show that with a = 0, the system (XII.41) transforms into a system of equations which describe the motion of a biplane in an infinite fluid. This result follows easily from the system (XII.41) written in a different way:

$$\bar{\Gamma}_1(\bar{y}) = \frac{a_{h_1}}{2\lambda(\bar{y})}\Big\{\alpha_1(\bar{y}) - \frac{\mathrm{Re}}{2\pi}\int_{-1}^{+1}\bar{\Gamma}_1'(\eta)\Big[\frac{1}{\bar{y}-\bar{\eta}} -$$

$$-\int_0^\infty e^{-4\lambda\bar{H}_1}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda + (1-a)\times$$

$$\times\int_{\bar{\omega}_1}^\infty e^{-4\lambda\bar{H}_1}\frac{\lambda}{\lambda-\bar{\omega}_1}\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\omega_1}{\lambda}}\,d\lambda\Big]d\eta -$$

$$-\frac{1}{2\pi}\int_{-1}^{+1}\bar{\Gamma}_2'\frac{2}{1+\bar{\varrho}}\int_{\bar{\omega}_2}^\infty e^{-2\lambda(\bar{H}_1+\bar{H}_2)k}\times$$

$$\times\frac{\lambda}{\lambda-\bar{\omega}_2}\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\bar{\omega}_2}{\lambda}}\,d\lambda d\eta\Big\}, \qquad \text{(XII.43)}$$

$$\bar{\Gamma}_2(\bar{y}) = \frac{a_{h_2}}{2\lambda(\bar{y})_2}\Big\{\alpha_2(\bar{y}) - \frac{1}{2\pi}\int_{-1}^{+1}\bar{\Gamma}_2'(\eta)\times$$

$$\times\Big[\frac{1}{\bar{y}-\eta} - \int_0^\infty e^{-4\lambda\bar{H}_2k}\sin\lambda(\bar{y}-\bar{\eta})\,d\lambda +$$ [499

$$+(1+a)\int_{\bar{\omega}_2}^\infty e^{-4\lambda\bar{H}_2k}\frac{\lambda}{\lambda-\bar{\omega}_2}\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\omega_2}{2}}\,d\lambda\Big]d\eta -$$

$$-\frac{1}{2\pi}\int_{-1}^{+1}\bar{\Gamma}_1'(\eta)\frac{2\bar{\varrho}}{1+\bar{\varrho}}\int_{\bar{\omega}_1}^\infty e^{-2\lambda(\bar{H}_1+H_2)}\times$$

$$\times\frac{\lambda}{\lambda-\bar{\omega}_1}\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\bar{\omega}_2}{\lambda}}\,d\lambda d\eta\Big\}.$$

The system of equations (XII.41) can be solved by the methods presented in Ch. X.

Let us introduce the following definitions:

$$G_1(y) = \mathrm{Re}\Big[\bar{\omega}_1(1-a)\,i\int_0^\infty\frac{\lambda+1}{\lambda}e^{-\bar{\omega}_1 i(\lambda+1)\left(y\sqrt{\frac{\lambda}{\lambda+1}}-4i\bar{H}_1\right)}d\lambda - \frac{1}{y-4i\bar{H}_1}\Big]\Big|$$

$$\left.\begin{aligned}
G_2(\bar{y}) &= \mathrm{Re}\left[\bar{\omega}_2(1+a)\, i\int_0^\infty \frac{\lambda+1}{\lambda} e^{-\bar{\omega}_2 i(\lambda+1)\left(y\sqrt{\frac{\lambda}{\lambda+1}}-4i\bar{H}_2 k\right)} - \frac{1}{y-4i\bar{H}_2 k}\right. \\
G_3(y) &= \mathrm{Re}\,\frac{2\bar{\omega}_2}{1+\bar{\varrho}}\, i\int_0^\infty \frac{\lambda+1}{\lambda} e^{-\bar{\omega}_2 i(\lambda+1)\left[y\sqrt{\frac{\lambda}{\lambda+1}}-2ik(\bar{H}_1+\bar{H}_2)\right]} d\lambda \\
G_4(y) &= \mathrm{Re}\,\frac{2\bar{\omega}_1\varrho_i}{1+\bar{\varrho}}\int_0^\infty \frac{\lambda+1}{\lambda} e^{\bar{\omega}_1 i(\lambda+1)\left[y\sqrt{\frac{\lambda}{\lambda+1}}-2i(\bar{H}_1+\bar{H}_2)\right.} d\lambda
\end{aligned}\right\} \quad \text{(XII.44)}$$

The expansion of the function $G_n(y)$ in powers of the parameter $\tau = \sqrt{4x^2+1} - 2x$ can be written as follows:

$$G_n(y) = \\
= \sum_{s=2,4,6}^{\infty} \tau_n^s \sum_{p=0}^{\frac{s}{2}-1} \frac{y^{s-1-2p}(s-1-p)\ldots(p+1)(-1)^{\frac{s}{2}-p+1}}{(s-1-2p)!} G_{sp}^n\left(\frac{\bar{\omega}_n}{\tau_n}\right), \quad \text{(XII.45)}$$

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$$\left.\begin{aligned}
& n = 1,2 \\
& G_{s,p}^n\left(\frac{\omega_n}{\tau_n}\right) = a_n \frac{e^{-\frac{\bar{\omega}_n}{\tau_n}}\left(\frac{\bar{\omega}_n}{\tau_n}\right)^{s-p}}{p!\,(s-1-p)\ldots(p+1)} \times \\
& \times \int_0^\infty e^{-\frac{\omega_n}{\tau_n}}(\lambda+1)^{\frac{s}{2}+1}\lambda^{\frac{s}{2}-p-\frac{3}{2}} d\lambda - 1 \\
& a_1 = 1-a \qquad a_2 = 1+a \\
& \tau_1 = \sqrt{4\bar{H}_1^2+1} - 2\bar{H}_1, \quad \tau_2 = \sqrt{4k_2^2\bar{H}_2^2+1} - 2k\bar{H}_2 \\
& n = 3,4 \\
& G_{sp}^n\left(\frac{\omega_n}{\tau_n}\right) = \frac{e^{-\frac{\omega_n}{\tau_n}}\left(\frac{\bar{\omega}_n}{\tau_n}\right)^{s-p}}{p!\,(s-1-p)\ldots(p+1)} \times \\
& \times \int_0^\infty e^{-\frac{\omega_n}{\tau_n}}(\lambda+1)^{\frac{s}{2}+1}\lambda^{\frac{s}{2}-p-\frac{3}{2}} \quad d\lambda \\
& a_3 = \frac{2}{1+\bar{\varrho}}, \qquad a_4 = \frac{2\bar{\varrho}}{1+\bar{\varrho}} \\
& \tau_3 = \sqrt{k^2(\bar{H}_1+\bar{H}_2)^2+1} - k(\bar{H}_1+\bar{H}_2) \\
& \tau_4 = \sqrt{(\bar{H}_1+\bar{H}_2)^2+1} - (\bar{H}_1+\bar{H}_2)
\end{aligned}\right\} \quad \text{(XII.46)}$$

For the hydrofoil with the elliptical distribution of circulation in an infinite flow we will obtain a simple approximate solution of the system (XII.41). This solution makes it possible to examine the type of interaction of the system, and this will be sufficient for the majority of practical problems.

As in Ch. X we will look for the solution of the system (XII.41) in the form of $\Gamma_i = \Phi_i \sqrt{1-\bar{y}^2}$.

Let us take the function $\frac{1}{\lambda_i(y)} = \frac{4}{\pi\lambda_i}\sqrt{1-\bar{y}^2}$ and integrate the equations with respect to y from -1 to +1. Then, from the system of (XII.41) we obtain a system of algebraic equations as follows:

$$\Phi_1 = \frac{2a_{h_1}}{\pi\lambda_1}\left\{\alpha_1 - \frac{\Phi_1}{2}\left[1 - \frac{2}{\pi^2}\int\limits_{-1}^{+1}\sqrt{1-\bar{y}^2}\times\right.\right.$$

$$\left.\times\, dy \int\limits_{-1}^{+1} \frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}} G_1(\bar{y}-\bar{\eta})\, d\bar{\eta}\right] +$$ [501

$$\left. + \frac{\Phi_2}{\pi^2}\int\limits_{-1}^{+1}\sqrt{1-y^2}\, d\bar{y}\int\limits_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}} G_3(\bar{y}-\bar{\eta})\, d\bar{\eta}\right\}, \qquad \text{(XII.47)}$$

$$\Phi_2 = \frac{2a_{h_2}}{\pi\lambda_2}\left\{\alpha_2 - \frac{\Phi_2}{2}\left[1 - \frac{2}{\pi^2}\int\limits_{-1}^{+1}\sqrt{1-\bar{y}^2}\times\right.\right.$$

$$\left.\times\, dy \int\limits_{-1}^{+1} \frac{\bar{\eta}}{\sqrt{1-\eta}} G_2(\bar{y}-\bar{\eta})\, d\bar{\eta}\right] +$$

$$\left. + \frac{\Phi_1}{\pi^2}\int\limits_{-1}^{+1}\sqrt{1-y^2}\, d\bar{y}\int\limits_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}} G_4(\bar{y}-\bar{\eta})\, d\bar{\eta}\right\};$$

from which we obtain

$$\Phi_1 = \frac{\frac{2a_{h_1}}{\pi\lambda_1}\alpha_1\left(1 - \frac{a_{h_2}}{\pi\lambda_2}\zeta_2\right) - \frac{2a_{h_2}}{\pi\lambda_2}\alpha_2\frac{a_{h_1}}{\pi\lambda_1}\zeta'_{12}}{\left(1+\frac{a_{h_1}}{\pi\lambda_1}\zeta_1\right)\left(1+\frac{a_{h_2}}{\pi\lambda_2}\zeta_2\right) - \frac{a_{h_2}}{\pi\lambda_2}\zeta'_{21}\frac{a_{h_1}}{\pi\lambda_1}\zeta'_{12}},$$

$$\text{(XII.48)}$$

$$\Phi_2 = \frac{\frac{2a_{h_2}}{\pi\lambda_2}\alpha_2\left(1 + \frac{a_{h_1}}{\pi\lambda_1}\zeta_1\right) - \frac{2a_{h_2}}{\pi\lambda_2}\alpha_1\frac{a_{h_2}}{\pi\lambda_2}\zeta'_{21}}{\left(1+\frac{a_{h_1}}{\pi\lambda_1}\zeta_1\right)\left(1+\frac{a_{h_2}}{\pi\lambda_2}\zeta_2\right) - \frac{a_{h_2}}{\pi\lambda_2}\zeta'_{21}\frac{a_{h_1}}{\pi\lambda_1}\zeta'_{12}},$$

where

$$\left.\begin{aligned}
\zeta_1 &= 1-\frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}}G_1(\bar{y}-\bar{\eta})\,d\bar{\eta}\\
\zeta_2 &= 1-\frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}}G_2(\bar{y}-\bar{\eta})\,d\bar{\eta}\\
\zeta'_{12} &= -\frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}}G_3(\bar{y}-\bar{\eta})\,d\bar{\eta}\\
\zeta'_{21} &= -\frac{2}{\pi^2}\int_{-1}^{+1}\sqrt{1-\bar{y}^2}\,d\bar{y}\int_{-1}^{+1}\frac{\bar{\eta}}{\sqrt{1-\bar{\eta}^2}}G_4(\bar{y}-\bar{\eta})\,d\bar{\eta}
\end{aligned}\right\}. \qquad \text{(XII.49)}$$

The coefficients of the lifting force and drag for [502
the hydrofoil system are defined by the formulas

$$C_{yl} = \lambda_l\int_{-1}^{+1}\Gamma_l(\bar{y})\,d\bar{y}, \qquad \text{(XII.50)}$$

$$C_{xl_1} = \frac{\lambda_1}{2\pi}\int_{-1}^{+1}\int_{-1}^{+1}\Gamma_1(y)\left\{\Gamma'_1(\eta)\left[\frac{1}{y-\eta}+G_1(y-\eta)\right]+\right.$$

$$\left.+\,\Gamma'_2(\eta)G_3(y-\eta)\right\}dy\,d\eta, \qquad \text{(XII.51)}$$

$$C_{xl_2} = \frac{\lambda_2}{2\pi}\int_{-1}^{+1}\int_{-1}^{+1}\Gamma_2(y)\left\{\Gamma'_2(\eta)\left[\frac{1}{y-\eta}+G_2(y-\eta)\right]+\right.$$

$$\left.+\,\Gamma'_1(\eta)G_4(y-\eta)\right\}dy\,d\eta. \qquad \text{(XII.52)}$$

Then, by using the results (XII.47)-(XII.49) we can write

$$C_{yl} = \frac{a_{hl}}{1+\frac{a_{hl}}{\pi\lambda_l}\zeta_{l\bar{l}}}\alpha_s, \qquad \text{(XII.53)}$$

$$C_{xl} = \frac{C_{yl}^2}{\pi\lambda}\zeta_{l\bar{l}}, \qquad \text{(XII.54)}$$

where

$$\zeta_{12} = \zeta_1+\frac{\zeta'_{12}}{\bar{\Phi}};$$

$$\zeta_{21} = \zeta_2+\bar{\Phi}\zeta'_{21}; \qquad \text{(XII.55)}$$

$$\overline{\Phi} = \frac{\Phi_1}{\Phi_2}.$$

The functions ζ_1 and ζ_2 determine the effect of the interface on the hydromechanical coefficients of hydrofoils, while the functions $\zeta_{ji}^{\prime}$ determine the interaction of hydrofoils in the system.

As in Ch. VI, it is easy to evaluate the order of the interaction functions:

with

$$P_1 \sim P_2 \quad \Phi_1 \sim \frac{1}{\overline{\varrho}} \Phi_2, \quad \overline{\Phi} \sim \frac{1}{\overline{\varrho}};$$

then [503

$$\zeta_{12}^{\prime} \sim \tau_3^2, \quad \zeta_{21}^{\prime} \sim \overline{\varrho} \tau_4^2$$

and

$$\zeta_{12} = \zeta_1 + \theta(\overline{\varrho}\tau_3^2) \quad \zeta_{21} = \zeta_2 + \theta(\tau_4^2). \qquad \text{(XII.56)}$$

It follows then, that for the three-dimensional case of fluid flow we obtain the same result as for the two-dimensional case.

For the motion of the system near the air/water interface the effect of the lower foil will be negligibly small, while the effect of the upper foil on the characteristics of the lower foil is considerable.

We obtain the final formulas for the function ζ by using the expansion (XII.45) as follows:

$$\begin{aligned}\zeta_l = {} & 1 + \frac{1}{2} G_{2,0}^l \tau^2 + (G_{4,1}^l - 0{,}75 G_{4,0}^l)\tau_l^4 + (1{,}5G_{6,2}^l - 3G_{6,1}^l + \\ & + 1{,}5625 G_{6,0}^l)\tau_l^6 + (2G_{8,3}^l - 7{,}5G_{8,2}^l + 9{,}375G_{81}^l - 3{,}82809G_{80}^l)\tau_l^8 + \\ & + (2{,}5G_{10,4}^l - 15G_{10,3}^l + 32{,}8125G_{10,2}^l - 30{,}62472G_{10,1}^l + 10{,}33587G_{10,0}^l)\tau_l^{10} + \\ & + (3G_{12,5}^l - 26{,}25G_{12,4}^l + 87{,}5G_{12,3}^l - 137{,}81124G_{12,2}^l + \\ & + 103{,}3587G_{12,1}^l - 29{,}77733G_{12,0}^l)\tau^{12}, \end{aligned} \qquad \text{(XII.57)}$$

$$\begin{aligned}\zeta_{jl}^{\prime} = {} & 0{,}5G_{2,0}^{l+2}\tau_{l+2}^2 + (G_{4,1}^{l+2} - 0{,}75G_{40}^{l+2})\tau_{l+2}^4 + (1{,}5G_{6,2}^{l+2} - 3G_{6,1}^{l+2} + \\ & + 1{,}5625G_{6,0}^{l+2})\tau_{l+2}^6 + (2G_{8,3}^{l+2} - 7{,}5G_{8,2}^{l+2} + 9{,}375G_{8,1}^{l+2} - \\ & - 3{,}82809G_{8,0})\tau_{l+2}^8 + (2{,}5G_{10,4}^{l+2} - 15G_{10,3}^{l+2} + 32{,}8125G_{10,2}^{l+2} - \end{aligned}$$

$$-30{,}62472G_{10,1}^{l+2}+10{,}33587G_{10,2}^{l+2})\tau_{l+2}^{10}+(3G_{12,5}^{l+2}-26{,}25G_{12,4}^{l+2}+$$
$$+87{,}5G_{12,3}^{l+2}-137{,}811G_{12,2}^{l+2}+103{,}358G_{12,1}^{l+2}+29{,}7773G_{12,0}^{l+2})\tau_{l+2}^{12}. \quad \text{(XII.58)}$$

In conclusion let us examine the limiting cases of motion. With

$$\frac{\omega}{\tau_n}\to 0 \quad G_{n,p}^1=-a, \quad G_{n,p}^2=+a, \quad G_{n,p}^3=\frac{2}{1+\varrho}, \quad G_{n,p}^4=\frac{2\bar{\varrho}}{1+\bar{\varrho}}.$$

With

$$\frac{\omega}{\tau_n}\to\infty \quad G_{n,p}^1=-1, \quad G_{n,p}^2=-1, \quad G_{n,p}^3=G_{n,p}^4=0.$$

Let us introduce the function N_n by writing the expression [504

$$N_n=0{,}5\tau_n^2+0{,}25\tau_n^4+0{,}0625\tau_n^6+0{,}0469\tau_n^8+0{,}0237\tau^{10}+0{,}0188\tau^{12}+\ldots \quad \text{(XII.59)}$$

For $\tau_n\to 0$ $N_n\to 0$, for $\tau_n\to 1$ $N_n\to 1$.

Then the formulas for the functions ζ in the form of $\frac{\omega}{\tau_n}\to\infty$ may be written as

$$\zeta_I=1-N_I; \quad \zeta'_{II}=0; \quad \zeta_{II}=\zeta_I. \quad \text{(XII.60)}$$

The formulas (XII.60) give the characteristics for the hydrofoils near a solid wall. The most interesting is the case when $\frac{\omega}{\tau}\to 0$

$$\zeta_1=1-aN_1, \quad \zeta_2=1+aN_2$$
$$\zeta'_{12}=\frac{2}{1+\bar{\varrho}}N_3, \quad \zeta'_{21}=\frac{2\bar{\varrho}}{1+\bar{\varrho}}N_4 \quad \text{(XII.61)}$$

$$\left.\begin{aligned}
&\zeta_{12}=1-aN_1+\frac{2\bar{\varrho}}{1+\varrho}\cdot\frac{N_3}{\bar{\bar{\Phi}}}\\
&\zeta_{21}=1+aN_2+\frac{2}{1+\bar{\varrho}}N_4\bar{\varphi}\\
&\bar{\bar{\Phi}}=\Phi\bar{\varrho}=\frac{p_1}{p_2}\cdot\frac{1}{k_1^2}
\end{aligned}\right\}. \quad \text{(XII.62)}$$

For $\varrho = 1$ and $a = 0$ the formulas (XII.62) will determine the characteristics of a biplane in an infinite flow as follows:

$$\zeta_{12} = 1 + \frac{N_3}{\bar{\bar{\Phi}}} \quad \zeta_{21} = 1 + N_4\bar{\bar{\Phi}}. \qquad \text{(XII.63)}$$

Of great importance is the significant increase in drag on the lower hydrofoil for low values of $\bar{\omega}$.

In determining the function ζ_{12} for a system moving near the interface between the air and water, we can neglect the term which takes into account the interaction, while for the function ζ_{21} we will have the formula

$$\zeta_{21} = 1 + N_2 + 2\bar{\bar{\Phi}}N_4. \qquad \text{(XII.64)}$$

It follows from formula (XII.64) that for $N_2 \sim N_4$ and $\bar{\bar{\Phi}} \approx 1$ the function ζ_{21} and, consequently, the drag increases significantly. As an example, let us examine the case in which $N_2 = N_4$. We have $\zeta_{21} = 1 + 1 + 2(\bar{\bar{\Phi}})N_2$, while for the [505
hydrofoil near the free surface the function ζ_2 will be determined from the formula $\zeta_2 = 1 + N_2$.

The values of the ζ_{21} function for a number of parameter values are presented in Tables 22-26 and their graphs are given in Figures 51-54. In Tables 22-26, $b = \frac{h_1}{h_2}$.

The results obtained in Ch. VI and XII concerning the interaction between the systems of hydrofoils near the interface of fluids with different densities allow us to draw some general conclusions.

1. For a low relative density of the upper fluid, the hydrodynamic characteristics of the hydrofoil above the interface differ only slightly from those of the hydrofoil near a solid wall in all modes of motion.

2. With a small relative density of the upper fluid the effect of the lower hydrofoil on the upper is negligible, while the effect of the upper hydrofoil on the lower is considerable.

3. The upper foil lowers considerably the magnitude of the lifting force of the lower foil and affects the stabilization effect which was used as a basis for the shallowly submerged hydrofoil configuration and which is considerable for the deep submergence of the lower hydrofoil

[506

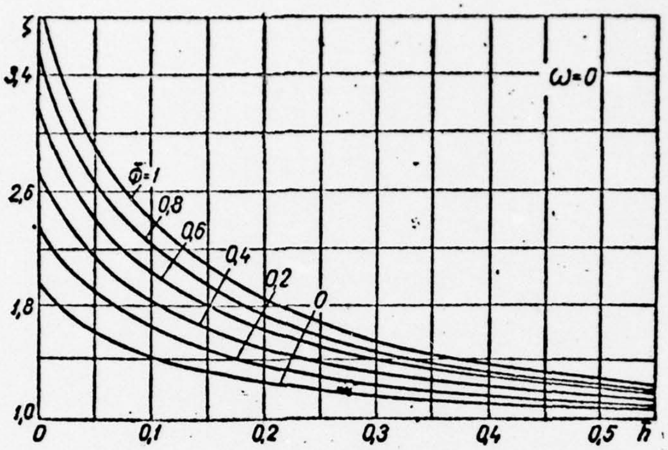

Fig. 51

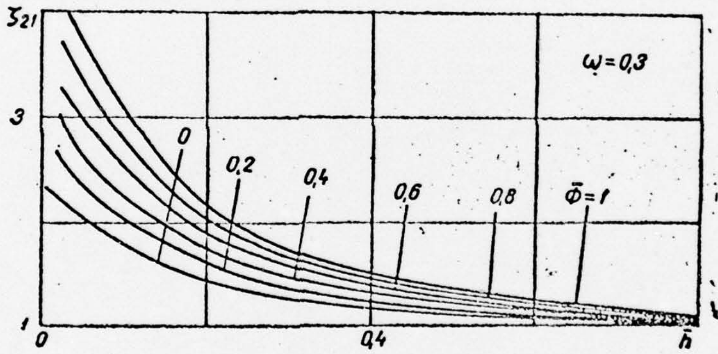

Fig. 52

as well. The necessary longitudinal stability of the hydrofoil can be achieved, in this case, with deep submergence. [505

4. As a result of the interaction between the hydrofoils in the three-dimensional flow the drag of the lower foil increases considerably.

12.4. Motion of the Lifting Surface Above the Interface Between Fluids with Different Densities and with the Lower Fluid Having a Finite Depth

Let us examine in greater detail the problem of motion of a three-dimensional source above the interface.

Let the three-dimensional source be located at the

Table 22

$\omega = 0,\ Fr = \infty$

$\bar{\Phi}$	$\bar{h}$	Values of ζ_{21} for b				
		0,6	0,8	1,0	1,2	1,4
0,0	0,025	1,1349	1,1349	1,1349	1,1349	1,1349
	0,05	1,6326	1,6326	1,6326	1,6326	1,6326
	0,1	1,4775	1,4775	1,4775	1,4775	1,4775
	0,2	1,2898	1,2898	1,2898	1,2898	1,2898
	0,4	1,1297	1,1297	1,1297	1,1297	1,1297
	0,6	1,0699	1,0699	1,0699	1,0699	1,0699
	0,8	1,0428	1,0428	1,0428	1,0428	1,0428
0,2	0,025	2,0380	2,0334	2,0288	2,0244	2,0200
	0,05	1,9011	1,8932	1,8857	1,8784	1,8714
	0,1	1,6906	1,6792	1,6685	1,6585	1,6492
	0,2	1,4300	1,4171	1,4057	1,3956	1,3867
	0,4	1,1994	1,1896	1,1816	1,1750	1,1695
	0,6	1,1098	1,1032	1,0980	1,0939	1,0906
	0,8	1,0680	1,0634	1,0599	1,0573	1,0552
0,4	0,025	2,3412	2,3319	2,3228	2,3139	2,3051
	0,05	2,1696	2,1538	2,1387	2,1242	2,1101
	0,1	1,9038	1,8809	1,8595	1,8396	1,8210
	0,2	1,5703	1,5444	1,5216	1,5015	1,4835
	0,4	1,2691	1,2495	1,2335	1,2203	1,2093
	0,6	1,1496	1,1363	1,1270	1,1178	1,1112
	0,8	1,0932	1,0840	1,0771	1,0718	1,0675
0,6	0,025	2,6443	2,6304	2,6167	2,6034	2,5903
	0,05	2,4380	2,4145	2,3918	2,3699	2,3489
	0,1	2,1169	2,0825	2,0505	2,0206	1,9927
	0,2	1,7105	1,6718	1,6375	1,6073	1,5804
	0,4	1,3388	1,3094	1,2854	1,2656	1,2491
	0,6	1,1894	1,1695	1,1540	1,1417	1,1318
	0,8	1,1183	1,1046	1,0943	1,0862	1,0799
0,8	0,025	2,9474	2,9289	2,9107	2,8929	2,8754
	0,05	2,7065	2,6751	2,6448	2,6159	2,5876
	0,1	2,3301	2,2842	2,2415	2,2016	2,1645
	0,2	1,8508	1,7991	1,7535	1,7131	1,6773
	0,4	1,4085	1,3693	1,3373	1,3109	1,2889
	0,6	1,2292	1,2027	1,1819	1,1656	1,1524
	0,8	1,1435	1,1252	1,1114	1,1007	1,0922
1,0	0,025	3,2506	3,2274	3,2047	3,1824	3,1605
	0,05	2,9750	2,9357	2,8979	2,8615	2,8264
	0,1	2,5433	2,4859	2,4325	2,3827	2,3363
	0,2	1,9910	1,9264	1,8594	1,8189	1,7742
	0,4	1,4782	1,4292	1,3892	1,3562	1,3287
	0,6	1,2690	1,2358	1,2100	1,1894	1,1730
	0,8	1,1686	1,1458	1,1285	1,1151	1,1046

Table 23

$\bar{\Phi}$	$\bar{h}$	Values of ξ_{21} for Fr and ω								
		5	4	1,9	1,34	0,79	0,64	0,52	0,36	0,28
		0,0199	0,0509	0,1499	0,2999	0,7999	1,1999	1,9999	3,9999	6,9999
					$b=0,6$					
0,2	0,025	2,0723	2,1428	2,3049	2,5744	3,3696	3,8224	4,1408	2,6674	0,6739
	0,05	1,9312	1,9930	2,1338	2,3631	2,9937	3,3045	3,4019	2,0165	0,5960
	0,1	1,7143	1,7626	1,8704	2,0377	2,4243	2,5412	2,3864	1,2885	0,5926
	0,2	1,4456	1,4771	1,5438	1,6348	1,7502	1,6937	1,4196	0,8691	0,7223
	0,4	1,2079	1,2243	1,2544	1,2804	1,2255	1,1301	1,9799	0,8769	0,8769
	0,6	1,1154	1,1256	1,1408	1,1436	1,0674	1,0065	0,9486	0,9310	0,9310
	0,8	1,0721	1,0792	1,0869	1,0804	1,0147	0,9814	0,9605	0,9533	0,9533
0,4	0,025	2,3813	2,4638	2,6535	2,9693	3,9024	4,4353	4,8145	3,0943	7,1879
	0,05	2,2049	2,2775	2,4429	2,7126	3,4571	3,8272	3,9519	2,3293	0,6427
	0,1	1,9317	1,9887	2,1162	2,3145	2,7776	2,9231	2,7519	1,4544	0,6103
	0,2	1,5889	1,6263	1,7061	1,8156	1,9617	1,9018	1,5828	0,9138	0,7244
	0,4	1,2793	1,2551	1,3355	1,3682	1,3089	1,1965	1,0131	0,8799	0,8799
	0,6	1,1563	1,1687	1,1874	1,1921	1,1036	1,0294	0,9556	0,9312	0,9312
	0,8	1,0981	1,1067	1,1165	1,1097	1,0315	0,9898	0,9621	0,9534	0,9534
0,6	0,025	2,6903	2,7848	3,0022	3,3642	4,4351	5,0482	5,4882	3,5211	8,2368
	0,05	2,4787	2,5619	2,7520	3,0621	3,9205	4,3498	4,5019	2,6421	0,6894
	0,1	2,1492	2,2149	2,3621	2,5914	3,1310	3,3051	3,1184	1,6202	0,6280
	0,2	1,7321	1,7756	1,8683	1,9964	2,1731	2,1099	1,7461	0,9585	0,7268
	0,4	1,3507	1,3737	1,4167	1,4561	1,3922	1,2629	1,0464	0,8828	0,8828
	0,6	1,1972	1,2117	1,2340	1,2406	1,1397	1,0523	0,9627	0,9314	0,9314
	0,8	1,1241	1,1342	1,1460	1,1390	1,0483	0,9982	0,9637	0,9535	0,9535
0,8	0,025	2,9993	3,1058	3,3509	3,7591	3,9679	5,6611	6,1619	3,9480	0,8985
	0,05	2,7524	2,8464	3,0611	3,4116	4,3839	4,8724	5,0518	2,9548	0,7361
	0,1	2,3666	2,4410	2,6079	2,8682	3,4843	3,6870	3,4849	1,7860	0,6457
	0,2	1,8753	1,9248	2,0305	2,1773	2,3846	2,3181	1,9093	1,0032	0,7287

Table 23 (cont.)

	0,4	1,4220	1,4485	1,4978	1,5440	1,4756	1,3293	1,0796	0,8857	0,8857
	0,6	1,2382	1,2548	1,2806	1,2692	1,1759	1,0752	0,9698	0,9317	0,9317
	0,8	1,1501	1,1617	1,1755	1,1683	1,0651	1,0065	0,9653	0,9536	0,9536
1,0	0,025	3,3083	3,4268	3,6996	4,1540	5,5006	6,2740	6,8357	4,3748	0,9734
	0,05	3,0261	3,1309	3,3702	3,7611	4,8473	5,3950	5,6018	3,2676	0,7827
	0,1	2,5840	2,6672	2,8537	3,1450	3,8376	4,0689	3,8514	1,9518	0,6634
	0,2	2,0185	2,0740	2,1928	2,3581	2,3960	2,5262	2,0725	1,0478	0,7309
	0,4	1,4934	1,5232	1,5789	1,6318	1,5590	1,3957	1,1129	0,8887	0,8887
	0,6	1,2791	1,2979	1,3272	1,3377	1,2121	1,0981	0,9769	0,9319	0,9319
	0,8	1,1761	1,1892	1,2050	1,1975	1,0819	1,0149	0,9668	0,9537	0,9537
					$b=0,8$					
0,2	0,025	2,0676	2,1379	2,2996	2,5683	3,3604	3,8103	4,1239	2,6512	0,6699
	0,05	1,9232	1,9847	2,1248	2,3528	2,9780	3,2842	3,3747	1,9929	0,5908
	0.4	1,7026	1,7505	1,8573	2,0226	2,4017	2,5127	2,3500	1,2639	0,5887
	0,2	1,4325	1,4635	1,5290	1,6178	1,7258	1,6651	1,3899	0,8564	0,7224
	0,4	1,1979	1,2140	1,2431	1,2675	1,2093	1,1142	0,9691	0,8755	0,8755
	0,6	1,1086	1,1180	1,1331	1,1349	1,0581	0,9992	0,9455	0,9309	0,9309
	0,8	1,0675	1,0743	1,0816	1,0744	1,0096	-0,9782	0,9597	0,9537	0,9537
0,4	0,025	2,3813	2,4638	2,6535	2,9693	3,9024	4,4353	4,8145	3,0943	0,7488
	0,05	2,1889	2,2609	2,4249	2,6920	3,4257	3,7866	3,8975	2,2821	0,6323
	0,1	1,9084	1,9645	2,0900	2,2843	2,7323	2,8663	2,6813	1,4050	0,6024
	0,2	1,5625	1,5090	1,6764	1,7815	1,9128	1,8446	1,5234	0,8884	0,7246
	0,4	1,2593	1,2783	1,3129	1,3425	1,2764	1,1648	0,9916	0,8770	0,8770
	0,6	1,1428	1,1546	1,1720	1,1747	1,0851	1,0148	0,9495	0,9311	0,9311
	0,8	1,0888	1,0969	1,1057	1,0977	1,0212	0,9833	0,9605	0,9542	0,9542
0,6	0,025	2,6762	2,7702	2,9864	3,3460	4,4073	5,0118	5,4376	3,4725	0,8118
	0,05	2,4547	2,5371	2,7251	3,0312	3,8734	4,2890	4,4202	2,5713	0,6737
	0,1	2,1141	2,1756	2,3226	2,5461	3,0631	3,2198	3,0124	1,5462	0,6162
	0,2	1,6926	1,7346	1,8238	1,9453	2,0998	2,0242	1,6568	0,9203	0,7269
	0,4	1,3207	1,3427	1,3828	1,4174	1,3436	1,2154	1,0141	0,8784	0,8784
	0,6	1,1769	1,1906	1,2109	1,2145	1,1121	1,0304	0,9535	0,9312	0,9318
	0,8	1,1104	1,1196	1,1299	1,1211	1,0329	0,9885	0,9612	0,9547	0,9547

Table 23 (cont.)

$\bar{\Phi}$	$\bar{h}$	Values of ζ_{21} for Fr and ω								
		5	4	1,9	1,34	0,79	0,64	0,52	0,36	0,28
		0,0199	0,0599	0,1499	0,2999	0,7999	1,1999	1,9999	3,9999	6,9999
					b=0,82					
0,6	0,025	2,9804	3,0863	3,3298	3,7348	4,9307	5,6126	6,0945	3,8833	0,8827
	0,05	2,7204	2,8133	3,0252	3,3704	4,3211	4,7913	4,9430	2,8605	0,7152
	0,1	2,3199	2,3926	2,5553	2,8078	3,3937	3,5733	3,3436	1,6873	0,6299
	0,2	1,8227	1,8702	1,9712	2,1091	2,2868	2,2037	1,7903	0,9523	0,7292
	0,4	1,3822	1,4070	1,4526	1,4924	1,4107	1,2660	1,0366	0,8799	0,8799
	0,6	1,2111	1,2267	1,2498	1,2543	1,1390	1,0461	0,9575	0,9314	0,9314
	0,8	1,1315	1,1422	1,1540	1,1444	1,0445	0,9936	0,9620	0,9552	0,9552
1,0	0,025	3,2847	3,4024	3,6732	4,1236	5,4542	6,2133	6,7513	4,2940	0,9537
	0,05	2,9861	3,0895	3,3253	3,7095	4,7688	5,2937	5,4658	3,1496	0,7567
	0,1	2,5956	2,6067	2,7880	3,0695	3,7244	3,9268	3,1747	1,8285	0,6437
	0,2	1,9527	2,0058	2,1186	2,2729	2,4738	2,3832	1,9238	0,9843	0,7314
	0,4	1,4436	1,4714	1,5225	1,5674	1,4779	1,3165	1,0591	1,8814	0,8814
	0,6	1,2453	1,2627	1,2886	1,2941	1,1659	1,0617	0,9616	0,9316	0,9316
	0,8	1,1528	1,1648	1,1782	1,1677	1,0562	0,9988	0,9627	0,9537	0,9557

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Table 24

$b=1$

$\bar{\Theta}$	$\bar{h}$	Values of ζ_{21} for Fr and ω								
		5	4	1,9	1,34	0,79	0,64	0,52	0,36	0,28
		0,02	0,06	0,15	0,30	0,80	1,20	2,00	4,00	7,00
0,2	0,025	2,0540	2,1238	2,2843	2,5508	3,3336	3,7753	4,0758	2,6061	0,6581
	0,05	1,9010	1,9617	2,0998	2,3241	2,9345	3,2283	3,3009	1,9320	0,5781
	0,1	1,6722	1,7189	1,8230	1,9832	2,3430	2,4402	2,2633	1,2094	0,5817
	0,2	1,4015	1,4313	1,4941	1,5776	1,6697	1,6019	1,3296	1,8359	0,7207
	0,4	1,1774	1,1927	1,2199	1,2410	1,1783	1,0866	0,9537	0,8742	0,8742
	0,6	1,0958	1,1052	1,1184	1,1184	1,0428	0,9887	0,9422	0,9313	0,9313
	0,8	1,0590	1,0654	1,0718	1,0637	1,0020	1,9743	0,9590	0,9847	0,9847
0,4	0,025	2,3447	2,4258	2,6124	2,9221	3,8303	4,3412	4,6845	2,9717	0,7172
	0,05	2,1445	2,2148	2,3749	2,6346	3,3386	3,6747	3,7498	2,1601	0,6068
	0,1	1,8474	1,9014	2,0214	2,2055	2,6151	2,7213	2,5077	1,2960	0,5685
	0,2	1,5006	1,5347	1,6065	1,7013	1,8006	1,7181	1,4029	0,8475	0,7214
	0,4	1,2184	1,2358	1,2665	1,2896	1,2145	1,1095	0,9608	0,8744	0,8744
	0,6	1,1171	1,1278	1,1425	1,1417	1,0545	0,9939	0,9430	0,9318	0,9318
	0,8	1,0719	1,0792	1,0861	1,0763	1,0061	0,9756	0,9591	0,9591	0,9591
0,6	0,025	2,6353	2,7279	2,9406	3,2935	4,3270	4,9070	5,2933	3,3373	0,7763
	0,05	2,3879	2,4680	2,6501	2,9451	3,7427	4,1211	4,1988	2,3883	0,6355
	0,1	2,0227	2,0838	2,2197	2,4278	2,8872	3,0023	2,7521	1,3826	0,5952
	0,2	1,5996	1,6382	1,7189	1,8250	1,9315	1,8344	1,4762	0,8591	0,7220
	0,4	1,2593	1,2788	1,3131	1,3381	1,2507	1,1324	0,9679	0,8746	0,8746
	0,6	1,1384	1,1504	1,1667	1,1650	1,0661	0,9990	0,9437	0,9323	0,9323
	0,8	1,0848	1,0929	1,1005	1,0889	1,0103	0,9768	0,9592	0,9592	0,9592
0,8	0,025	2,9260	3,0299	3,2687	3,6648	4,8237	5,4729	5,9020	3,7029	0,8354
	0,05	2,6314	2,7211	2,9252	3,2556	4,1468	4,5675	4,6477	2,6165	0,6642
	0,1	2,1980	2,2663	2,4181	2,6502	3,1593	3,2833	2,9965	1,4692	0,6019
	0,2	1,6937	1,7416	1,8313	1,9487	2,0623	1,9507	1,5495	0,8707	0,7227
	0,4	1,3062	1,3219	1,3596	1,3866	1,2869	1,1553	0,9750	0,8748	0,8748
	0,6	1,1598	1,1731	1,1908	1,1883	1,0778	1,0042	0,9445	0,9328	0,9328
	0,8	1,0977	1,1067	1,1148	1,1015	1,0143	0,9781	0,9593	0,9593	0,9593
1,0	0,025	3,2167	3,3320	3,5968	4,0361	5,3204	6,0387	6,5107	4,0685	0,8944
	0,05	2,8749	2,9743	3,2003	3,5660	4,5509	5,0139	5,0967	2,8147	0,6929
	0,1	2,3733	2,4488	2,6164	2,8725	3,4314	3,5643	3,2409	1,5558	0,6087
	0,2	1,7977	1,8450	1,9437	2,0723	2,1932	2,0669	1,6228	0,8823	0,7233
	0,4	1,3412	1,3649	1,4062	1,4352	1,3231	1,1782	0,9821	0,8750	0,8750
	0,6	1,1811	1,1957	1,2150	1,2116	1,0894	1,0093	0,9452	0,9333	0,9333
	0,8	1,1106	1,1204	1,1291	1,1142	1,0185	0,9793	0,9594	0,9594	0,9594

Table 25

$b = 1.2$

$\bar{\Phi}$	$\bar{h}$	Values of ξ_{21} for Fr and ω								
		5	4	1,9	1,34	0,79	0,64	0,52	0,36	0,28
		0,0199	0,0599	0,1499	0,2999	0,7999	1,1999	1,9992	3,9999	6,9999
0,2	0,025	2,0584	2,1284	2,2893	2,5565	3,3423	3,7867	4,0914	2,6206	0,6619
	0,05	1,9081	1,9690	2,1078	2,3333	2,9184	3,2462	3,3242	1,9508	0,5818
	0,1	1,6816	1,7287	1,8336	1,9954	2,3612	2,4625	2,2893	1,2247	0,5825
	0,2	1,4106	1,4408	1,5043	1,5894	1,6859	1,6198	1,3458	0,8406	0,7225
	0,4	1,1831	1,1985	1,2262	1,2482	1,1864	1,0934	0,9570	0,8744	0,8744
	0,6	1,0991	1,1087	1,1222	1,1227	1,0465	0,9910	0,9428	0,9312	0,9312
	0,8	1,0612	1,0677	1,0743	1,0664	1,0037	1,9750	0,9591	0,9571	0,9571
0,4	0,025	2,3536	2,4351	2,6224	2,9336	3,8478	4,3640	4,7158	3,0007	0,7248
	0,05	2,1587	2,2296	2,3910	2,6530	3,3665	3,7105	3,7966	2,1978	0,6144
	0,1	1,8664	1,9210	2,0427	2,2300	2,6514	2,7657	2,5597	1,3258	0,5900
	0,2	1,5188	1,5537	1,6270	1,7249	1,8331	1,7540	1,4352	0,8569	0,7249
	0,4	1,2296	1,2474	1,2792	1,3040	1,2307	1,1231	0,9674	0,8748	0,8748
	0,6	1,1238	1,1349	1,1503	1,1503	1,0618	0,9984	0,9441	0,9316	0,9316
	0,8	1,0762	1,0837	1,0912	1,0818	1,0095	0,9770	0,9593	0,9611	0,9611
0,6	0,025	2,6487	2,7417	2,9555	3,3106	4,3532	4,9412	5,3402	3,3808	0,7876
	0,05	2,4093	2,4902	2,6741	2,9727	3,7846	4,1747	4,2689	2,4448	0,6469
	0,1	2,0511	2,1133	2,2517	2,4646	2,9416	3,0689	2,8302	1,4288	0,5976
	0,2	1,6270	1,6666	1,7497	1,8604	1,9802	1,8882	1,5247	0,8732	0,7273
	0,4	1,2761	1,2963	1,3322	1,3598	1,2749	1,1528	0,9778	0,8752	0,8752
	0,6	1,1485	1,1610	1,1783	1,1779	1,0771	1,0058	0,9454	0,9321	0,9321
	0,8	1,0913	1,0998	1,1080	1,0971	1,0152	0,9790	0,9595	0,9651	0,9651
0,8	0,025	2,9438	3,0484	3,2887	3,6877	4,8586	5,5186	5,9645	3,7608	0,8506
	0,05	2,6600	2,7507	2,9573	3,2924	4,2027	4,6390	4,7412	2,6917	0,6794
	0,1	2,2358	2,3055	2,4607	2,6991	3,2318	3,3722	3,1006	1,5308	0,6052
	0,2	1,7351	1,7794	1,8724	1,9958	2,1273	2,0223	1,6141	0,8895	0,7298
	0,4	1,3226	1,3452	1,3851	1,4155	1,3192	1,1825	0,9882	0,8755	0,8755
	0,6	1,1733	1,1871	1,2063	1,2056	1,0924	1,0132	0,9467	0,9325	0,9325
	0,8	1,1064	1,1158	1,1249	1,1124	1,0210	0,9810	0,9557	0,9690	0,9690
1,0	0,025	3,2389	3,3550	3,6218	4,0647	5,3641	6,0957	6,5889	4,1409	0,9135
	0,05	2,9106	3,0113	3,2404	3,6121	4,6208	5,1033	5,2135	2,9397	0,7119
	0,1	2,4206	2,4978	2,6697	2,9337	3,5220	3,6754	3,3711	1,6328	0,6127
	0,2	1,8433	1,8923	1,9951	2,1313	2,2745	2,1565	1,7036	0,9057	0,7322
	0,4	1,3692	1,3941	0,4381	1,4713	1,3634	1,2122	0,9986	0,8759	0,8759
	0,6	1,1980	1,2133	1,2344	1,2332	1,1077	1,0206	0,9480	0,9329	0,9329
	0,8	1,1214	1,1318	1,1418	1,127	1,0268	0,9830	0,9598	0,9730	0,9730

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Table 26

$b = 1,4$

$\bar{\Phi}$	$\bar{h}$	Values of ζ_{21} for Fr and ω								
		5	4	1,9	1,34	0,79	0,64	0,52	0,36	0,28
		0,02	0,06	0,15	0,30	0,80	1,20	2,00	4,00	7,00
0	0,025	1,7633	1,8218	1,9562	2,1795	2,8369	3,2095	3,4671	2,2406	0,5990
	0,05	1,6575	1,7085	1,8247	2,0136	2,5304	2,7819	2,8519	1,7033	0,5493
	0,1	1,4969	1,5364	1,6246	1,7608	2,0709	2,1592	2,0188	1,1227	0,5749
	0,2	1,3024	1,3279	1,3816	1,4540	1,5388	1,4856	1,2564	0,8244	0,7201
	0,4	1,1365	1,1496	1,1733	1,1925	1,1421	1,0637	0,9466	0,8740	0,8740
	0,6	1,0744	1,0826	1,0942	1,0950	1,0312	0,9836	0,9415	0,9308	0,9308
	0,8	1,0461	1,0517	1,0574	1,0511	0,9979	0,9739	0,9589	0,9532	0,9532
0,2	0,025	2,0630	2,1331	2,2944	2,5624	3,3513	3,7984	4,1075	2,6356	0,6655
	0,05	1,9155	1,9767	2,1161	2,3429	2,9629	3,2648	3,3488	1,9711	0,5859
	0,1	1,6917	1,7392	1,8451	2,0085	2,3806	2,4866	2,3181	1,2428	0,5860
	0,2	1,4209	1,4514	1,5159	1,6027	1,7045	1,6407	1,3656	0,8472	0,7236
	0,4	1,1808	1,2055	1,2338	1,2569	1,1965	1,1024	0,9619	0,8743	0,8743
	0,6	1,1033	1,1131	1,1241	1,1281	1,0514	0,9943	0,9437	0,9310	0,9310
	0,8	1,0639	1,0706	1,0775	1,0699	1,0061	0,9762	0,9593	0,9547	0,9547

Table 26 (cont.)

$b = 1,42$

$\overline{\Phi}$	$\overline{h}$	Values of ξ_{21} for Fr and ω								
		5	4	1,9	1,34	0,79	0,64	0,52	0,36	0,28
		0,02	0,06	0,15	0,30	0,80	1,20	2,00	4,00	7,00
0,4	0,025	2,3626	2,4445	2,6326	2,9453	3,8656	4,3873	4,7479	3,0307	0,7320
	0,05	2,1736	2,2450	2,4077	2,6722	3,3956	3,7477	3,8457	2,2383	0,6226
	0,1	1,8866	1,9420	2,0655	2,2562	2,6904	2,8139	2,6174	1,3628	0,5971
	0,2	1,5393	1,5760	1,6502	1,7515	1,8702	1,7958	1,4748	0,8701	0,7270
	0,4	1,2430	1,2614	1,2945	1,3214	1,2509	1,1410	0,9771	0,8755	0,8755
	0,6	1,1322	1,1436	1,1599	1,1611	1,0717	1,0050	0,9460	0,9312	0,9312
	0,8	1,0317	1,0895	1,0975	1,0887	1,0142	0,9794	0,9597	0,9561	0,9561
0,6	0,025	2,6623	2,7558	2,9708	3,3281	4,3799	4,9761	5,3883	3,4258	0,7985
	0,05	2,4316	2,5132	2,6991	3,0014	3,8281	4,2306	4,3426	2,5056	0,6592
	0,1	2,0815	2,1448	2,2359	2,5039	3,0000	3,1413	2,9166	1,4829	0,6081
	0,2	1,6578	1,6985	1,7845	1,9003	2,0359	1,9568	1,5841	0,8929	0,7305
	0,4	1,2963	1,3173	1,3551	1,3858	1,3053	1,1797	0,9924	0,8763	0,8763
	0,6	1,1611	1,1741	1,1927	1,1941	1,0919	1,0157	0,9483	0,9314	0,9314
	0,8	1,0995	1,1084	1,1175	1,1075	1,0224	0,9826	0,9601	0,9576	0,9576
0,8	0,025	2,9619	3,0671	3,3090	3,7110	4,8943	5,5650	6,0287	3,8209	0,8650
	0,05	2,6896	2,7814	2,9906	3,3307	4,2607	4,7135	4,8395	2,7729	0,6958
	0,1	2,2764	2,3476	2,5063	2,7516	3,3097	3,4686	3,2159	1,6029	0,6191
	0,2	1,7752	1,8220	1,9188	2,0490	2,2016	2,1060	1,6933	0,9157	0,9157
	0,4	1,3495	1,3732	1,4157	1,4503	1,3597	1,2184	1,0077	0,8770	0,8770
	0,6	1,1900	1,2046	1,2256	1,2271	1,1121	1,0264	0,9506	0,9317	0,9317
	0,8	1,1173	1,1274	1,1377	1,1263	1,0306	0,9858	0,9604	0,9591	0,9591
1,0	0,025	3,2616	3,3785	3,6472	4,0939	5,4087	6,1539	6,6691	4,2160	0,9315
	0,05	2,9476	3,0496	3,2821	3,6599	4,6934	5,1965	5,3365	3,0402	0,7325
	0,1	2,4713	2,5504	2,7268	2,9992	3,6194	3,7959	3,5152	1,7229	0,6302
	0,2	1,8947	1,9456	2,0531	2,1978	2,3674	2,2611	1,8026	0,9386	0,7374
	0,4	1,4028	1,4290	1,4763	1,5147	1,4141	1,2571	1,0230	0,8778	0,8778
	0,6	1,2189	1,2351	1,2584	1,2601	1,1324	1,0372	0,9529	0,9319	0,9319
	0,8	1,1351	1,1463	1,1577	1,1451	1,0387	0,9890	0,9608	0,9605	0,9605

[514

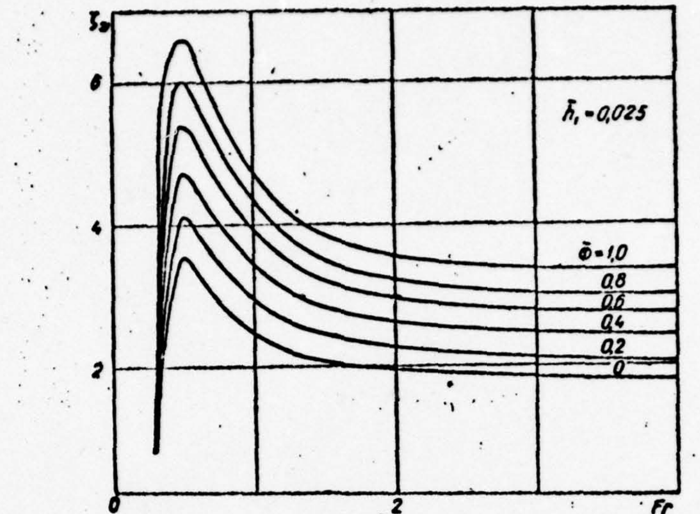

Fig. 53 [515

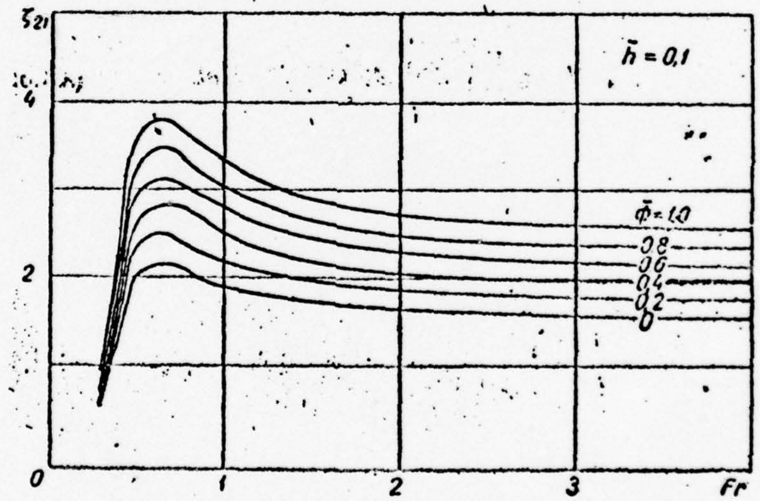

Fig. 54

point $p(0, 0, \zeta)$ in the upper half-space. [505

We will look for the velocity potentials of the produced motion in the form of

$$G_1 = G_1^0 + F(x, y, z),$$

where G_1^0 - the harmonic function in the upper half-space;

$F(x, y, z)$ - the harmonic function in the upper half-space with the exception of the point $p(0, 0, \zeta)$;

G_2 - the harmonic function in the region bounded by the surfaces $z = 0$ and $z = -h_0$.

Let us determine the function $F(x, y, z)$ from the condition

$$F_z = 0, \quad z = 0. \qquad \text{(XII.65)}$$

This condition is satisfied by the function [506

$$F = \frac{1}{r} + \frac{1}{r_1'},$$

where

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2},$$

$$r_1' = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta+2h_0)^2}.$$

From the conditions (XII.1)-(XII.3) for determining [516
functions G_1^0 and G_2, we have the system of equations

$$\left.\begin{aligned} &\bar{\varrho}\,(G^0_{1xx} - \mu G^0_{1x} + \nu G^0_{1z}) - (G_{2xx} - \mu G_{2x} + \nu G_{2z}) = N(x, \xi) \\ &\qquad\qquad z = 0. \\ &G^0_{1z} = G_{2z} \\ &G_{2z} = 0, \quad z = -h_0 \\ &N_1 = -\bar{\varrho}\,(F_{xx} + \nu F_z) \end{aligned}\right\}. \qquad \text{(XII.66)}$$

Let us use the integral expression for functions $\frac{1}{r}$ and $\frac{1}{r_1}$:

$$\frac{1}{r} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{\lambda[(z-\zeta)+i\omega]}d\lambda \qquad z-\zeta < 0,$$

$$\frac{1}{r_1} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{\lambda[-(z+\zeta)+i\omega]}d\lambda \qquad z+\zeta > 0,$$

then we obtain

$$N = \bar{\varrho}\,\frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{\lambda[-\zeta+i\omega]}\lambda^2 \cos^2\theta\, d\lambda. \qquad \text{(XII.67)}$$

Let us look for G_1^0 and G_2 in the following form:

$$G_1^0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty A(\lambda, \theta)\, e^{\lambda[-(z+\zeta)+i\omega]}d\lambda,$$

$$G_2 = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty B(\lambda, \theta)\, e^{i\lambda\omega} e^{-\lambda(h_0+\zeta)} \operatorname{ch} \lambda (z+\zeta)\, d\lambda, \tag{XII.68}$$

then from the system (XII.66) we obtain

$$B(\lambda, \theta) = \frac{\bar{\varrho}\lambda \cos^2\theta\, e^{\lambda h_0}}{[\bar{\varrho}(\lambda\cos^2\theta + \nu) + (\lambda\cos^2\theta + i\mu\cos\theta - \nu \operatorname{th} \lambda h_0) \operatorname{cth} \lambda h_0] \operatorname{sh} \lambda h_0}.$$

$$A(\lambda, \theta) = -B(\lambda, \theta) \operatorname{sh} \lambda h_0 e^{-\lambda h_0}.$$

Combining these results, we obtain [517

$$C_1(x, y, z, \xi\, \eta, \zeta) = \frac{1}{r} + \frac{1}{r'} -$$

$$-\frac{\bar{\varrho}}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty \frac{e^{\lambda[-(z+\zeta)+i\omega]}\ \lambda\cos^2\theta\, d\lambda}{[\bar{\varrho}(\lambda\cos^2\theta + \nu) + (\lambda\cos^2\theta - i\mu\cos\theta - \nu \operatorname{th}\lambda h_0) \times \operatorname{cth}\lambda h_0]}, \tag{XII.69}$$

$$C_2(x, y, z, \xi, \eta, \zeta) =$$

$$= \frac{\bar{\varrho}}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty \frac{\frac{1}{\operatorname{sh}\lambda h_0} e^{i\lambda\omega} e^{-\lambda\zeta} \operatorname{ch}\lambda(z+h_0)\, \lambda\cos^2\theta\, d\lambda}{[\bar{\varrho}(\lambda\cos^2\theta + \nu) + (\lambda\cos^2\theta + i\mu\cos\theta - \nu\operatorname{th}\lambda h_0)\operatorname{cth}\lambda h_0]}. \tag{XII.70}$$

With

$$\bar{\varrho} = 1 \quad G_1 = C_2 = \frac{1}{r} + \frac{1}{r_1},$$

where

$$r_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta+2h_0)}.$$

Specific points of the integrand expressions (XII.69) and (XII.70) will be found as the roots of the equation

$$\frac{\nu}{\cos^2\theta}\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\operatorname{th}\lambda h_0}\right)\operatorname{th}\lambda_0 \bar{h}_0 = \lambda_0. \tag{XII.71}$$

The roots will be real with

$$\frac{\nu h_0}{\cos^2\theta}\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\operatorname{th} u}\right) > 1. \tag{XII.72}$$

Separating the remainders at points λ_0, the formulas (XII.69) and (XII.70) will be expressed as follows:

$$G_1(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + \frac{1}{r'} -$$

$$-\frac{\bar{\varrho}}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{e^{[-(z+\zeta)+i\omega]\lambda_0}\lambda_0 \cos^2\theta d\lambda}{[\bar{\varrho}(\lambda\cos^2\theta + \nu) + (\lambda\cos^2\theta - \nu \operatorname{th}\lambda_0 h)\operatorname{cth}\lambda h_0]} +$$

$$+ \operatorname{Re} 2i\bar{\varrho} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{[-(z+\zeta)+i\omega]\lambda_0}\lambda_0 \cos^2\theta \operatorname{sh}\lambda_0 h_0 \operatorname{ch}\lambda_0 h_0 \, d\theta}{\begin{gathered}\operatorname{ch}^2\lambda_0 h_0 \cos^2\theta + \bar{\varrho}h_0(\lambda_0\cos^2\theta + \nu) + \\ + \bar{\varrho}\cos^2\theta \operatorname{ch}^2\lambda_0 h \operatorname{sh}\lambda_0 h - \nu h_0]\end{gathered}} \quad \text{(XII.73)}$$

$$C_2(x, y, z, \xi, \eta, \zeta) =$$ [518

$$= \frac{\bar{\varrho}}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{e^{i\lambda\omega} e^{-\lambda\zeta} \operatorname{ch}\lambda(z + h_0)\lambda\cos^2\theta d\lambda}{[\bar{\varrho}(\lambda\cos^2\theta + \nu) + (\lambda\cos^2\theta - \nu \operatorname{th}\lambda_0 h_0)\operatorname{cth}\lambda_0 h_0]} -$$

$$- \operatorname{Re} 2i\bar{\varrho} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{i\lambda_0\omega} e^{-\lambda_0\zeta} \operatorname{ch}\lambda_0(z + h_0)\lambda_0 \cos^2\theta \operatorname{sh}\lambda_0 h_0 \operatorname{ch}\lambda_0 h_0}{\begin{gathered}[\operatorname{ch}^2\lambda_0 h_0 \cos^2\theta + \bar{\varrho}h_0(\lambda_0\cos^2\theta + \nu) + \\ + \bar{\varrho}\cos^2\theta \operatorname{ch}^2\lambda_0 h_0 \operatorname{sh}\lambda_0 h_0 - \nu h_0]\end{gathered}} d\theta;$$

The integration with respect to θ in single integrals extends only to θ_1 which satisfy the conditions (XII.72).

From formulas (VII.31) and (VII.34), after the transformation, we obtain the following:

$$\theta_1 = \frac{v_0}{4\pi}\iint_s \gamma(\theta)\left\{\frac{(z-\zeta)}{r^{\frac{3}{2}}} - \frac{(z+\zeta+2h_0)}{r'^{\frac{3}{2}}} -\right.$$

$$- 2\bar{\varrho}\operatorname{Re}: \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{[-(z+\zeta)+i\omega]\lambda_0}\lambda_0^2 \cos^2\theta \operatorname{sh}\lambda_0 h_0 \operatorname{ch}\lambda_0 h_0 d\theta}{\begin{gathered}[\operatorname{ch}^2\lambda_0 h_0 \cos^2\theta + \bar{\varrho}h_0(\lambda_0\cos^2\theta + \nu) + \\ + \bar{\varrho}\cos^2\theta \operatorname{ch}\lambda h_0 \operatorname{sh}\lambda h_0 - \nu h_0]\end{gathered}} +$$

$$\left. + \frac{\bar{\varrho}}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{e^{\lambda[-(z+\zeta)+i\omega]}\lambda^2\cos^2\theta}{[\bar{\varrho}(\lambda\cos^2\theta + \nu) + (\lambda\cos^2\theta - \nu \operatorname{th}\lambda_0 h_0)\operatorname{cth}\lambda h_0]} d\lambda\right\} ds, \quad \text{(XII.74)}$$

$$Q_2 = \frac{v_0}{4\pi}\iint_s \gamma(\theta) \times$$

$$\times\left(-\mathrm{Re}\,2i\bar{\varrho}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{i\lambda_0\omega}e^{-\lambda_0\zeta}\,\mathrm{ch}\,\lambda_0(z+h_0)\lambda_0^2\cos^2\theta\,\mathrm{sh}\,\lambda_0h_0\,\mathrm{ch}\,\lambda_0h_0}{[\mathrm{ch}^2\lambda_0h_0\cos^2\theta+\bar{\varrho}h_0(\lambda_0\cos^2\theta+\nu)+\bar{\varrho}\cos^2\theta\,\mathrm{ch}\,\lambda_0h_0\,\mathrm{sh}\,\lambda h_0-\nu h_0]}-\right.$$

$$\left.-\frac{\bar{\varrho}}{\pi}\int\limits_{-\pi}^{+\pi}d\theta\int\limits_0^{\infty}\frac{e^{i\lambda\omega}e^{-\lambda\zeta}\,\mathrm{ch}\,\lambda(z+h_0)\lambda_0\cos^2\theta\,d\lambda}{[\bar{\varrho}(\lambda\cos^2\theta+\nu)+(\lambda\cos^2\theta-\nu\,\mathrm{th}\,\lambda_0h_0)\,\mathrm{cth}\,\lambda_0h_0]}\right)ds, \qquad \text{(XII.75)}$$ [519

$$\varphi_1=-\frac{1}{4\pi}\iint\limits_s\gamma(\theta)\left[\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{(x-\xi)}{r}-1\right)-\right.$$

$$-\frac{(z+\zeta)}{(y-\eta)^2+(z+\zeta)^2}\left(\frac{(x-\xi)}{r_1}-1\right)\times$$

$$\times\,2\bar{\varrho}\mathrm{Re}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{[-(z+\zeta)+i\omega]\lambda_0}\lambda_0\cos^2\theta\,\mathrm{sh}\,\lambda_0h_0\,\mathrm{ch}\,\lambda_0h_0\,d\theta}{[\mathrm{ch}^2\lambda_0h_0\cos^2\theta+\bar{\varrho}h_0(\lambda_0\cos^2\theta+\nu)+\bar{\varrho}\cos^2\theta\,\mathrm{ch}\,\lambda h_0\,\mathrm{sh}\,\lambda h_0-\nu h_0]}-$$

$$\left.-\frac{\bar{\varrho}i}{\pi}\int\limits_{-\pi}^{+\pi}d\theta\int\limits_0^{\infty}\frac{e^{\lambda[-(z+\zeta)+i\omega]}\lambda\cos\theta\,d\lambda}{[\bar{\varrho}(\lambda\cos^2\theta+\nu)+(\lambda\cos^2\theta-\nu\,\mathrm{th}\,\lambda_0h_0)\,\mathrm{cth}\,\lambda h_0]}\right]ds, \qquad \text{(XII.76)}$$

$$\varphi_2=-\frac{1}{4\pi}\iint\limits_s\gamma(\theta)\left[-\mathrm{Re}2\bar{\varrho}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{e^{i\lambda_0\omega}e^{-\lambda_0\zeta}\lambda_0\,\mathrm{ch}\,\lambda_0(z+h_0)\cos^2\theta\,\mathrm{sh}\,\lambda_0h_0\,\mathrm{ch}\lambda_0h}{[\mathrm{ch}^2\lambda_0h_0\cos^2\theta+\bar{\varrho}h_0(\lambda_0\cos^2\theta+\nu)+\bar{\varrho}\cos^2\theta\,\mathrm{ch}\,\lambda_0h_0\,\mathrm{sh}\,\lambda_0h_0-\nu h_0]}\times\right.$$

$$\left.\times\,d\theta-\frac{\bar{\varrho}i}{\pi}\int\limits_{-\pi}^{+\pi}d\theta\int\limits_0^{\infty}\frac{e^{i\lambda\omega}e^{-\lambda\zeta}\,\mathrm{ch}\,\lambda(z+h_0)\lambda\cos\theta\,d\lambda}{[\bar{\varrho}(\lambda\cos^2\theta+\nu)+(\lambda\cos^2\theta-\nu\,\mathrm{th}\,\lambda_0h_0)\,\mathrm{cth}\,\lambda_0h_0]}\right]ds, \qquad \text{(XII.77)}$$

The integral equation of the lifting surface of an arbitrary shape will be in the form:

$$\frac{1}{4\pi}\iint\limits_s\gamma(\theta)\left\{\frac{\partial}{\partial z}\left[\frac{(z-\zeta)}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{(x-\xi)}{r}-1\right)-\right.\right.$$

$$-\frac{(z+\zeta)}{(y-\eta)^2+(z+\zeta)^2}\left[\frac{(x-\xi)}{r_1}-1\right]+2\bar{\varrho}\mathrm{Re}\int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}e^{[-(z+\zeta)+i\omega]\lambda_0}\times$$

$$\times \frac{\lambda_0^2 \cos^2\theta \operatorname{sh} \lambda_0 h_0 \operatorname{ch} \lambda_0 h_0}{\operatorname{ch}^2 \lambda_0 h_0 \cos^2\theta + \bar{\varrho} h_0 (\lambda_0 \cos^2\theta + \nu) + \bar{\varrho} \cos^2\theta \operatorname{ch} \lambda_0 h_0 \operatorname{sh} \lambda h_0 - \nu h_0} \times$$ [520

$$\times d\theta + \frac{\bar{\varrho} i}{\pi} \int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{e^{\lambda[-(z+\zeta)-i\omega]} \lambda^2 \cos\theta \, d\lambda}{\bar{\varrho}(\lambda \cos^2\theta + \nu) + (\lambda \cos^2\theta - \nu \operatorname{th} \lambda_0 h) \operatorname{cth} \lambda h_0} \Bigg\} ds = v_0 a.$$

$$(z = \zeta = h) \qquad \text{(XII.78)}$$

In the approximations of the Prandtl's lifting line theory the nucleus of the equation (VIII.15) will be as follows:

$$G(y - \eta) = - \frac{(\bar{y} - \bar{\eta})}{(\bar{y} - \bar{\eta})^2 + 16\bar{h}^2} +$$

$$+ 2\bar{\varrho} \operatorname{Re} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{+\lambda[-4h + i(\bar{y} - \bar{\eta}) \sin\theta]} \lambda_0^2 \cos^2\theta \times$$

$$\times \frac{\operatorname{sh} 2\lambda_0 \bar{h}_0 \operatorname{ch} 2\lambda_0 \bar{h}_0 d\theta}{[\operatorname{ch}^2 2\lambda_0 \bar{h}_0 \cos^2\theta + \bar{\varrho} 2\bar{h}_0 (\lambda_0 \cos^2\theta + \omega) + \bar{\varrho} \cos^2\theta \operatorname{ch} 2\lambda_0 \bar{h}_0 \operatorname{sh} 2\lambda_0 \bar{h}_0 - 2\omega \bar{h}_0]}, \qquad \text{(XII.79)}$$

where λ_0 is the root of the equation

$$\frac{\omega}{\cos^2\theta} \left(\frac{1 - \bar{\varrho}}{1 + \bar{\varrho} \operatorname{th} 2\lambda_0 \bar{h}_0} \right) \operatorname{th} 2\lambda_0 \bar{h}_0 = \lambda_0. \qquad \text{(XII.80)}$$

If we introduce a new variable

$$\lambda = \frac{\nu (1 - \bar{p}) \operatorname{th} \lambda \bar{h}_0}{\cos^2\theta} - \bar{\varrho} \lambda \operatorname{th} \lambda h_0,$$

$$\frac{d\lambda}{2\nu (1 - \bar{\varrho}) \operatorname{sh} \lambda h_0 \operatorname{ch} \lambda h_0 \sin\theta} =$$

$$= \frac{d\theta}{\cos^3\theta \left[\operatorname{ch}^2 \lambda h_0 + \frac{\bar{\varrho} h_0 \nu}{\cos^2\theta} + \bar{\varrho} h_0 \lambda + \bar{\varrho} \operatorname{ch} \lambda_0 h_0 \operatorname{sh} \lambda h_0 - \frac{\nu h_0}{\cos^2\theta} \right]},$$

then the expression (XII.79) can be written as follows: [521

$$G(y - \eta) = - \frac{(\bar{y} - \bar{\eta})}{(\bar{y} - \bar{\eta})^2 + 16\bar{h}^2} +$$

$$+2\varrho\int\limits_{\bar\lambda_0}^{\infty}\frac{e^{-4\lambda\bar h}\,\mathrm{th}\,2\lambda\bar h_0\sin\lambda(y-\eta)\sqrt{1-\frac{\omega\,\mathrm{th}\,2\lambda\bar h_0}{\lambda}\frac{(1-\bar\varrho)}{1+\bar\varrho\,\mathrm{th}2\lambda\bar h_0}}}{\left[1-\frac{\omega\,\mathrm{th}\,2\lambda\bar h_0}{\lambda}\left(\frac{1-\bar\varrho}{1+\bar\varrho\,\mathrm{th}\,2\lambda\bar h_0}\right)\right]1+\varrho\,\mathrm{th}\,2\lambda\bar h_0}\,d\lambda, \qquad (XII.81)$$

where $\bar\lambda_0$ is the root of the equation (XII.81) with $\cos\theta =$ $= 1$.

With $h_0 \to \infty$, the formula (XII.81) transforms into (XII.14).

With $\bar\omega \to \infty$, the function $G_\infty(y - \eta)$ tends to

$$-\frac{(\bar y-\bar\eta)}{(y-\bar\eta)^2+16\bar h^2}.$$

Thus, with small velocities, the bottom does not affect the characteristics of the upper hydrofoil. However, the depth of the fluid alters the wave formation at the interface between fluids and in another limiting case the function $G(y - \eta)$ will depend on h_0. For $\omega \to 0$, however, this function is of the order of $\theta(\bar\varrho)$:

$$G_0(\bar y-\bar\eta)=-\frac{(\bar y-\bar\eta)}{(\bar y-\bar\eta)+16\bar h^2}+2\bar\varrho\int\limits_0^{\infty}\frac{e^{-4\lambda\bar h}\,\mathrm{th}\,2\lambda\bar h_0\sin\lambda(\bar y-\bar\eta)d\lambda}{1+\bar\varrho\,\mathrm{th}2\lambda\bar h_0}. \qquad (XII.82)$$

12.5. Motion of the Lifting Surface Under the Interface Between Fluids of Different Densities and with the Lower Fluid Having a Finite Depth

This problem is solved, in general terms, in the same way as the preceding problem.

The potentials of the three-dimensional source located at the point $P(0, 0, \zeta)$ in the lower half-space will be found in the form of G_1 and

$$G_2 = G_2^0 + F_2(x, y, z), \qquad (XII.83)$$

where G_1 and G_2^0 are harmonic in the region limited by the planes $z = 0$ and $z = -h_0$; $F_2(x, y, z)$ is the harmonic function in the upper half-space.

Let us choose $F_2(x, y, z)$ according to the condition

$F_{2z} = 0$ with $z = -h_0$. Then, for determining functions G_1 and G_2^0 we will have a system of equations

$$\bar{\varrho}(G_{1xx} - \mu G_{1x} + \nu G_{1z}) - (G^0_{2xx} - \mu G^0_{2x} + \nu G^0_{2z})_{z=0} = N_2(x, y).$$
$$G_{1z} = G^0_{2z} + F_{2z}, \quad z = 0, \quad G_{2z} = 0, \quad z = -h_0, \qquad \text{(XII.84)}$$

where [522

$$F_2(x, y, z) = \frac{1}{r} + \frac{1}{r_1};$$
$$r_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta+2h_0)^2};$$
$$N_2(x, y, z) = F_{2xx} + \nu F_{2z}.$$

Using the already known integral equations we obtain

$$F_2(x, y, z) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{-\lambda h_0} e^{i\lambda\omega} e^{-\lambda z} \operatorname{ch}\lambda(\zeta + h_0)\, d\lambda,$$
$$N_2(x, y, z) = -\frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{-\lambda h_0} e^{i\lambda\omega} e^{-\lambda z} \lambda(\lambda\cos^2\theta + \nu) \operatorname{ch}\lambda(\zeta + h_0)\, d\lambda. \qquad \text{(XII.85)}$$

Let us look for the solution in the following form:

$$G_1 = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty A_1(\lambda, \Theta) e^{-\lambda h_0} e^{i\lambda\omega} e^{-\lambda z} \operatorname{ch}\lambda(\zeta + h_0)\, d\lambda, \quad \text{(XII. (XII.86)}$$

$$G_2^0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty B_2(\lambda, \theta) e^{-\lambda h_0} e^{i\lambda\omega} e^{-\lambda z} \frac{\operatorname{ch}\lambda(\zeta + h_0)\operatorname{ch}\lambda(z + h_0)}{\operatorname{ch}\lambda h_0}\, d\lambda. \qquad \text{(XII.87)}$$

The expression (XII.87) satisfies the condition $G_{2z} = 0$ $z = -h_0$, while functions $A_1(\lambda, \theta)$ and $B_2(\lambda, \theta)$ will be found from the first two conditions:

$$A(\lambda, \theta) = \frac{\lambda\cos^2\theta - \nu\operatorname{th}\lambda h_0 - (\lambda\cos^2\theta + \nu)\operatorname{th}\lambda h}{\lambda\cos^2\theta + i\mu\cos\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\cos^2\theta + \nu)\operatorname{th}\lambda h_0},$$
$$B(\lambda, \theta) = \frac{(\lambda\cos^2\theta + \nu)(1 - \bar{\varrho})}{\lambda\cos^2\theta + i\mu\cos\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda\cos^2\theta + \nu)\operatorname{th}\lambda h_0}. \qquad \text{(XII.88)}$$

and thus functions G_1 and G_2 will be as follows:

$$G_1(x, y, z, \xi, \eta, \zeta) = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{-\lambda h_0} e^{i\lambda\omega} e^{-\lambda z} \operatorname{ch}\lambda(\zeta + h_0) \times$$

$$\times \frac{\lambda \cos^2\theta(1+\operatorname{th}\lambda h)}{\lambda\cos^2\theta + i\mu\cos\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda\cos^2\theta+\nu)\operatorname{th}\lambda h_0}\,d\lambda, \qquad \text{(XII.89)}$$

$$G_2(x,y,z,\xi,\eta,\zeta) = \frac{1}{r} + \frac{1}{r_1} - \frac{(1-\bar{\varrho})}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{-\lambda h_0} e^{i\lambda\omega} \operatorname{ch}\lambda(\zeta+h_0) \times$$ [523

$$\times \frac{(\lambda\cos^2\theta+\nu)\operatorname{ch}\lambda(z+h_0)}{\operatorname{ch}\lambda h_0[\lambda\cos^2\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda\cos^2\theta+\nu)\operatorname{th}\lambda h_0]}\,d\lambda. \qquad \text{(XII.90)}$$

The specific points of the integrand expressions (XII.89) and (XII.90) will be found as the roots of the transcendental equation (XII.71).

Determining the remainders at these points we obtain

$$G_1(x,y,z,\xi,\eta,\zeta) = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{-\lambda h_0} e^{i\lambda\omega} e^{-\lambda z} \operatorname{ch}\lambda(\zeta+h_0) \times$$

$$\times \frac{\lambda\cos^2\theta(1+\operatorname{th}\lambda h_0)}{\lambda\cos^2\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda\cos^2\theta+\nu)\operatorname{th}\lambda h_0}\,d\lambda -$$

$$-\operatorname{Re}2i\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_0 h_0} e^{i\lambda\omega} e^{-\lambda_0 z}\operatorname{ch}\lambda_0(\zeta+h_0)\lambda_0\cos^2\theta(1+\operatorname{th}\lambda_0 h_0)\operatorname{ch}^2\lambda_0 h_0\,d\theta}{[\operatorname{ch}^2\lambda_0 h_0\cos^2\theta + \varrho\bar{h}_0(\lambda_0\cos^2\theta+\nu) + \bar{\varrho}\cos^2\theta\operatorname{ch}\lambda_0 h_0\operatorname{sh}\lambda_0 h_0 - \nu h_0]}, \qquad \text{(XII.91)}$$

$$G_2 = \frac{1}{r} + \frac{1}{r_1} - \frac{1-\bar{\varrho}}{\pi} \times$$

$$\times \int_{-\pi}^{+\pi} d\Theta \int_0^\infty \frac{e^{-\lambda h_0} e^{i\lambda\omega}\operatorname{ch}\lambda(\zeta+h_0)(\lambda\cos^2\theta+\nu)\operatorname{ch}\lambda(z+h_0)}{\operatorname{ch}\lambda h_0[\lambda\cos^2\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda\cos^2\theta+\nu)\operatorname{th}\lambda h_0]}\,d\lambda -$$

$$-\operatorname{Re}2(1-\bar{\varrho})i \times$$

$$\times \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_0 h_0} e^{i\lambda\omega}\operatorname{ch}\lambda_0(\zeta+h_0)(\lambda_0\cos^2\theta+\nu)\operatorname{ch}\lambda_0(z+h_0)\operatorname{ch}\lambda_0 h_0\,d\theta}{[\operatorname{ch}^2\lambda_0 h_0\cos^2\theta + \bar{\varrho}h_0(\lambda\cos^2\theta+\nu) + \bar{\varrho}\cos^2\theta\operatorname{ch}\lambda_0 h_0\operatorname{sh}\lambda_0 h_0 - \nu h_0]}. \qquad \text{(XII.92)}$$

In the case of fluids moving with different velocities, N. Ye. Kochin [55] has obtained the following formula for the potentials of the three-dimensional source: [524

$$G_1(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + \frac{1}{r_1} - \frac{2}{\pi} \int\limits_{-\pi}^{+\pi} \int\limits_{0}^{\infty} \Big\{ L(\lambda, \theta) \operatorname{ch} \lambda_0 z \operatorname{ch} \lambda \zeta \times$$

$$\times \cos[\lambda (x - \xi) \cos\theta + \lambda (y - \eta) \sin\theta] - \frac{\nu}{2\lambda (m_1 \cos^2\theta - \nu h_0)} \Big\} d\lambda d\theta +$$

$$+ \operatorname{Re} \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} [A_0(\theta) e^{\lambda_0 z} e^{\lambda(x\cos\theta + y\sin\theta)i} + \bar{A}_0(\theta) e^{-\lambda z} e^{-\lambda_0 (x\cos\theta + y\sin\theta)i}] d\theta. \qquad \text{(XII.93)}$$

$$G_2(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + \frac{1}{r_1} + \operatorname{Re} \frac{1}{\pi} \int\limits_{-\pi}^{+\pi} \int\limits_{0}^{\infty} L(\lambda, \theta) (e^{2\lambda h} - 1) \times$$

$$\times \operatorname{ch} \lambda \zeta e^{-\lambda z} e^{-\lambda[(x-\xi)\cos\theta + (y-\eta)\sin\theta]i} d\lambda d\theta -$$

$$- \operatorname{Re} \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \bar{A}_0(\theta) (e^{2\lambda_0 h} - 1) e^{-\lambda_0 z} e^{-\lambda_0 (x\cos\theta + y\sin\theta)i} d\theta \qquad \text{(XII.94)}$$

where

$$r_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2},$$

$$L(\lambda, \theta) = \frac{(m_1 - m_2)\lambda \cos^2\theta + \nu}{\vartheta(\lambda, \theta)};$$

$$\vartheta(\lambda, \theta) = (m_1 - m_2)\lambda\cos^2\theta + \nu + (\lambda\cos^2\theta - \nu) e^{2\lambda h};$$

$$m_k = \frac{\varrho_k v_{0k}^2}{\varrho_1 v_{01}^2 + \varrho_2 v_{02}^2}; \qquad \nu = \frac{g(\varrho_1 - \varrho_2)}{\varrho_1 v_{01}^2 + \varrho_2 v_{02}^2}.$$

The formulas (XII.93) and (XII.94) are satisfied by the boundary conditions of (XII.1) and (XII.2) for the arbitrarily selected function $A_0(\theta)$. In the absence of disturbances, ahead at infinity, the functions $A_0(\theta)$ will be found from the condition

$$\nabla\varphi \underset{x \to +\infty}{\to} 0$$

If $v_{01} = v_{02}$ and function $A_0(\theta)$ is properly determined, formulas (XII.93) and (XII.94) can be transformed into (XII.91) and (XII.92).

With $\bar{\varrho} = 0$, formula (XII.92) transforms into the formula for the potential of the three-dimensional source under the free surface of a fluid of finite depth, obtained by M. D. Khaskind [154] (see Ch. IX).

Let us write the expressions for G_j, θ_j, Φ_j as follows: [525

$$\Theta_1 = \frac{v_0}{4\pi}\iint\limits_s \gamma(\theta)\left(\frac{1}{\pi}\int\limits_{-\pi}^{+\pi} d\theta \int\limits_0^{\infty} e^{-\lambda h_0} e^{i\lambda\omega} e^{-\lambda z} \operatorname{sh}\lambda(\zeta+h_0) \times \right.$$

$$\times \frac{\lambda^2 \cos^2\theta (1+\operatorname{th}\lambda h_0)}{\lambda\cos^2\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda\cos^2\theta+\nu)\operatorname{th}\lambda h_0}\,d\lambda -$$

$$\left. -\operatorname{Re} 2i \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_0 h_0} e^{i\lambda_0\omega} e^{-\lambda_0 z} \operatorname{sh}\lambda_0(\zeta+h_0)\lambda_0^2\cos^2\theta(1+\operatorname{th}\lambda_0 h_0)\operatorname{ch}^2\lambda_0 h_0\,d\theta}{[\operatorname{ch}^2\lambda_0 h_0\cos^2\theta + \bar{\varrho}h_0(\lambda_0\cos^2\theta+\nu) + \bar{\varrho}\cos^2\theta\operatorname{ch}\lambda_0 h_0\operatorname{sh}\lambda_0 h_0 - \nu h_0]}\right) ds, \quad \text{(XII.95)}$$

$$\theta_2 = \frac{v_0}{4\pi}\iint\limits_s \gamma(Q)\left(\frac{z-\zeta}{r^{\frac{3}{2}}} - \frac{z+\xi+2h_0}{r_1^{\frac{3}{2}}} - \right.$$

$$-\frac{(1-\bar{\varrho})}{\pi}\int\limits_{-\pi}^{+\pi} d\theta \int\limits_0^{\infty} \frac{e^{-\lambda h_0} e^{i\lambda\omega}\lambda\operatorname{sh}(\zeta+h_0)\operatorname{ch}\lambda(z+h_0)(\lambda\cos^2\theta+\nu)}{\operatorname{ch}\lambda h_0[\lambda\cos^2\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda\cos^2\theta+\nu)\operatorname{th}\lambda h_0]}\,d\lambda +$$

$$\left. +\operatorname{Re} 2(1-\bar{\varrho})i \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_0 h_0} e^{i\lambda_0\omega}\lambda_0\operatorname{sh}\lambda_0(\zeta+h_0)\operatorname{ch}\lambda_0(z+h_0) \times (\lambda_0^2\cos^2\theta+\nu)\operatorname{ch}\lambda_0 h_0\,d\theta}{[\operatorname{ch}^2\lambda_0 h_0\cos^2\theta - \bar{\varrho}h_0(\lambda_0\cos^2\theta+\nu) + \bar{\varrho}\cos^2\theta\operatorname{ch}\lambda_0 h_0\operatorname{sh}\lambda_0 h_0 - \nu h_0]}\right). \quad \text{(XII.96)}$$

$$\varphi_1 = -\frac{1}{4\pi}\iint\limits_s \gamma(Q) \times$$

$$\times\left(-\frac{\bar{\varrho}i}{\pi}\int\limits_{-\pi}^{+\pi} d\theta \int\limits_0^{\infty} \frac{e^{-\lambda h_0} e^{i\lambda\omega} e^{-\lambda z}\operatorname{sh}\lambda(\zeta+h_0)\lambda\cos\theta(1+\operatorname{th}\lambda h_0)}{\lambda\cos^2\theta - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda\cos^2\theta+\nu)\operatorname{th}\lambda h_0}\,d\lambda - \right.$$

$$\left. -\operatorname{Re} 2 \int\limits_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_0 h_0} e^{i\lambda_0\omega} e^{-\lambda_0 z}\operatorname{sh}\lambda_0(\zeta+h_0)\lambda_0\cos^2\theta(1+\operatorname{th}\lambda_0 h_0) \times \operatorname{ch}^2\lambda_0 h_0\,d\theta}{[\operatorname{ch}^2\lambda_0 h_0\cos^2\theta + \bar{\varrho}h_0(\lambda_0\cos^2\theta+\nu) + \bar{\varrho}\cos^2\theta\operatorname{ch}\lambda_0 h_0\operatorname{sh}\lambda_0 h_0 - \nu h_0]}\right). \quad \text{(XII.97)}$$

$$\varphi_2 = -\frac{1}{4\pi}\iint\limits_s \gamma(Q)\left[\frac{z-\zeta}{(y-\eta)^2+(z-\zeta)^2}\left(\frac{x-\xi}{r}-1\right) - \right.$$ [526

$$-\frac{z+\zeta+2h_0}{(y-\eta)^2+(z+\zeta+2h_0)^2}\left(\frac{x-\xi}{r_1}-1\right) + \frac{(1-\bar{\varrho})}{\pi}\,i\int\limits_{-\pi}^{+\pi}\frac{d\theta}{\cos\theta}\times$$

$$\times \int_0^\infty \frac{e^{-\lambda h_0} e^{i\lambda\omega} \operatorname{sh} \lambda (\zeta + h_0) \operatorname{ch} \lambda (z + h_0) (\lambda \cos^2 \theta + \nu)\, d\lambda}{\operatorname{ch} \lambda h_0 (\lambda \cos^2 \theta - \nu \operatorname{th} \lambda h_0 + \bar{\varrho} (\lambda \cos^2 \theta + \nu) \operatorname{th} \lambda h_0)} +$$

$$+ \operatorname{Re} 2(1 - \bar{\varrho}) \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\begin{array}{c} e^{-\lambda_0 h_0} e^{i\lambda_0\omega} \operatorname{sh} \lambda_0 (\zeta + h_0) \operatorname{ch} \lambda_0 (z + h_0) \times \\ \times (\lambda \cos^2 \theta + \nu) \operatorname{ch} \lambda_0 h_0 \end{array}}{\begin{array}{c} [\operatorname{ch}^2 \lambda_0 h_0 \cos^2 \theta + \bar{\varrho} h_0 (\lambda \cos^2 \theta + \nu) + \\ + \bar{\varrho} \cos^2 \theta \operatorname{ch} \lambda_0 h_0 \operatorname{sh} \lambda_0 h_0 - \nu h_0] \end{array}} d\theta -$$

$$\left. - \operatorname{Re} 2 \int_0^\infty \frac{e^{-\lambda h_0} e^{i\lambda(y - \eta)} \operatorname{sh} \lambda (\zeta + h_0) \operatorname{ch} \lambda_0 (z + h_0)}{\operatorname{sh} \lambda h_0} d\lambda \right]. \qquad \text{(XII.98)}$$

With $\bar{\varrho} = 0$, formula (XII.98) transforms into the formula (IX.8). The integral equation of the problem will be as follows:

$$\frac{1}{4\pi} \iint_S \gamma(Q) \left\{ \frac{\partial}{\partial z} \left[\frac{z - \zeta}{(y - \eta)^2 + (z - \zeta)^2} \left(\frac{x - \xi}{r} - 1 \right) - \right.\right.$$

$$\left. - \frac{z + \zeta + 2h_0}{(y - \eta)^2 + (z + \zeta + 2h_0)^2} \left(\frac{x - \xi}{r_1} - 1 \right) \right] + \frac{(1 - \bar{\varrho})}{\pi} i \times$$

$$\times \int_{-\pi}^{+\pi} \frac{d\theta}{\cos \theta} \int_0^\infty \frac{e^{-\lambda h_0} e^{i\lambda\omega} \lambda \operatorname{sh} \lambda (\zeta + h_0) \operatorname{sh} \lambda (z + h_0) (\lambda \cos^2 \theta + \nu)\, d\lambda}{\operatorname{ch} \lambda h_0 [\lambda \cos^2 \theta - \nu \operatorname{th} \lambda h_0 + \bar{\varrho} (\lambda \cos^2 \theta + \nu) \operatorname{th} \lambda h_0]} +$$

$$+ \operatorname{Re} 2(1 - \bar{\varrho}) \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\begin{array}{c} e^{-\lambda_0 h_0} e^{i\lambda_0\omega} \lambda_0 \operatorname{sh} \lambda_0 (\zeta + h_0) \operatorname{sh} \lambda_0 (z + h_0) \times \\ \times (\lambda_0 \cos^2 \theta + \nu) \operatorname{ch} \lambda_0 h_0\, d\theta \end{array}}{\begin{array}{c} \operatorname{ch}^2 \lambda_0 h_0 \cos^2 \theta + \bar{\varrho} h_0 (\lambda_0 \cos^2 \theta + \nu) + \\ + \bar{\varrho} \cos^2 \theta \operatorname{ch} \lambda_0 h_0 \operatorname{sh} \lambda_0 h_0 - \nu h_0] \end{array}} -$$

$$\left. - \operatorname{Re} 2 \int_0^\infty \frac{e^{-\lambda_0 h_0} e^{i\lambda(y - \eta)} \operatorname{sh} \lambda (\zeta + h_0) \operatorname{sh} \lambda_0 (z + h_0)}{\operatorname{sh} \lambda h_0} d\lambda \right\}_{z = \zeta = -h} = v_0 \alpha. \qquad \text{(XII.99)}$$

The function $G(\bar{y} - \bar{\eta})$ of this problem in the equation (VIII.15) is as follows: [527

$$G(\bar{y} - \bar{\eta}) = - \frac{(\bar{y} - \bar{\eta})}{(\bar{y} - \bar{\eta})^2 + 16(\bar{h}_0 - \bar{h})^2} -$$

$$- 2 \int_0^\infty \frac{e^{-2\lambda \bar{h}_0} \operatorname{sh}^2 2\lambda (\bar{h}_0 - \bar{h}) \sin \lambda (\bar{y} - \bar{\eta})}{\operatorname{sh} 2\lambda \bar{h}_0} d\lambda +$$

$$+\operatorname{Re} 2(1-\bar{\varrho}) \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{e^{-\lambda_0 h_0} e^{i\lambda_0(\bar{y}-\bar{\eta})} \lambda_0 \operatorname{sh}^2 2\lambda_0(\bar{h}_0-\bar{h}) \times \times(\lambda_0 \cos^2\theta+\omega) \operatorname{ch} 2\lambda_0 h_0 d\theta}{[\operatorname{ch}^2 2\lambda_0 h_0 \cos^2\theta + 2\bar{\varrho}\bar{h}_0(\lambda_0\cos^2\theta+\omega) + +\bar{\varrho}\cos^2\theta \operatorname{ch} 2\lambda_0 h_0 \operatorname{sh} 2\lambda_0 h_0 - \omega h_0]}, \qquad \text{(XII.100)}$$

and using the new variables (XII.100),

$$G(\bar{y}-\bar{\eta}) = -\frac{(\bar{y}-\bar{\eta})}{(\bar{y}-\bar{\eta})^2+16(\bar{h}_0-\bar{h})^2} -$$

$$-2\int_0^{\infty} \frac{e^{-2\lambda\bar{h}_0}\operatorname{sh}^2 2\lambda_0(\bar{h}_0-\bar{h}) \sin\lambda(\bar{y}-\bar{\eta})}{\operatorname{sh} 2\lambda h_0} d\lambda +$$

$$+2(1-\bar{\varrho}) \int_{\bar{\lambda}_0}^{\infty} \frac{e^{-2\lambda h_0}\operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h}) \sin\lambda(\bar{y}-\bar{\eta}) \times \times \sqrt{1-\frac{\omega \operatorname{th} 2\lambda\bar{h}_0}{\lambda}\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\operatorname{th} 2\lambda\bar{h}_0}\right)}}{\operatorname{sh} 2\lambda\bar{h}_0\left[1-\frac{\omega \operatorname{th} 2\lambda\bar{h}_0}{\lambda}\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\operatorname{th} 2\lambda\bar{h}_0}\right)\right]} \times$$

$$\times\left[1+\operatorname{th} 2\lambda\bar{h}_0\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\operatorname{th} 2\lambda\bar{h}_0}\right)\right] d\lambda. \qquad \text{(XII.101)}$$

With $\omega \to \infty$, formulas (XII.100) and (XII.101) transform into the formula (IX.201): with $\omega \to 0$ the function $G(\bar{y} - \bar{\eta})$ approaches the value of

$$G(\bar{y}-\bar{\eta}) = -\frac{(\bar{y}-\bar{\eta})}{(\bar{y}-\bar{\eta})^2+16(\bar{h}_0-\bar{h})^2} +$$

$$+2\int_0^{\infty} \frac{e^{-2\lambda h_0}\operatorname{sh}^2 2\lambda(\bar{h}_0-\bar{h})\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\operatorname{th} 2\lambda\bar{h}_0}\right)}{\operatorname{ch} 2\lambda\bar{h}_0} \sin\lambda(\bar{y}-\bar{\eta})\, d\lambda.$$

12.6. Motion of a System of Hydrofoils Near the Interface Between Fluids of Different Densities, with the Lower Fluid Having a Finite Depth [528

The problem of interaction of hydrofoils can be extended to include the system of hydrofoils moving near the interface of fluids, with the lower fluid having a finite depth.

The functions $G_{ij}(x, y, z, \xi, \eta, \zeta)$ in formulas (XII.35) and (XII.36) will be determined by formulas (XII.72),

(XII.73), (XII.91) and (XII.92). The system of integral equations of the problem is also given by formulas (XII.37).

The system of integro-differential equations was obtained in the following form:

$$\Gamma_1(\bar{y}) = \frac{a_{\bar{h}_1}}{2\lambda(\bar{y})_1}\left\{\alpha_1(y) - \frac{1}{2\pi}\int_{-1}^{+1}\Gamma_1'(\eta)\left[\frac{1}{\bar{y}-\bar{\eta}} + G_1(\bar{y}-\bar{h})\right]d\eta - \right.$$

$$\left. - \frac{1}{2\pi}\int_{-1}^{+1}\Gamma_2'(\eta)\, G_3(\bar{y}-\bar{\eta})\, d\eta\right\},$$

$$\Gamma_2(\bar{y}) = \frac{a_{\bar{h}_2}}{2\lambda(\bar{y})_2}\left\{\alpha_2(y) - \frac{1}{2\pi}\int_{-1}^{+1}\Gamma_2'(\eta)\left[\frac{1}{y-\bar{\eta}} + G_2(\bar{y}-\bar{\eta})\right]d\eta - \right.$$

$$\left. - \frac{1}{2\pi}\int_{-1}^{+1}\Gamma_1'(\eta)\, G_4(\bar{y}-\bar{\eta})\, d\bar{\eta}\right\}, \qquad \text{(XII.102)}$$

where

$$G_1(\bar{y}-\bar{\eta}) = -\frac{y-\bar{\eta}}{(y-\eta)^2+16\bar{h}_1^2} +$$

$$+2\bar{\varrho}\int_{\lambda_{01}}^{\infty}\frac{e^{-4\lambda\bar{h}_1}\,\mathrm{th}\,2\lambda\bar{h}_0 \sin\lambda(\bar{y}-\bar{\eta})\times \sqrt{1-\frac{\omega}{\lambda}\,\mathrm{th}\,2\lambda\bar{h}_0\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,2\lambda\bar{h}_0}\right)}}{\left[1-\frac{\omega_1\,\mathrm{th}\,2\lambda\bar{h}_0}{\lambda}\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,2\lambda\bar{h}_0}\right)\right]\left(1+\bar{\varrho}\,\mathrm{th}2\lambda\bar{h}_0\right)}d\lambda$$

$$G_2(y-\bar{\eta}) = \frac{-(\bar{y}-\bar{\eta})}{(\bar{y}-\bar{\eta})^2+16[(\bar{h}_0-\bar{h}_2)k]^2} -$$

$$-\int_0^{\infty} e^{-2\lambda\bar{h}_0 k}\,\frac{\mathrm{sh}^2\,2\lambda k(\bar{h}_0-\bar{h}_2)\sin\lambda(\bar{y}-\bar{\eta})}{\mathrm{sh}\,2\lambda\bar{h}_0 k}\,d\lambda +$$

$$+\frac{2(1-\varrho)\int_{\lambda_{02}}^{\infty} e^{-2\lambda\bar{h}_0 k}\,\mathrm{sh}^2\,2\lambda k(\bar{h}_0-\bar{h}_2)\left[1+\mathrm{th}\,2\lambda k\bar{h}_0\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,2\lambda h_0}\right)\right]}{\mathrm{sh}\,2\lambda\bar{h}_0 k\left[1-\frac{\omega_2\,\mathrm{th}\,2\lambda_0 h_0 k}{\lambda}\times\left(\frac{1-\varrho}{1+\bar{\varrho}\,\mathrm{th}\,2\lambda\,\bar{h}_0 k}\right)\right]}\times$$

$$\times\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\frac{\omega_2\,\mathrm{th}\,2\lambda\bar{h}_0 k(1-\bar{\varrho})}{\lambda(1+\bar{\varrho}\,\mathrm{th}\,2\lambda\bar{h}_0)}}\,d\lambda, \qquad [529$$

$$G_3(\bar{y}-\bar{\eta}) = 2\int\limits_{\lambda_0}^{\infty} \frac{e^{-2\lambda(\bar{h}_0+\bar{h}_1)k}\,\mathrm{sh}\,2\lambda k(\bar{h}_0-\bar{h}_2)(1+\mathrm{th}\,2\lambda\bar{h}_0 k)\times \times\sin\lambda(\bar{y}-\bar{\eta})\sqrt{1-\dfrac{\omega_2\,\mathrm{th}\,2\lambda\bar{h}_0 k(1-\bar{\varrho})}{\lambda(1+\bar{\varrho}\,\mathrm{th}\,2\lambda\bar{h}_0)}}}{(1+\bar{\varrho}\,\mathrm{th}\,2\lambda_0\bar{h}_0 k)\left(1-\dfrac{\omega_2}{\lambda}\,\mathrm{th}\,2\lambda\bar{h}_0 k\right)\times\times\left(\dfrac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,2\lambda h_0 k}\right)}\,d\lambda;$$

$$G_4(\bar{y}-\bar{\eta}) = 2\bar{\varrho}\int\limits_{\lambda_0}^{\infty} \frac{e^{-2\lambda\bar{h}_0}\,\mathrm{sh}\,2\lambda(\bar{h}_0-\bar{h}_1)\sin\lambda(\bar{y}-\bar{\eta})\times\times\sqrt{1-\dfrac{\omega_1\,\mathrm{th}\,2\lambda\bar{h}_0}{\lambda}\left(\dfrac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,2\lambda\bar{h}_0}\right)}}{\mathrm{ch}\,2\lambda\bar{h}_0\left[1-\dfrac{\omega_1\,\mathrm{th}\,2\lambda\bar{h}_0}{\lambda}\left(\dfrac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,2\lambda_0\bar{h}}\right)\right]\times\times(1+\bar{\varrho}\,\mathrm{th}\,2\lambda_0\bar{h}_0)}\,d\lambda. \quad \text{(XII.103)}$$

For the hydrofoil with the elliptical distribution of circulation in an infinite fluid, formulas (XII.47)-(XII.55) will be valid if functions $G_j(y - \eta)$ are determined from formulas (XII.103).

Functions a_{h_1} and a_{h_2} are determined from the solution of the two-dimensional motion problem for this system. We will obtain a system of integral equations for the two-dimensional problem from (XII.37). The potentials will be

$$\varphi_1 = \frac{1}{4\pi}\int\limits_{L_1}\gamma_1(\xi)\,d\xi\int\limits_{\infty}^{x}d\tau\int\limits_{-\infty}^{\infty}G_{11}(x,y,z,\xi,\eta,\zeta)\,d\eta + \frac{1}{4\pi}\int\limits_{L_2}\gamma_2(\xi)\,d\xi\int\limits_{\infty}^{x}d\tau\int\limits_{-\infty}^{+\infty}G_{21}(x,y,z,\xi,\eta,\zeta)\,d\eta, \quad \text{(XII.104)}$$

$$\varphi_2 = \frac{1}{4\pi}\int\limits_{L_2}\gamma_2(\xi)\,d\xi\int\limits_{\infty}^{x}d\tau\int\limits_{-\infty}^{+\infty}G_{12}(x,y,z,\xi,\eta,\zeta)\,d\eta + \quad [530$$

$$+\frac{1}{4\pi}\int\limits_{L_1}\gamma_1(\xi)\,d\xi\int\limits_{\infty}^{x}d\tau\int\limits_{-\infty}^{+\infty}G_{22}(x,y,z,\xi,\eta,\zeta)\,d\eta. \quad \text{(XII.105)}$$

The computations produce the following:

$$\varphi_1 = \frac{1}{2\pi}\int\limits_{L_1}\gamma_1(\xi)\left(-\mathrm{arctg}\left(\frac{x-\xi}{z-\zeta_1}\right)+\mathrm{arctg}\left(\frac{x-\xi}{z+\zeta_1}\right)-\right.$$

$$- 2\bar{\varrho}\int_0^\infty \frac{e^{\lambda[-(z+\zeta_1)}\sin\lambda(x-\xi)}{[\bar{\varrho}(\lambda+\nu)\operatorname{th}\lambda h_0 + (\lambda - \nu\operatorname{th}\lambda h_0)]\operatorname{cth}\lambda h_0}\, d\lambda +$$

$$+ 2\pi\bar{\varrho}\, \frac{e^{-\lambda_0(z+\zeta_1)}\operatorname{sh}\lambda_0 h_0 \operatorname{ch}\lambda h_0 \cos\lambda(x-\xi)}{[\operatorname{ch}^2\lambda_0 h_0 + \bar{\varrho}h_0(\lambda+\nu) + \bar{\varrho}\operatorname{ch}^2\lambda_0 h_0 \operatorname{th}\lambda_0 h_0 - \nu h_0]}\Bigg) d\xi +$$

$$+ \frac{1}{2\pi}\int_{L_2} \gamma_2(\xi)\left(-2\int_0^\infty \frac{e^{-\lambda h}e^{-\lambda z}\operatorname{sh}\lambda(\zeta+h_0)(1+\operatorname{th}\lambda h_0)\sin\lambda(x-\xi)}{\lambda - \nu\operatorname{th}\lambda h_0 + \bar{\varrho}(\lambda+\nu)\operatorname{th}\lambda h_0}\, d\lambda + \right.$$

$$\left. + 2\pi\, \frac{e^{-\lambda_0 h_0}\operatorname{sh}\lambda_0(\zeta+h_0)(1+\operatorname{th}\lambda_0 h_0)\operatorname{ch}^2\lambda_0 h_0 \cos\lambda(x-\xi)\, e^{-\lambda_0 z}}{\operatorname{ch}^2\lambda_0 h_0 + \bar{\varrho}h_0(\lambda_0+\nu) + \bar{\varrho}\operatorname{sh}\lambda_0 h_0 \operatorname{ch}\lambda_0 h - \nu h_0}\right) d\xi. \qquad \text{(XII.106)}$$

$$\varphi_2 = \frac{1}{2\pi}\int_{L_2}\gamma_2(\xi)\left[-\operatorname{arctg}\left(\frac{x-\xi}{z+\zeta_2}\right) + \operatorname{arctg}\left(\frac{x-\xi}{z+\zeta_2+2h_0}\right) + \right.$$

$$+ \int_0^\infty \frac{e^{-\lambda h_0}\operatorname{sh}\lambda(\zeta_2+h_0)\operatorname{ch}\lambda(z+h_0)(\lambda+\nu)(1-\bar{\varrho})\sin\lambda(x-\xi)}{[\bar{\varrho}(\lambda+\nu)\operatorname{th}\lambda h_0 + (\lambda-\nu\operatorname{th}\lambda h_0)]\operatorname{cth}\lambda h_0}\, d\lambda -$$

$$\left. - 2\pi\, \frac{e^{-\lambda_0 h_0}\operatorname{sh}\lambda_0(\zeta_2+h_0)\operatorname{sh}\lambda(z+h_0)(\lambda_0+\nu)\operatorname{ch}\lambda_0 h_0 \cos\lambda_0(x-\xi)}{[\operatorname{ch}^2\lambda_0 h_0 + \bar{\varrho}h_0(\lambda_0+\nu) + \bar{\varrho}\operatorname{ch}^2\lambda_0 h_0 \operatorname{th}\lambda_0 h_0 - \nu h_0]\lambda_0}\right] d\xi +$$

$$+ \int_{L_1}\gamma_1(\xi)\left(+\bar{\varrho}\int_0^\infty \frac{e^{-\lambda\zeta_1}\operatorname{ch}\lambda(z+h_0)\sin\lambda(x-\xi)\, d\lambda}{[\bar{\varrho}(\lambda+\nu)\operatorname{th}\lambda h_0 + (\lambda-\nu\operatorname{th}\lambda h_0)]\operatorname{cth}\lambda h_0} - \right.$$

$$\left. - 2\pi\, \frac{e^{-\lambda_0\zeta}\operatorname{ch}\lambda_0(z+\zeta)\operatorname{ch}\lambda_0 h_0 \cos\lambda_0(x-\xi)}{[\operatorname{ch}^2\lambda_0 h_0 + \bar{\varrho}h_0(\lambda_0+\nu) + \bar{\varrho}\operatorname{ch}^3\lambda_0 h_0 \operatorname{th}\lambda_0 h_0 - \nu h_0]}\right) d\xi. \qquad \text{(XII.107)}$$

At this point it is easy to obtain a system of integral equations [531

$$\int_{L_1}\gamma_1(\xi)\left(\frac{1}{x-\xi} - \frac{x-\xi}{(x-\xi)^2+4h_1^2} + 2\bar{\varrho}\int_0^\infty \times \right.$$

$$\times \frac{e^{-2\lambda h_1}\sin\lambda(x-\xi)\lambda\, d\lambda}{[\bar{\varrho}(\lambda+\nu)\operatorname{th}\lambda h_0 + (\lambda-\nu\operatorname{th}\lambda h_0)]\operatorname{cth}\lambda h_0} -$$

$$\left. - \frac{2\pi\bar{\varrho}\, e^{-2\lambda_0 h_1}\lambda_0 \operatorname{sh}\lambda_0 h_0 \operatorname{ch}\lambda_0 h_0 \cos\lambda_0(x-\xi)}{[\operatorname{ch}^2\lambda_0 h_0 + \bar{\varrho}h_0(\lambda_0+\nu) + \bar{\varrho}\operatorname{sh}\lambda_0 h_0 \operatorname{ch}\lambda_0 h_0 - \nu h_0]}\right) d\xi +$$

$$+ \int_{L_2}\gamma_2(\xi)\left(+2\int_0^\infty \frac{e^{-\lambda_1 h_0}e^{-\lambda h_1}\lambda \operatorname{sh}\lambda(h_0-h_2)(1+\operatorname{th}\lambda h_0)\sin\lambda(x-\xi)}{\bar{\varrho}(\lambda+\nu)\operatorname{th}\lambda h_0 + (\lambda-\nu\operatorname{th}\lambda h_0)} - \right.$$

$$\left. - 2\pi\, \frac{e^{-\lambda_0 h_0}e^{-\lambda_0 h_1}\lambda_0 \operatorname{sh}\lambda_0(h_0-h_2)(1+\operatorname{th}\lambda_0 h_0)\operatorname{ch}^2\lambda_0 h_0 \cos\lambda_0(x-\xi)}{[\operatorname{ch}^2\lambda_0 h_0 + \bar{\varrho}h_0(\lambda_0+\nu) + \bar{\varrho}\operatorname{ch}\lambda_0 h_0 \operatorname{sh}\lambda_0 h_0 - \nu h_0]}\right\} d\xi =$$

$$= -2\pi v_0 f_1'(x),$$

$$\int\limits_{L_2} \gamma_2(\xi) \left\{ \frac{1}{x-\xi} - \frac{(x-\xi)}{(x-\xi)^2 + 4(h_0-h_2)^2} + 2\int\limits_0^\infty e^{-\lambda h_0} \operatorname{sh}^2 \lambda (h_0 - h_2) \times \right.$$

$$\times \frac{(\lambda+\nu)(1-\bar{\varrho}) \sin\lambda(x-\xi)\, d\lambda}{[\bar{\varrho}(\lambda+\nu) \operatorname{th} \lambda h_0 + \lambda - \nu \operatorname{th} \lambda h_0] \operatorname{ch} \lambda h_0} -$$

$$\left. - \frac{2\pi e^{-\lambda h_0} \operatorname{sh}^2 \lambda (h_0 - h_2)(\lambda+\nu) \operatorname{ch} \lambda h_0 \cos\lambda(x-\xi)}{[\operatorname{ch}^2 \lambda_0 h_0 + \bar{\varrho} h_0 (\lambda+\nu) + \bar{\varrho} \operatorname{ch}^2 \lambda h_0 \operatorname{th} \lambda h_0 - \nu h_0]} \right\} d\xi +$$

$$+ \int\limits_{L_1} \gamma_1(\xi) \left(+ 2\bar{\varrho} \int\limits_0^\infty \frac{e^{-\lambda h_1} \operatorname{sh} \lambda (h_0 - h_2) \sin\lambda(x-\xi)\, \lambda\, d\lambda}{[\bar{\varrho}(\lambda+\nu) \operatorname{th} \lambda h_0 + (\lambda - \nu \operatorname{th} \lambda h_0)] \operatorname{ch} \lambda h_0} - \right.$$

$$\left. - \frac{2\pi e^{-\lambda h_1} \operatorname{sh} \lambda (h_0 - h_2) \operatorname{ch} \lambda h_0 \cos\lambda(x-\xi)}{[\operatorname{ch}^2 \lambda h_0 + \bar{\varrho} h_0 (\lambda+\nu) + \bar{\varrho} \operatorname{sh} \lambda h_0 \operatorname{ch} \lambda h_0 - \nu h_0]} \right) d\xi =$$

$$= -2\pi v_0 f_2'(x). \qquad \text{(XII.108)}$$

where λ_0 is the real and positive root of the equation

$$\nu \left(\frac{1-\bar{\varrho}}{1+\bar{\varrho} \operatorname{th} \lambda_0 h_0} \right) \operatorname{th} \lambda_0 h_0 = \lambda_0.$$

Precisely this system solves the problem of the system of hydrofoils moving near the interface of fluids with different densities. [532

From the system (XII.108) we can obtain a series of new special cases studied earlier.

1. The hydrofoil under the interface of fluids of different densities, with the lower layer having a finite depth:

$$\int\limits_{-a}^{+a} \gamma(\xi) \left(\frac{1}{x-\xi} - \frac{(x-\xi)}{(x-\xi)^2 + 4(h_0-h_1)^2} + 2\int\limits_0^\infty e^{-\lambda h_0} \operatorname{sh}^2 \lambda (h_0 - h_1) \times \right.$$

$$\times \frac{(\lambda+\nu)(1-\bar{\varrho}) \sin\lambda(x-\xi)}{[\bar{\varrho}(\lambda+\nu) \operatorname{th} \lambda h_0 + \lambda - \nu \operatorname{th} \lambda h_0] \operatorname{ch} \lambda h_0} d\lambda -$$

$$\left. - \frac{2\pi e^{-\lambda h_0} \operatorname{sh}^2 \lambda (h_0 - h_2)(\lambda+\nu) \operatorname{ch} \lambda h_0 \cos\lambda(x-\xi)}{(\operatorname{ch}^2 \lambda_0 h_0 + \bar{\varrho} h_0 (\lambda+\nu) + \bar{\varrho} \operatorname{ch}^2 \lambda h_0 \operatorname{th} \lambda h_0 - \nu h_0]} \right) d\xi =$$

$$= -2\pi v_0 f'(x). \qquad \text{(XII.109)}$$

With $\varrho = 0$ we obtain from this equation an equation for a hydrofoil submerged under the free surface of a fluid

of finite depth (XI.116).

2. The hydrofoil above the interface of fluids of different densities, with the lower layer having a finite depth:

$$\int_{-a}^{+a} \gamma(\xi)\left(\frac{1}{x-\xi} - \frac{(x-\xi)}{(x-\xi)^2+4h_1^2} + \right.$$

$$+\bar{\varrho}\int_0^\infty \frac{e^{-2\lambda h_1}\sin\lambda(x-\xi)\,\lambda d\lambda}{[\bar{\varrho}(\lambda+\nu)\,\mathrm{th}\,\lambda h_0 + (\lambda-\nu\,\mathrm{th}\,\lambda h_0)]\,\mathrm{cth}\,\lambda h_0} -$$

$$\left. - \frac{2\pi\bar{\varrho}e^{-2\lambda_0 h_1}\lambda_0\,\mathrm{sh}\,\lambda_0 h_0\,\mathrm{ch}\,\lambda_0 h_0 \cos\lambda_0(x-\xi)}{[\mathrm{ch}^2\lambda_0 h_0 + \bar{\varrho}h_0(\lambda_0+\nu) + \bar{\varrho}\,\mathrm{sh}^2\lambda_0 h_0\,\mathrm{ch}\,\lambda_0 h_0 - \nu h_0]}\right)d\xi =$$

$$= -2\pi v_0 f'(x). \qquad \text{(XII.110)}$$

For $h_0 \to \infty$ the equation (XII.110) will describe a hydrofoil moving above the interface of fluids with the lower layer of infinite depth:

$$\int_{-a}^{+a} \gamma(\xi)\left(\frac{1}{x-\xi} - \frac{x-\xi}{(x-\xi)^2+4h_1^2} + \frac{2\bar{\varrho}}{1+\bar{\varrho}}\int_0^\infty \frac{e^{-2\lambda h_1}\sin\lambda(x-\xi)\lambda}{(\lambda-\bar{\nu})}d\lambda - \right.$$

$$\left. - 2\pi\bar{\nu}e^{-\bar{\nu}h_1}\cdot\frac{\bar{\varrho}}{1+\bar{\varrho}}\cos\bar{\nu}(x-\xi)\right)d\xi = -2\pi v_0 f'(x). \qquad \text{(XII.111)}$$

3. The system of foils near the interface of fluids with different densities ($h_0 \to \infty$): [533

$$\int_{L_1} \gamma_1(\xi)\left(\frac{1}{x-\xi} - \frac{(x-\xi)}{(x-\xi)^2+4h_1^2} + \frac{2\bar{\varrho}}{1+\bar{\varrho}}\int_0^\infty \frac{e^{-2\lambda h_1}\sin\lambda(x-\xi)\lambda}{\lambda-\bar{\nu}}d\lambda - \right.$$

$$\left. - 2\pi\frac{\bar{\varrho}}{1+\bar{\varrho}}e^{-2\bar{\nu}h_1}\cos\bar{\nu}(x-\xi)\right)d\xi +$$

$$+ \int_{L_2} \gamma_2(\xi)\left(\frac{2}{1+\bar{\varrho}}\int_0^\infty \frac{e^{-\lambda(h_1+h_2)}\lambda\sin\lambda(x-\xi)}{\lambda-\bar{\nu}}d\lambda - \right.$$

$$\left. - \frac{2\pi\bar{\nu}}{1+\bar{\varrho}}e^{-\bar{\nu}(h_1+h_2)}\cos\nu(x-\xi)\right)d\xi = -2\pi v_0 f_1'(x). \qquad \text{(XII.112)}$$

$$\int_{L_2} \gamma_1(\xi)\left(\frac{1}{x-\xi} + a\int_0^\infty e^{-2\lambda h_2}\frac{\lambda+\nu}{\lambda-\nu}\sin\lambda(x-\xi)\,d\lambda - \right.$$

$$-2\pi\frac{\bar{\nu}}{1+\bar{\varrho}}e^{-2\bar{\nu}h_1}\cos\bar{\nu}(x-\xi)\Bigg)d\xi+$$

$$+\int_{L_1}\gamma_2(\xi)\Bigg\{\gamma_2(\xi)\Bigg(\frac{2\bar{\varrho}}{1+\bar{\varrho}}\int_0^{\infty}\frac{e^{-\lambda(h_1+h_2)}\lambda\sin\lambda(x-\xi)}{\lambda-\bar{\nu}}d\lambda-$$

$$-2\pi\frac{\bar{\varrho}}{1+\bar{\varrho}}\bar{\nu}e^{-\nu(h_1+h_2)}\cos\nu(x-\xi)\Bigg)d\xi=-2\pi v_0 f_2'(x). \qquad \text{(XII.113)}$$

With this we will conclude our study of the steady-state motion of lifting systems near the interface of fluids with different densities. For solving the systems of integral equations derived we can use the methods discussed in Ch. I and II.

12.7. The Potentials of Moving and Pulsating Three-Dimensional Sources Near the Interface of Fluids with Different Densities

The velocity potentials of a moving and pulsating source, located at point $p(0, 0, \zeta)$ in the upper half-space, may, as in the case of the steady-state motion, be found in the following form:

$G_1(x, y, z) = G_1^0(x, y, z) + F_1(x, y, z)$ and $G_2(x, y, z)$, [534

where $G_1^0(x, y, z)$ and $G_2(x, y, z)$ are the harmonic functions in the upper half-space; $F_1(x, y, z)$ is the harmonic function in the upper half-space, with the exception of point $p(0, 0, \zeta)$.

Selecting $F_1(x, y, z)$ from the expression $F_{1z} = 0$ with $z = 0$ we obtain

$$F_1(x, y, z) = \frac{1}{r} + \frac{1}{r_1};$$

From the condition at the interface we have the following:

$$\bar{\varrho}\left[G_{1z}^0-2i\tau_0(1-i\beta)G_{1x}^0-\nu_1(1-2i\beta)G_1^0+\frac{\tau_1^2}{\nu_1}G_{1xx}^0\right]-$$

$$-\left[G_{2z}-2i\tau_0(1-i\beta)G_{2x}-\nu_1(1-2i\beta)G_2+\frac{\tau_0^2}{\nu_1}\tau_{2xx}\right]=$$

$$= \bar{\varrho} N_1(x, y, 0), \qquad \text{(XII.114)}$$

$$G^0_{1z} = G_{2z},$$

where

$$N_1(x, y, 0) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \int_{0}^{\infty} e^{\lambda[-h+i\omega]} \left[-\frac{\tau_0^2}{\nu_1} \lambda^2 \cos^2\theta + 2\tau_0 \lambda \cos\theta + \lambda - \nu \right] d\lambda.$$

From the expression (XII.114)

$$G_1(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + \frac{1}{r_1} + \frac{\bar{\varrho}}{\pi(1-\bar{\varrho})} \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} e^{\lambda[-(z-\zeta)+i\omega]} \times$$

$$\times \frac{\frac{\tau^2}{\nu_1} \lambda^2 \cos^2\theta - 2\tau_0 \cos\theta\lambda + \nu_1}{\frac{\tau^2}{\nu_1} \lambda^2 \cos^2\theta - 2\tau_1 (1-i\beta)\cos\theta - \lambda a + \nu_1(1-2i\beta)} d\lambda. \qquad \text{(XII.115)}$$

$$G_2(x, y, z, \xi, \eta, \zeta) = -\frac{\bar{\varrho}}{\pi(1+\bar{\varrho})} \times$$

$$\times \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} \frac{e^{\lambda[z+\zeta+i\omega]} \left(\frac{\tau_0^2}{\nu} \lambda_1^2 \cos^2\theta - 2\tau_0 \cos\theta\lambda + \lambda + \nu_1 \right) d\lambda}{\frac{\tau_0^2}{\nu_1} \lambda^2 \cos^2\theta - 2\tau_1 (1-i\beta)\cos\theta\lambda - \lambda a + \nu_1(1-2i\beta)}. \qquad \text{(XII.116)}$$

For the three-dimensional source below the interface of fluids, the potentials G_1 and G_2 were obtained as follows: [535

$$G_1(x, y, z, \xi, \eta, \zeta) = \frac{1}{\pi(1+\bar{\varrho})} \times$$

$$\times \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} \frac{e^{\lambda[-(z-\zeta)+i\omega]} \frac{\tau_0^2}{\nu} \lambda^2 \cos^2\theta - 2\tau_0 \cos\theta\lambda + \nu_1 + \lambda}{\frac{\tau_0^2}{\nu_1} \lambda^2 \cos^2\theta - 2\tau_0 (1-2i\beta)\cos\theta\lambda - \lambda a + \nu_1(1+2i\beta)} d\lambda, \qquad \text{(XII.117)}$$

$$G_2(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + \frac{1}{r_1} - \frac{a}{\pi} \times$$

$$\times \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} \frac{e^{\lambda[z+\zeta+i\omega]} \frac{\tau_0^2}{\nu_1} \lambda^2 \cos^2\theta - 2\tau_0 \cos\theta\lambda + \nu_1 + \lambda}{\frac{\tau_0^2}{\nu_1} \lambda^2 \cos^2\theta - 2\tau_0 (1-i\beta)\cos\theta\lambda - \lambda a + \nu_1(1-2i\beta)} d\lambda. \qquad \text{(XII.118)}$$

The specific points of the expressions (XII.114)-(XII.118), located on the positive side of the real axis, are

determined as the roots of the equation

$$\frac{\tau_0}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0(1-i\beta)\cos\theta\lambda - \lambda a + \nu_1(1-2i\beta) = 0. \quad \text{(XII.119)}$$

These roots are in the form

$$\lambda'_{12} = \nu\frac{a + 2\tau\cos\theta(1-i\beta) \pm \sqrt{a^2 + 4\tau_0\cos\theta a - 4\tau_0^2\beta^2\cos^2\theta a - 4\tau_0 i\beta\cos\theta a}}{2\tau_0^2\cos^2\theta}$$

or more accurately

$$\lambda'_{12} = \lambda_{12} - \frac{\nu_1\beta_i}{\tau\cos\theta}\frac{\sqrt{a^2 + 4\tau_0\cos\theta a} \pm a}{\sqrt{a^2 + 4\tau_0\cos\theta a}},$$

$$\lambda_{12} = \frac{\nu_1}{2\tau_0^2\cos^2\theta}(a + 2\tau\cos\theta \pm \sqrt{a^2 + 4\tau\cos\theta a}), \quad \text{(XII.120)}$$

from which

$$\operatorname{Sign}\operatorname{Im}\lambda'_1 = -\operatorname{Sign}\cos\theta, \quad \operatorname{Sign}\lambda'_2 = -1.$$

With $\cos\theta > 0$, the roots $\lambda_{1,2}$ will be real for all values of τ_0, while with $\cos\theta < 0$ the roots $\lambda_{1,2}$ will be real only for $\tau_0 < \frac{a}{4}$. [536

With $\cos\theta < 0$, it is necessary to bypass the specific point λ from above along the path L_1, while with $\cos\theta < 0$ from below along the path L_2. The specific point λ_2 is bypassed from above along the path L_1 with both positive and negative values of $\cos\theta$. By separating the remainders we can transform the formulas (XII.115)-(XII.118) into another form.

With $k = 0$, formulas (XII.117), (XII.118), (XII.91) and (XII.92) are congruent. In determining the velocity potentials for the moving and pulsating source near the interface of two fluids with different densities and with the lower layer having a finite depth, it is necessary to use the boundary condition at the bottom. The potentials G_1 and G_2 for the source above the interface of fluids can also be found in the form of (XII.113), where functions $G_1^0(x, y, z)$ and $G_2(x, y, z)$ will be harmonic in the region bounded by the planes $z = 0$ and $z = h$.

The function $F_1(x, y, z)$ can be selected from the expression $F_{1z} = 0$, $z = 0$ or $F_{1z} = 0$, $z = -h_0$. Let us take the function F_{1z} from the condition $F_{1z} = 0$, $z = -h_0$. If one uses the integral expression for the function above and takes $\frac{1}{r}$ in the form of $\frac{1}{r_1}$, then the F_1 function will be as follows:

$$F_1(x, y, z) = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{-\lambda h} e^{-\lambda\zeta} \operatorname{ch}\lambda(z+h_0)\, e^{\lambda i\omega}\, d\lambda.$$

The function $N(x, y, \theta)$ in the relationship (XII.114) will be determined by the formula

$$N = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty e^{-\lambda(\zeta+h_0)} e^{\lambda i\omega} \operatorname{ch}\lambda(z+h_0) \times$$
$$\times\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0\lambda\cos\theta + \operatorname{th}\lambda h_0\lambda + \nu_1\right)d\lambda.$$

Let us look for the solution in the following form:

$$G_1^0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty A(\lambda, \theta)\, e^{\lambda[-(z+\zeta)+i\omega]}\, d\lambda\, d\theta,$$

$$G_2^0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^\infty B(\lambda, \theta)\, e^{-\lambda(h_0+\zeta)} e^{\lambda i\omega} \operatorname{ch}\lambda(z+h_0)\, d\lambda.$$

and in the regular way we obtain for the functions $A(\lambda, \theta)$ [537
and $B(\lambda, \theta)$ the following:

$$A(\lambda, \theta) = \frac{e^{-\lambda h_0}\operatorname{sh}\lambda h_0\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau\lambda\cos\theta + \nu_1 - \lambda\operatorname{th}\lambda h_0\right)(1-\bar{\varrho})}{\left[\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0(1-i\beta)\lambda\cos\theta + \nu_1(1-2i\beta)\right]\times (1-\bar{\varrho}\operatorname{th}\lambda h_0) - \lambda(1-\bar{\varrho})\operatorname{th}\lambda h_0},$$

$$B(\lambda, \theta) = \frac{\bar{\varrho}\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0\lambda\cos\theta + \nu_1\right)(1+\operatorname{th}\lambda h_0)}{\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0(1-i\beta)\lambda\cos\theta + \nu_1(1-2i\beta)\right]\times (1+\bar{\varrho}\operatorname{th}\lambda h_0) - \lambda(1-\bar{\varrho})\operatorname{th}\lambda h_0}.$$

Then, the potentials G_1 and G_2 will be in the form

$$G_1(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + \frac{1}{r_1} + \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} e^{\lambda[-(z+\zeta)+i\omega]} \times$$

$$\times \frac{e^{-\lambda h_0} \operatorname{sh} \lambda h_0 \left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0\lambda\cos\theta + \nu_1 - \lambda \operatorname{th}\lambda h_0\right)(1-\bar{\varrho})}{\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0(1-i\beta)\lambda\cos\theta + \nu_1(1-2i\beta)\right]\times (1+\bar{\varrho}\operatorname{th}\lambda h_0) - \lambda(1-\bar{\varrho})\operatorname{th}\lambda h_0}, \quad \text{(XII.121)}$$

$$G_2(x, y, z, \xi, \eta, \zeta) = \frac{\bar{\varrho}}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} e^{-\lambda(h_0+\zeta)} e^{\lambda i\omega} \operatorname{ch}\lambda(z+h_0) \times$$

$$\times \frac{\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0\lambda\cos\theta + \nu_1\right](1+\operatorname{th}\lambda h_0)}{\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0(1-i\beta)\lambda\cos\theta + \nu_1(1-2i\beta)\right]\times (1+\bar{\varrho}\operatorname{th}\lambda h_0) - \lambda(1-\bar{\varrho})\operatorname{th}\lambda h_0}, \quad \text{(XII.122)}$$

where

$$r_1 = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta+2h_0)^2}.$$

The potentials of the moving and pulsating source under the interface with a finite layer are obtained in the form: [538

$$G_1(x, y, z, \xi, \eta, \zeta) = \frac{1}{\pi}\int_{-\pi}^{+\pi} d\theta \int_0^{\infty} e^{-\lambda h_0} e^{\lambda i\omega} e^{-\lambda z} \operatorname{ch}\lambda(\zeta+h_0) \times$$

$$\times \frac{\left(\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0\cos\theta\lambda + \nu_1\right)(1+\operatorname{th}\lambda h_0)}{(1+\bar{\varrho}\operatorname{th}\lambda h_0)\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta - 2\tau_0(1-i\beta)\cos\theta\lambda + \nu_1(1-2i\beta) - \lambda\operatorname{th}\lambda h_0\frac{(1-\bar{\varrho})}{(1+\bar{\varrho}\operatorname{th}\lambda h_0)}\right]} d\lambda, \quad \text{(XII.123)}$$

$$G_2(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + \frac{1}{r_1} - \frac{1-\bar{\varrho}}{\pi} \times$$

$$\times \int_{-\pi}^{+\pi} d\theta \int_0^{\infty} e^{-\lambda h_0} e^{\lambda i\omega} \operatorname{ch}\lambda(\zeta+h_0)\frac{\operatorname{ch}\lambda(z+h_0)}{\operatorname{ch}\lambda h_0} \times$$

$$\times \frac{\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta-(2\tau_0\cos\theta-1)\lambda+\nu_1\right]}{(1+\bar{\varrho}\,\mathrm{th}\,\lambda h_0)\left[\frac{\tau_0^2}{\nu_1}\lambda^2\cos^2\theta-2\tau_0(1-i\beta)\cos\theta\lambda+\right.} d\lambda. \qquad \text{(XII.124)}$$

$$\left.+\nu_1(1-2i\beta)-\lambda\frac{(1-\bar{\varrho})}{(1+\bar{\varrho}\,\mathrm{th}\,\lambda h_0)}\mathrm{th}\,\lambda h_0\right]$$

Let us examine the roots of the equation:

$$\tau_0^2\lambda^2\cos^2\theta-\lambda\nu\left[2\tau_0(1-i\beta)+\mathrm{th}\,\lambda h_0\left(\frac{1-\bar{\varrho}}{1+\varrho\,\mathrm{th}\,\lambda h_0}\right)\right]+ \\ +\nu_1^2(1-2i\beta)=0, \qquad \text{(XII.125)}$$

$$\lambda_i=\frac{\nu_1}{2\tau_0^2\cos^2\theta}\left[\mathrm{th}\lambda_1 h_0\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,\lambda_1 h_0}\right)+2\tau(1-i\beta)\cos\theta\pm\right.$$

$$\pm\sqrt{\mathrm{th}^2\lambda_i h_0\left(\frac{1-\bar{\varrho}}{1-\bar{\varrho}\,\mathrm{th}\,\lambda_i h_0}\right)^2+4\tau_0\cos\theta\times}$$

$$\left.\overline{\times\left[\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\lambda_i h_0}\right]-4\tau i\beta\cos\theta-4\tau_0^2\beta^2\cos^2\theta}\right].$$

From here we obtain two equations for determining λ_1 and λ_2: [539

$$\lambda_1=\frac{\nu_1}{2\tau_0^2\cos^2\theta}\left[\mathrm{th}\,\lambda_1 h_0\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,\lambda_1 h_0}\right)+2\tau_0\cos\theta+\right.$$

$$\left.+\sqrt{\mathrm{th}^2\lambda_1 h_0\left[\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,\lambda_1 h_0}\right]^2+4\tau_0\cos\theta\left[\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,\lambda h_0}\right]}\right], \qquad \text{(XII.126)}$$

$$\lambda_2=\frac{\nu}{2\tau_0^2\cos^2\theta}\left[\mathrm{th}\,\lambda_2 h_0\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,\lambda_2 h_0}\right)+2\tau_0\cos\theta-\right.$$

$$\left.-\sqrt{\mathrm{th}^2\lambda_2 h_0\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,\lambda_2 h_0}\right)^2+4\tau_0\cos\theta\left(\frac{1-\bar{\varrho}}{1+\bar{\varrho}\,\mathrm{th}\,\lambda_2 h_0}\right)}\right]. \qquad \text{(XII.127)}$$

With $\cos\theta > 0$, the roots of the equations will be real and positive for all values of τ_0; when $\cos\theta < 0$ the roots will be real only for those values of τ_0 whose upper limit is given by the equation

$$\mathrm{th}\,\lambda h_0=\frac{|4\tau\cos\theta|}{(1-\bar{\varrho})(1-\bar{\varrho}|4\tau\cos\theta|)}, \qquad \text{(XII.128)}$$

and then from equations (XII.126) and (XII.127) follows the second condition for the existence of the real roots:

$$\frac{\nu h_0}{4\cos\theta}\left[\frac{(1-\bar{\varrho})-\bar{\varrho}^2|4\tau_0\cos\theta|}{(1-\bar{\varrho})+\bar{\varrho}|4\tau\cos\theta|}\right](1-\bar{\varrho})(1-\bar{\varrho}|4\tau_0\cos\theta|)>1. \quad \text{(XII.129)}$$

Using the same technique as in equation (XII.119), we obtain

$$\operatorname{Sign}\operatorname{Im}\lambda_1' = -\operatorname{Sign}\cos\theta, \ \operatorname{Sign}\lambda_2' = -1,$$

and then the direction for bypassing the specific points will be determined in the same way as in the case of the fluid of infinite depth.

The general problem of the unsteady motion of the lifting surface near the interface of fluids with different densities can also be thoroughly studied by using the methods developed in this book. From the general formulas one may obtain the expression for the potentials θ and φ, and then proceed to the general integral equations and various approximations.

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